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On the Index of Pullback Vector Fields

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Let V be a vector field on $\mathbb{R}^n$. Suppose that M^n is a smooth manifold of dimension n, and suppose that $f : M \to \mathbb{R}^n$ is a smooth map. We ask the question: Is there a vector field V^* on M which *lifts* V? That means that $f_*(V^*(m)) = V(f(m))$ for every $m \in M$. By f_* we mean the differential map of f induced between the tangent bundles $TM \to T\mathbb{R}^n$.

Such a lifting vector field need not exist. Its existence is related to the index of a pullback vector field f^*V. This vector field is dual to a pullback by f of the one–form on $\mathbb{R}^n$ dual to V. Stated another way, we define f^*V by using the following equation where $\langle\ ,\ \rangle$ represents some choice of Riemannian innerproduct.

$$\langle f^*V(m), v_m \rangle = \langle V(f(m)), f_*(v_m) \rangle$$

If a lifting V^* for the vector field V exists, then its index must be equal to the index of $f^*(V)$. We can calculate the index of $f^*(V)$ locally in $\mathbb{R}^n$ for the case when f has no singularities on the boundary of M. We need some notation to express the theorem. We suppose that V has isolated zeroes denoted by x_i. Then x_i has a local index v_i and a winding number w_i. The winding number w_i is defined as follows. Draw a ray from x_i to infinity. At each intersection of the ray with the image under f of the boundary ∂M there is a local inside and outside. If the ray passes from the inside to the outside add a $+1$, if the ray passes from the outside to the inside, add a -1. The sum of these ± 1 equals the winding number w_i. Next we define the normal degree $\deg \hat{N}$. For each $m \in \partial M$, there is a unit outward pointing normal N based at $f(m)$. Then the map $\hat{N} : \partial M \to S^{n-1}$ is defined by parallel translating each of these vectors to the origin. This is the famous Gauss map, or normal map. Then $\deg \hat{N}$ is simply its degree where the orientation of M and S^{n-1} are chosen consistent with the orientation of $\mathbb{R}^n$ and the notion of outward pointing normals.

THEOREM. *Let M be a compact smooth manifold with orientable boundary. Suppose f is a smooth map from M^n to $\mathbb{R}^n$ so that the singular set does not intersect the boundary and so that no zero of the vector field V lies on the image of the boundary. Then if $n > 1$,*

$$Ind\ (f^*V) = \sum w_i v_i + (\chi(M) - \deg \hat{N}).$$

Remarks. 1) There is no need for M to be orientable. Also, it can happen that a zero x_i is not in the image of f. Thus the term $\Sigma w_i v_i$ is *not* the degree of the composition of f and V at 0.

2) The boundary of M must be orientable, hence it is stablely parallelizable. This follows since f is a codimension 1 immersion of ∂M into $\mathbb{R}^n$.

3) For n odd, the dimension of ∂M is even. Hence, by a theorem of [Hopf], the normal degree of the immersion must equal half of the Euler characteristic $\chi(\partial M)$. It follows that $(\chi(M) - \deg \hat{N}) = 0$. Thus for odd dimensions the theorem becomes

$$\text{Ind } f^*(V) = \sum w_i v_i$$

4) The condition that V has isolated zeroes is not necessary. If x_i denotes a connected component of the zeroes of V, the integer v_i still will denote the index of the set x_i and w_i will denote its winding number. These still remain well–defined integers.

The zeroes of $f^*(V)$ are of two types. There are the *ordinary* zeroes, which occur at nonsingular points and the *latent* zeroes which occur on the singular set of f. The image of an ordinary zero is always a zero of the vector field V. The image of a latent zero need not be zero. In fact a latent zero occurs at m precisely when the vector $V(f(m))$ is orthogonal to the image of the tangent space at m under the derivative map f_*. If we assume that V has isolated zeroes x_i and that $f^*(V)$ has latent zeroes which indices r_j we get the following formula. Here n_i denotes the number of nonsingular points in $f^{-1}(x_i)$.

$$\text{Ind } f^*(V) = \sum n_i v_i + \sum r_j$$

Now suppose that V has no zeroes. Then

$$\text{Ind } f^*(V) = \sum r_j = \chi(M) - \deg \hat{N}$$

For odd dimensional M, we see that Ind $f^*(V) = \Sigma r_j = 0$.

