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A DE MOIVRE FORMULA FOR FIXED POINT THEORY

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1. *Introduction*

De Moivre's formula is $e^{i\theta} = \cos\theta + i\sin\theta$. It was discovered, or perhaps defined, in the seventeenth century. But whether a definition or theorem, it contains within itself an amazing amount of information. The key to this information is the standard algebra of Complex Numbers.

In this lecture I will propose a formula relating the index of vectorfield which, like De Moivre's formula, contains a large amount of information.

Let M be a smooth manifold with boundary ∂M. Let V be a continuous vectorfield on M with no zeros on ∂M. Then

$$\text{Ind}\, V + \text{Ind}(\partial_- V) = \chi(M)$$

Here $\chi(M)$ is the Euler-Poincaré number of M, and Ind stands for the Index of a vectorfield. The vectorfield $\partial_- V$ is defined on part of the boundary ∂M as follows. Let $\partial_- M$ consist of all $m \in \partial M$ such that the vector V_m of V at m points into M. Then $\partial_- V$ is defined from V by

$$V \to V|\partial M \to \partial V \to \partial V|\partial_- M = \partial_- V$$

In words, first restrict V to the boundary ∂M; then using the normal vectorfield N on the boundary, project $V|\partial M$ onto its component tangent to ∂M. This we call ∂V. Then restrict ∂V to $\partial_- M$ to get $\partial_- V$.

The formula has a very simple proof given Hopf's result that Ind $V = \chi(M)$ on a closed manifold. A version of it was known to Morse who noted that critical points on the boundary where the gradient points into the manifold should be added into his inequalities for Morse theory for bounded manifolds.

There is a philosophical reason why this index formula should have a large number of consequences: It serves as an inductive "definition" of Index. $\partial_- V$ is defined on a space which is one dimension lower than M. Thus any result about indicies of vector fields should "follow" from the formula.

First we shall prove the formula. Then we shall show some examples of its use to prove known facts about vectorfields. Then we shall construct vectorfields in various topological situations and obtain formulas: we shall consider the case of 'immersions of manifolds with boundary into other manifolds; Actions of Lie Groups on Manifolds; and a formula relating the Euler-Poincaré Number, the Lefschetz number and the coincidence number of two maps.

2. *Proof of formula*

Let V be a vectorfield on M. Let $M_+ = M \cup (\partial M \times I)$ where $\partial M \times I$ is a collar. We extend V to V_+ by adding the outward pointing normal vector $t\, N_m$ to V_m for $t \in I$.

We choose N so large that $N_m + V_m$ always points out of ∂M_+.

Then by the theorem of Hopf

$\mathrm{Ind}(V_+) = \chi(M)$

The new zero's of V_+ created by this process arise from those vectors of V which point normally inward. The index around a zero of $\partial_- V$ is equal to the index of the zero created in V_+ since in the normal direction the index on $m \times I$ is 1 and the index at the point in V_+ is equal to the product of the index in the tangent direction and the normal direction. Thus $\mathrm{Ind}\, V_+ = \mathrm{Ind}\, V + \mathrm{Ind}(\partial_- V) \times 1 =$ $= \chi(M)$.

3. *Examples of known consequences*

a) If $\partial M = \phi$ then $\partial_- V$ is empty and $\mathrm{Ind}\, \partial_- V = \phi$.

Thus $\mathrm{Ind}\, V = 0$.

b) If V always points outward then $\partial_- V$ is empty and

$\mathrm{Ind}\, \partial_- V = 0$ and $\mathrm{Ind}\, V = \chi(M)$.

c) If V is tangent to ∂M, then $\partial_- V$ is empty and $\mathrm{Ind}\, \partial_- V = 0$ and $\mathrm{Ind}\, V = \chi(M)$.

d) If V is transverse to ∂M, then $\partial_o M = \phi$, where $\partial_o M$ is the set set of points in ∂M where the vectorfield is tangent to ∂M.

If M is even dimensional, then ∂M, is an odd dimensional closed manifold and every component has Euler-Poincaré number equal to zero. Thus $\mathrm{Ind}(\partial_- M) = \chi(\partial_- M) = 0$.

hence $\mathrm{Ind}\, V = \chi(M)$.

If M is odd dimensional, then $\chi(\partial M) = 2\chi(M)$. Now $\partial M = \partial_+ M \cup \partial_- M$ since $\partial_0 M = \phi$. So $\chi(\partial M) = \chi(\partial_+ M) + \chi(\partial_- M)$. Now since $\text{Ind}\ \partial_- V = \chi(\partial_- M)$ we see that $\text{Ind}\ V + \text{Ind}(\partial_- M) = \chi(M)$ gives $\text{Ind}\ V + \chi(\partial_- M) = \frac{1}{2}(\chi(\partial_+ M) + \chi(\partial_- M)$.

Hence $\text{Ind}\ V = \frac{1}{2}(\chi(\partial_+ M) - \chi(\partial_- M))$

e) If $M = I$ and V assigns to t the vector t, then

$\text{Ind}\ V = \chi(M) - \text{Ind}(\partial_- V) = 1-0 = 1$

$\text{Ind}(-V) = \chi(M) - \text{Ind}(\partial_- V) = 1-2 = -1$

$\text{Ind}\ C = \chi(M) - \text{Ind}(\partial_- I) = 1-1 = 0$ where C is a constant vectorfield.

f) We can prove that $\text{Ind}(-V) = (-1)^n \text{Ind}(V)$. This is true from the formula because $\chi(\partial M) = 0$ if n is even and $\chi(\partial M) = 2\chi(M)$ is n is odd. Here $n = \dim M$.

