LEMMA 2.
(a) $A^1_{1,1}=A^1_{1,0}=A^1_{0,0}=A^0_{0,0}=1$,
(b) $A^k_{m,m}=I^k_m$, *the* $C_{K,m}\times C_{K,m}$ *identity matrix.*

In order to simplify the hypothesis for the following recurrence formula, we shall extend the definition of $A^K_{m,n}$ for all integers K, m, n.

CONVENTIONS.

(a) Let $\varnothing$ stand for the empty matrix and let $O^K_{m,n}$ stand for the $C_{K,m}\times C_{K,n}$ matrix of all zeros. Sometimes $O^K_{m,n}$ may be abbreviated by 0.

(b) $A^K_{m,n}=\varnothing$ if $K<m$ or n; or if K, m or n is less than zero.

(c) $A^K_{m,n}=O^K_{m,n}$ if $m<n$.

It is now easy to see that the following formula is true.

LEMMA 3.

$$A^K_{m,n}=\left(\begin{array}{c|c} A^{K-1}_{m-1,n-1} & A^{K-1}_{m-1,n} \\ \hline O^{K-1}_{m,n-1} & A^{K-1}_{m,n}\end{array}\right).$$

The $A^K_{m,n}$ matrices satisfy a multiplication formula which gives a simple proof of our main result.

LEMMA 4. *Let* $K\geqq m\geqq p\geqq n\geqq 0$. *Then*

$$A^K_{m,p}A^K_{p,n}=C_{m-n,p-n}A^K_{m,n}.$$

PROOF. For $K=0$ or $K=1$ (see Lemma (2a)) the lemma is true.

Now the proof proceeds easily by induction. We multiply $A^K_{m,p}$ by $A^K_{p,n}$ using the partition formula of Lemma 3. The resulting matrix is partitioned into four parts. Three parts clearly agree with the stated formula. For the upper right hand submatrix we have

$$\begin{aligned} A^{K-1}_{m-1,p-1}A^{K-1}_{p-1,n}+A^{K-1}_{m-1,p}A^{K-1}_{p,n} &= C_{m-1-n,p-1-n}A^{K-1}_{m-1,n}+C_{m-1-n,p-n}A^{K-1}_{m-1,n} \\ &= C_{m-n,p-n}A^{K-1}_{m-1,n}\end{aligned}$$

which is the desired submatrix.

4. Before we state our theorems, it is convenient to establish some notation. From now on, $K\geqq m\geqq n\geqq 0$. By $\mathrm{RN}^K_{m,n}$, we denote the row null space of $A^K_{m,n}$. Similarly, $\mathrm{CN}^K_{m,n}$ denotes the column null space of $A^K_{m,n}$. If M^m and N^n are two vector spaces whose vectors are regarded as tuples of numbers, then $M^m\oplus N^n$ denotes the $m+n$ dimensional vector space given by those $(m+n)$-tuples of numbers such that the first m positions form a vector of M^m and the last n positions form a vector of N^n.