On fibre spaces and the evaluation map

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1. Introduction

In [6], the author studied a subgroup $G(X)$ of the fundamental group $\pi_1(X)$. This group may be defined by means of the evaluation map $\omega: L(A, B) \to B$ where $L(A, B)$ is the space of continuous maps from A to B. The definition of $G(X)$, called the *evaluation subgroup* of X, along with some of its properties, appears in § 2 for the convenience of the reader. The main theme of this paper is the investigation of the role played by the evaluation subgroup in the theory of Hurewicz fibrations.

In § 3, given a Hurewicz fibration $p: E \to B$, with B a CW-complex and with fibre F, we classify the set of fibre homotopy equivalences from E to E up to homotopies of fibre homotopy equivalences. In fact, this set forms a group, $\mathfrak{L}(E)$, with multiplication induced by composition. Then if B_∞ is the classifying space for Hurewicz fibrations with fibre F, and $k: B \to B_\infty$ is the classifying map for $p: E \to B$ (see [1] or [3]), we have Theorem 1, which says that $\mathfrak{L}(E) \cong \pi_1(L(B, B_\infty); k)$.

We use the results of § 3 to study the evaluation subgroup of the fibre F in § 4. With every fibration $p: E \to B$ there is a homomorphism $d: \pi_2(B) \to \pi_1(F)$ which arises from the homotopy exact sequence for p. The union, over all fibrations with fibre F, of the images of d will be shown to equal $G(F)$. As a consequence, given a Hurewicz fibration $p: E \to B$, every map $f: S^2 \to B$ can be lifted to $\bar{f}: S^2 \to E$ if the fibre F is a compact, connected, polyhedron with Euler-Poincaré number different from zero. In fact, even a much more general statement is true.

In § 5, we take a very different tack. For every fibration $p: E \to B$ with fibre F, we may ask the following question. Which homotopy equivalences of F into itself can be extended to fibre homotopy equivalences of E into itself? That is, which homotopy equivalences can be extended to fibre preserving maps from E to E which send each fibre of E into itself? The homotopy classes of these homotopy equivalences form a subgroup of the group of homotopy equivalences from F to F. We denote this group by $\mathcal{F}(E)$.

Using the result of § 3, we show (Theorem 3) that $\mathcal{F}(E_\infty)$ is isomorphic to $G(B_\infty)$. Here $p_\infty: E_\infty \to B_\infty$ is a universal fibration for the fibre F.

Using the results of § 2, we show there exists a connected F such that

$F(E_\infty)$ is not trivial. We also use the above result in an opposite manner to calculate $G(B_\infty)$ for some F's.

In § 6, we indicate how to extend our results to the case of fibre bundles and principal fibrations.

2. Preliminaries

Let A and B be topological spaces. Then $L(A, B)$ will denote the space of continuous maps from A to B with the compact open topology. Let $k: A \to B$ be a continuous map. Then $L(A, B; k)$ will denote the path component of $L(A, B)$ which contains k.

Let L' be a subspace of $L(A, B)$. Let $f: X \to L'$ be any function. We shall say that f is *quasi-continuous* if the associated function $X \times A \to B$ is continuous. This concept made its first appearance in [10]. Note that if A is Hausdorff, then a quasi-continuous map is continuous. Two quasi-continuous maps are *quasi-homotopic* if there exists a homotopy $H: X \times I \to L'$ between them, such that H is quasi-continuous. We call H a *quasi-homotopy*.

There is a quasi-homotopy extension theorem which says: Let E^n be the unit n-ball, if $f: E^n \to L'$ is a quasi-continuous map and $h_t: S^{n-1} \to L'$ is a quasi-homotopy such that $f \mid S^{n-1} = h_0$, then there is a quasi-homotopy $f_t: E^n \to L'$ such that $f_0 = f$ and $f_t \mid S^{n-1} = h_t$.

Let $L'' \subset L' \subset L(A, B)$. Then we define $Q_i(L', L''; f)$ to be the set of quasi-homotopy classes of quasi-continuous maps $(E^n, S^{n-1}, x_0) \to (L', L'', f)$ where $f \in L''$ and x_0 is the base point of S^{n-1}. It is easy to see that $Q_i(L', L''; f)$ forms a group in the same manner as $\pi_i(L', L''; f)$ forms a group. We shall call these groups quasi-homotopy groups.

