

We define the following $C_{K,m} \times C_{K,m}$ matrix.

$$T = \left(\begin{array}{c|c} I_{m-1} & 0 \\ \hline -1 & A_{m,m-1}^{K-1} \\ m-n & I_m \end{array} \right)$$

where I_{m-1} and I_m are the $C_{K-1,m-1}$ and $C_{K-1,m}$ dimensional identity matrices respectively. T is a nonsingular matrix. In fact

$$T^{-1} = \left(\begin{array}{c|c} I_{m-1} & 0 \\ \hline 1 & A_{m,m-1}^{K-1} \\ m-n & I_m \end{array} \right)$$

THEOREM 1. *The nonsingular matrix T , operating on the right, induces a linear isomorphism from $\text{RN}_{m-1,n}^{K-1} \oplus \text{RN}_{m,n-1}^{K-1}$ to $\text{RN}_{m,n}^{K-1}$.*

PROOF.

$TA_{m,n}^K$

$$= \left(\begin{array}{c|c} A_{m-1,n-1}^{K-1} & A_{m-1,n}^{K-1} \\ \hline -\frac{1}{m-n} A_{m,m-1}^{K-1} A_{m-1,n-1}^{K-1} & -\frac{1}{m-n} A_{m,m-1}^{K-1} A_{m-1,n}^{K-1} + A_{m,n}^{K-1} \end{array} \right).$$

Since $A_{m,m-1}^{K-1} A_{m-1,n-1}^{K-1} = (m-n+1)A_{m,n-1}^{K-1}$ and $A_{m,m-1}^{K-1} A_{m-1,n}^{K-1} = (m-n)A_{m,n}^{K-1}$ (see Lemma 4) we have

$$TA_{m,n}^K = \left(\begin{array}{c|c} A_{m-1,n-1}^{K-1} & A_{m-1,n}^{K-1} \\ \hline -\frac{(m-n+1)}{m-n} A_{m,n-1}^{K-1} & 0 \end{array} \right).$$

Since T is nonsingular, right multiplication by T carries the row null space of $TA_{m,n}^K$ isomorphically to $\text{RN}_{m,n}^K$, the row null space of $A_{m,n}^K$. Thus all we need do is study the row null space of $TA_{m,n}^K$.

Consider the row vector $v = (x, y)$ where x and y are vectors of appropriate length such that

$$vTA_{m,n}^K = \left(xA_{m-1,n-1}^{K-1} - \frac{(m-n+1)}{m-n} yA_{m,n-1}^{K-1}, xA_{m-1,n}^{K-1} \right).$$

Then $vTA_{m,n}^K = 0$ if and only if

$$(1) \quad xA_{m-1,n-1}^{K-1} - \frac{(m-n+1)}{m-n} yA_{m,n-1}^{K-1} = 0$$