If $f^*(V)$ has no zeroes then

$$\text{Ind } f^*(V) = 0 = \sum w_i v_i + (\chi(M) - \deg \hat{N})$$

In the case when both $f^*(V)$ and V have no zeroes we see that $\deg \hat{N} = \chi(M)$. This occurs when a nonzero V has a lifting V^*. As a corollary we obtain the theorem of [Haefliger] that the normal degree of the boundary equals the Euler characteristic of the manifold when the map f is an immersion.

In general when V has no zeroes on the image of the singular set, a necessary condition for the existence of a lifting V^* is

$$\sum w_i v_i + (\chi(M) - \deg \hat{N}) = \sum n_i v_i$$

For odd dimensional M we have $\Sigma w_i v_i = \Sigma n_i v_i$.

Let Δ denote $f(\Sigma)$, the image of the singular set under f. If V is a vector field which is zero on Δ, then there is an automatic lifting V^*. If Δ has a tubular neighborhood T so that $f^{-1}(T)$ is a tubular neighborhood of $f^{-1}(\Delta)$, then we can calculate Σr_j when the vectors of V are outward pointing normal to ∂T. It is simply equal to $\chi(f^{-1}(\Delta))$. Such a vector field can always be constructed in the case when $\Delta \cap f(\partial M)$ is empty, so we obtain the following result.

THEOREM. *Suppose that $\Delta \cap f(\partial M)$ is empty and there are tubular neighborhood as above. Then*

$$\chi(f^{-1}(\Delta)) = \sum w_i \chi(\Delta_i) + (-1)^n \sum (w_j - n_j)\chi(D_j) + \chi(M) - \deg \hat{N}$$

where Δ_i are the components of Δ and the D_j are the bounded components of $R^n - \Delta$. For each D_j we select a point not in $f(\partial M)$ and using this point *we calculate w_j and n_j. The numbers w_j and n_j vary as we vary the chosen point, but their difference is constant.*

These theorems follow from the following index formula. Let M be a compact manifold with or without boundary. Let V be a tangent vector field on M which has no zeroes on ∂M. Then

$$\text{Ind } V + \text{ Ind } \partial_- V = \chi(M)$$

where the vector field $\partial_- V$ is defined as follows. Let $\partial_- M$ be the open set on the boundary where V is pointing inward. Then we first consider V restricted to $\partial_- M$ and then we project each $V(m)$ onto its component $\partial_- V(m)$ tangent to $\partial_- M$.

Remarks. 1) the set of points $\partial_- M$ may very well be empty. In that case the index of $\partial_- V$ is zero. Then for closed M or for vector fields pointing out of the boundary we have the familiar fact that the index is equal to the Euler Characteristic.

2) The formula is certainly not well known. Since Marston Morse discovered it sixty years ago it has been rediscovered at least three times: [Morse], [Pugh], [Koschorke], [Gottlieb 1] and [Gottlieb 2].

3) The formula is virtually an inductive definition of index. If we assume that the index of a zero dimensional point is one, the equation will tell us how to compute the index for vector fields on one dimensional spaces. Then the one dimensional index gives the two dimensional index and so on. The only technical difficulty is that $\partial_- M$ is not a compact manifold. But despite this, every fact about the index of vector fields should follow from this equation.

Now the proof of the first theorem proceeds as follows. The fact that f is an immersion on ∂M and that there are no zeroes of V on $f(\partial M)$ allows us to define a "Gauss map" $\hat{V} : \partial M \to S^{n-1}$. The coincidence number of the Gauss maps $\hat{N}$ and $\hat{V}$ is equal to $Ind(\partial_-(f^*V))$ up to a sign. But the coincidence number in this situation is the sum of degrees and so one gets Ind $(\partial_-(f^*V)) = \deg \hat{N} - \deg \hat{V}$. The degree of $\hat{V}$ can be calculated using winding numbers and indices of zeroes since each component of ∂M is an oriented manifold which bounds. Then we insert the value of Ind $(\partial_{-}f^{*}V)$ into the index formula above.

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