Because of the quasi-homotopy extension theorem, we obtain an exact sequence for the pair (L', L'') which is similar to the homotopy exact sequence

$$\cdots \longrightarrow Q_i(L'') \xrightarrow{i_*} Q_i(L) \xrightarrow{j_*} Q_i(L', L'') \xrightarrow{d} Q_{i-1}(L'') \longrightarrow \cdots$$

We shall use this quasi-homotopy exact sequence in the sequel.

Let $L' \subset L(A, B)$. Suppose $f: X \to L'$ is continuous. Suppose $X \times A$ is a k-space, see [4, p. 249]. That is, $X \times A$ is Hausdorff, and has the weak topology determined by the family of its compact subspaces. Then f is quasi-continuous, see [4, p. 263].

Suppose A is a k-space. (This will be true for a large class of space, in particular when A is first countable or locally compact or, most importantly for our purposes, when A is a CW-complex.) Then it is known that $X \times A$ is a k-space whenever X is locally compact, or if both X and A are first countable.

Thus by the above remarks we have the following fact.

PROPOSITION 1. *If X is compact and A is a k-space, then every function $f: X \to L' \subset L(A, B)$ is continuous if and only if it is quasi-continuous.*

COROLLARY. *$Q_i(L') = \pi_i(L')$ for $L' \subset L(A, B)$ if A is first countable or a* CW-*complex.*

Let $* \in A$ be a base point for any Hausdorff space A. There is a function $\omega: L(A, B) \to B$ given by $\omega(f) = f(*)$. We shall call ω the *evaluation map.* We shall always denote an evaluation map by ω in this paper.

Definition. Let X be a Hausdorff space. Then the evaluation subgroup, $G(X)$, is the image of $\omega_*: Q_1(L(X, X; 1_X)) \to \pi_1(X)$ contained in $\pi_1(X)$. In case X is a k-space, $G(X) = \omega_* \pi_1(L(X, X; 1_X))$.

The following properties of $G(X)$ are true for CW-complexes, and their proofs may be found in [5] and [6].

(a) $G(X)$ is an invariant of homotopy type.

(b) $G(X \times Y) \cong G(X) \oplus G(Y)$.

(c) $G(X) \subseteq Z(\pi_1(X))$, the center of $\pi_1(X)$.

(d) $G(X) = \pi_1(X)$ if X is a connected H-space.

(e) $G(X) = Z(\pi)$, the center of π, if X is a $K(\pi, 1)$ space.

(f) $G(X)$ is trivial if X has the homotopy type of a compact, connected, polyhedron with Euler-Poincaré number $\chi(X) \neq 0$.

(g) Finally, it was shown in [6] that if $K = K(\pi, 1)$, then $L(K, K; 1_K)$ is a $K(Z(\pi), 1)$.

For the purposes of comparison with (g) and with § 3, we record the result of R. Thom in [12, Th. 2 and preceding remarks].

(h) If $K = K(\pi, m)$, where $m > 0$ and π abelian, then $\pi_i(L(A, K)) \cong H^{m-i}(A; \pi)$ for $i > 0$. Thus $L(K, K; 1_K)$ is a $K(\pi, m)$.

3. The group of fibre homotopy equivalences

Let F be any space. There exists a Hurewicz fibration $p_\infty: E_\infty \to B_\infty$ with fibre F, which is universal for every Hurewicz fibration with fibre the homotopy type of F, and base space a CW-complex. By that we mean that the homotopy classes of maps from a CW-complex X to B_∞, written $[X, B_\infty]$, are in $1 - 1$ correspondence to the fibre homotopy equivalence classes of all the fibre spaces over X, with fibre the homotopy type of F. The correspondence is given by associating with every map $k: X \to B_\infty$ the induced fibre space $k^*(E_\infty)$ over X.

Stasheff established this result in [11] for F a compact CW-complex. Dold [3] has removed the restrictions on F. See also [1].