Now $\text{Ind}(\partial_-(-V)) = \text{Ind}(-\partial_+ V) = (-1)^{n-1}\text{Ind}(\partial_+ V)$

Also $\chi(\partial M) = \text{Ind}(\partial V) = \text{Ind}(\partial_+ V) + \text{Ind}(\partial_- V)$

Then $\text{Ind}(-V) = \chi(M) - \text{Ind}(\partial_-(-V)) = \chi(M) - (-1)^{n-1}\text{Ind}(\partial_+ V) =$

$$= \chi(M) + (-1)^n(\chi(\partial M) - \text{Ind}(\partial_- V))$$

If n is even $\text{Ind}(-V) = \chi(M) + (0-\text{Ind}\,\partial V) = \text{Ind}\ V$

If n is odd $\text{Ind}(-V) = \chi(M) - (2\chi(M) - \text{Ind}(\partial_- V))$

$$= -(\chi(M) - \text{Ind}(\partial_- V)) = -\text{Ind}\ V.$$

4. *Immersions*

Let $f : M^n \to R^n$ be a smooth immersion. Let $a \in R^n$ and let V be the vectorfield given by the gradient of the distance function $d_a = R^n \to R : x \to ||x-a||$. Since f is an immersion we have a pullback vectorfield f^*V on M. If we assume that $a \notin f(\partial M)$, we can apply the formula to get

$$\text{Ind}(f^*V) + \text{Ind}(\partial_- f^*V) = \chi(M)$$

Since Ind V = 1 and f is an immersion, Ind f^*V = number of points in $f^{-1}(a)$. Thus can be calculated by sending a ray out from a and counting +1 for each point where the ray intesects $f(\partial M)$ pointing out and -1 for each point where it points in. The sum we will call the winding numbers of $f(\partial M)$. The $\text{Ind}(\partial_- f^*V)$ can be shown to equal the sum of the indicies of the critical points of the function $f : \partial M \to R$ where the gradient points inward.

Thus if $f : D \to R^2$ is an immersion of the Disk D into the plane we have

$$(\text{Winding number about a of } f|\partial D) - (\# \text{ of maxima of dof on } \partial_- D) + (\# \text{ of minima of dof on } \partial_- D) = 1$$

5. $e^{\pi i}+1 = 0$

The formula in the title of this section unites the most used constants in one equation and follows from De Moivre's formula.

If $f : M \to M$ is a self map of a compact manifold M^n contained in R^n with no fixed points on ∂M, then

$$\Lambda_f + \Lambda_{-V,\gamma} = \chi(M)$$

Here the Euler-Poincaré number, the Lefschetz number Λ_f and the coincidence number are united into one formula. $\Lambda_{-V,\gamma}$ is the coincidence number of the Gauss map $\gamma : \partial M \to S^{n-1}$ and $-V: \partial M \to S^{n-1}$ is given by $m \to \frac{f(m)-m}{||f(m)-m||}$.

Proof: Let V_f be the vector field on M given by $m \to m-f(m)$. Then $\text{Ind } V_f + \text{Ind } \partial_- V_f = \chi(M)$. Now $\text{Ind } V_f = \Lambda_f$. So $\text{Ind}(\partial_- V_f) = \chi(M) - \Lambda_f = \deg \gamma - \deg V_f$. Now $\gamma : \partial M \to S^{n-1}$ is the Gauss map and let $V : \partial M \to S^{n-1}$ be the "Gauss" map for the vectorfield $V_f | \partial M$. Then $\Lambda_{V,\gamma} = \Sigma(-1)^i \text{ trace } (\gamma^! V^*) = \deg \gamma + (-1)^{n-1} \deg V$. So $\Lambda_{-V,\gamma} = \deg \gamma + (-1)^{n-1} \deg(-v) = \deg \gamma + (-1)^{n-1}(-1)^n \deg V = \deg \gamma - \deg V$. Then $\Lambda_{-V,\gamma} = \text{Ind}(\partial_- V_f)$.

6. Lie Group actions

Let G be a compact Lie Group acting smoothly on a compact manifold M^n. Every vector $v \in T_e(G) = g$ in the Lie Algebra g of G gives rise to a vector field V_v on M as follows. The vector v gives rise to a one parameter subgroup $R \xrightarrow{h} G$. Then $R \times M \xrightarrow{h \times 1} G \times M \xrightarrow{\hat{w}} M$ is an R-action $t \to \psi_t$. This gives rise to a vectorfield on M by $m \to \frac{d}{dt}(\psi_t(m))\Big|_{t=0}$.

This vectorfield has the property that $w_m \in V_v$ is a zero if and only if v is tangent to the isotropy subgroup of m at e.

THEOREM. *Let T be a torus and let S be a subtorus acting on a manifold M . Suppose that N is a submanifold of the same dimension as M and that N is invariant under S and that $\chi(N)$ is different from zero. Then either S has a fixed point on ∂M or T has a fixed point on N.*

proof: Let v be a vector tangent to S . Then the vector field V_v is tangent to the boundary of M . Assume that V_v has no zeros on the boundary of N , then

$$Ind(V_v) + 0 = \chi(M)$$

so V_v has a nonzero index. Now choose a vector w close to v and tangent to a dense subgroup R of T . Then V_v is close to V_w and so they must have equal indicies. Thus the index of V_w is non zero and so V_w has a zero in N . Hence T must have a zero in N .

COROLLARY: tr(G,M) $\neq$ 0, hence the orbit map $\omega : G \to M$ is trivial on integral homology.

Here tr(G,M) denotes the trace of the action (G,M).

It is defined in [1] and its properties are developed there.

Bibliography:

1. Daniel Henry Gottlieb, The trace of an action and the degree of a map. Transactions of the American Mathematical Society,Vol.293(1986).