and

(2) $$xA_{m-1,n}^{K-1} = 0.$$

The first equation is equivalent to

$$yA_{m,n-1}^{K-1} = 0$$

since equation (2) and Lemma 4 imply that

$$xA_{m-1,n-1}^{K-1} = \frac{1}{m-n}\,(xA_{m-1,n}^{K-1})\,A_{n,n-1}^{K-1} = 0.$$

Thus the row null space of $TA_{m,n}^{K}$ is identical with $RN_{m-1,n}^{K-1} \oplus RN_{m,n-1}^{K-1}$. This proves the theorem.

Similarly, if we define

$$L = \left(\begin{array}{c|c} \dfrac{I_{n-1}}{-\dfrac{1}{m-n}A_{n,n-1}^{K-1}} & \begin{array}{c} 0 \\ \hline I_n \end{array} \end{array}\right)$$

we get

THEOREM 2. L, *operating on the left, induces a linear isomorphism from* $CN_{m,n-1}^{K-1} \oplus CN_{m-1,n}^{K-1}$ *to* $CN_{m,n}^{K}$.

5. In developing the consequences of the above theorems it is helpful to keep the following in mind.

REMARK. Suppose $K \geq m \geq n \geq 0$, then $C_{K,m} \geq C_{K,n}$ if and only if either $m = n$ or $m+n \leq K$.

COROLLARY 1.

(a)
$$\begin{aligned} \dim (RN_{m,n}^{K}) &= 0 && if\ m+n \geq K, \\ &= C_{K,m} - C_{K,n} && if\ m+n \leq K. \end{aligned}$$

(b)
$$\begin{aligned} \dim (CN_{m,n}^{K}) &= C_{K,n} - C_{K,m} && if\ m+n \geq K, \\ &= 0 && if\ m+n \leq K. \end{aligned}$$

PROOF. The theorem is true for $K=1$ (see Lemma 2). Assume the theorem is true for $K-1$. From the above theorems we see that

(1) $$\dim (RN_{m,n}^{K}) = \dim (RN_{m-1,n}^{K-1}) + \dim (RN_{m,n-1}^{K-1})$$

(2) $$\dim (CN_{m,n}^{K}) = \dim (CN_{m,n-1}^{K-1}) + \dim (CN_{m-1,n}^{K-1}).$$

Suppose $m+n \geq K$. Then $(m-1)+n \geq K-1$ and $m+(n-1) \geq K-1$. So by induction $\dim (RN_{m-1,n}^{K-1}) = \dim (RN_{m,n-1}^{K-1}) = 0$. Hence