Let $p: E \to B$ be a Hurewicz fibration with fibre F. A map $f: E \to E$ is said to be *over B* if the following diagram is commutative.

$$\begin{array}{ccc} E & \xrightarrow{f} & E \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ B & \xrightarrow[1_B]{} & B\,. \end{array}$$

Thus $f\colon E \to E$ is over B if and only if it carries every fibre into itself. A homotopy. $h_t\colon E \to E$ is *over B* if each stage is a map over B. *A fibre homotopy equivalence* is a map $f\colon E \to E$ over B for which there exists a map $g\colon E \to E$ over B such that both $f \circ g$ and $g \circ f$ are homotopic to 1_E by homotopies which are over B. Dold has characterized fibre homotopy equivalences in [2, Th. 6.3] by showing that $f\colon E \to E$ is a fibre homotopy equivalence when B is a CW-complex if and only if f carries every fibre into itself by a homotopy equivalence. In light of this fact, a homotopy $h_t\colon E \to E$ over B between two fibre homotopy equivalences has the property that every stage of h_t is a fibre homotopy equivalence.

Let f and g both be fibre homotopy equivalences. Then the relation $f \cong g$, read f is homotopic to g over B, is an equivalence relation on the set of all fibre homotopy equivalences from $E \to E$. Let$\{f\}$ denote the equivalence class containing $f\colon E \to E$. We define a multiplication on the set of these equivalence classes given by $\{f\}\cdot\{g\} = \{f \circ g\}$. This multiplication is well defined and gives rise to a group.

Definition. The group defined above will be called the group of fibre homotopy equivalences of the Hurewicz fibration $p\colon E \to B$ *and will be denoted by* $\mathfrak{L}(E)$.

The purpose of this section is to classify $\mathfrak{L}(E)$ in terms of the Universal Classifying Space B_∞, and classifying map $k\colon B \to B_\infty$ for the Hurewicz fibration $p\colon E \to B$. This is done in Theorem 1.

Let $L^{**}(E, E)$ denote the space consisting of all the fibre homotopy equivalences from E to E topologized by the compact open topology. We wish to compute the quasi-homotopy groups $Q_i\big(L^{**}(E, E)\big)$. To do this we must define a few more notions.

From this point on, we shall consider a fixed Hurewicz fibration $p\colon E \to B$ with fibre F. Here B is a CW-complex. We shall let $p_\infty\colon E_\infty \to B_\infty$ be the universal fibration with fibre F, and $k\colon B \to B_\infty$ shall be a classifying map for $p\colon E \to B$. In fact, we shall choose B_∞ and E_∞ such that k is an inclusion and the restriction of $p_\infty\colon E_\infty \to B_\infty$ to B is just $p\colon E \to B$. Such a $p_\infty\colon E_\infty \to B_\infty$ may be chosen because of [9, p. 410, Th. 11] and the fact that Hurewicz fibrations give rise to a homotopy functor; see [1].

Let $L^*(E, E_\infty)$ be the space of fibre preserving maps from E to E_∞ which carry each fibre of E into a fibre of E_∞ by a homotopy equivalence. Let $L^*(E, E_\infty; k)$ be the set of maps in $L^*(E, E_\infty)$ with the additional property that every map $\tilde{f} \in L^*(E, E_\infty; k)$ covers a map $f\colon B \to B_\infty$ which is homotopic to $k\colon B \to B_\infty$. We define a function $\Phi\colon L^*(E, E_\infty; k) \to L(B, B_\infty; k)$ as follows. Let $\tilde{f} \in L^*(E, E_\infty; k)$. Then there is a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\tilde{f}} & E_\infty \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p_\infty} \\ B & \xrightarrow[f]{} & B_\infty \end{array}$$

Set

$$\Phi(\tilde{f}) = f .$$

Define $L^{**}(E, E_\infty; k)$ to be the subspace of $L^*(E, E_\infty; k)$ consisting of those maps which cover k. Thus $L^{**}(E, E_\infty; k) = \Phi^{-1}(k)$.

Since k is an inclusion and $p\colon E \to B$ is induced by k, we see that $L^{**}(E, E) = L^{**}(E, E_\infty; k)$.

LEMMA 1. *Φ is continuous.*

PROOF. Let K be a compact subspace of B, and let U be an open subspace of B_∞. Let $M(K, U)$ be the set of maps of $L(B, B_\infty; k)$ which carry K into U. These sets, of course, form a sub-basis for the compact open topology of $L(B, B_\infty; k)$. We shall show that Φ is continuous by showing $\Phi^{-1}(M(K, U))$ is open in $L^*(E, E_\infty; k)$ for all compact K and open U.

Let $M^*(C, D)$ be the set of fibre preserving maps of $L^*(E, E_\infty; k)$ which carry $C \subset E$ into $D \subset E_\infty$. Then the set of $M^*(C, D)$, for C compact and D open, is a sub-basis for $L^*(E, E_\infty; k)$ induced by the compact open topology of $L(E, E_\infty)$.

Now $\Phi^{-1}(M(K, U)) = M^*(p^{-1}(K), p_\infty^{-1}(U))$. Suppose C is any subset of $p^{-1}(K)$ which covers K, that is $p(C) = K$. Then

$$M^*(C, p_\infty^{-1}(U)) = M^*(p^{-1}(K), p_\infty^{-1}(U)).$$

If we can find a compact C which covers K, then $M^*(C, p_\infty^{-1}(U))$ will be a sub-basic set of $L^*(E, E_\infty)$, hence open. Since K is compact in the CW-complex B, it is contained in a finite sub-complex of B. Thus there is a finite number of closed cells D_i and maps $\alpha_i\colon D_i \to B$. Each α_i gives rise to a fibre preserving map $\tilde{\alpha}_i\colon D_i \times F \to E$ which lifts α_i. Since $D_i \times *$ is compact, $\tilde{\alpha}_i(D_i \times *)$ is compact in E. Let

$$C = \left(\textstyle\bigcup_i \tilde{\alpha}_i(D_i \times *)\right) \cap p^{-1}(K) .$$

Then C is compact, and $p(C) = K$. This proves the lemma.

LEMMA 2. *$Q_i(L^*(E, E_\infty; k))$ is trivial for all i.*

PROOF. Let X be any compact CW-complex. Let $\bar{g}: X \to L^*(E, E_\infty; k)$ be a quasi-continuous map. We wish to show that $\bar{g}$ may be factored through CX, the cone over X, i.e., $\bar{g}: X \xrightarrow{i} CX \to L^*(E, E_\infty; k)$ where i is the natural inclusion of X into CX. Let $\bar{\varphi} = \Phi \circ \bar{g}$. Then we have the commutative triangle

$$\begin{array}{ccc} & & L^*(E, E_\infty; k) \\ & \nearrow \bar{g} & \downarrow \Phi \\ X & \xrightarrow[\bar{\varphi}]{} & L(B, B_\infty, k) \end{array}$$

Let $g: X \times E \to E_\infty$ and $\varphi: X \times B \to B_\infty$ be the associated maps to $\bar{g}$ and $\bar{\varphi}$. Then we obtain the commutative diagram

$$\begin{array}{ccc} X \times E & \xrightarrow{g} & E_\infty \\ \downarrow 1 \times p & & \downarrow p_\infty \\ X \times B & \xrightarrow[\varphi]{} & B_\infty \end{array}$$

Consider

$$\begin{array}{ccc} \varphi^*(E_\infty) & \xrightarrow{\tilde{\varphi}} & E_\infty \\ \downarrow \varphi^*(p_\infty) & & \downarrow p_\infty \\ X \times B & \xrightarrow[\varphi]{} & B_\infty \end{array}$$

Here $\tilde{\varphi}$ is the naturally induced map, i.e., $\tilde{\varphi}(x, e) = e$ for $x \in X \times B$ and $e \in E_\infty$. There exists a unique $g': X \times E \to \varphi(E_\infty)$ such that $\tilde{\varphi} \circ g' = g$. This is given by $g'(x, e) = (x, g(e))$.

Now $1 \times p: CX \times E \to CX \times B$ is a fibration with fibre F and hence there exists a map $A: CX \times B \to B_\infty$ such that

$$\begin{array}{ccc} CX \times E & \underset{h}{\overset{h^{-1}}{\rightleftarrows}} & A^*(E_\infty) \\ \downarrow 1 \times p & & \downarrow A^*(p_\infty) \\ CX \times B & \underset{1}{\longleftrightarrow} & CX \times B \end{array}$$

is commutative where h and h^{-1} are inverse fibre homotopy equivalences. We may select A such that $(i \times 1)^*(A^* E_\infty)$ is the same $\varphi^*(E_\infty)$ (recall $i: X \to CX$) since $\varphi: X \times B \to B_\infty$must be homotopic to A restricted to $X \times B$. We now have the following commutative diagram, where $j = i \times 1$.

$$\begin{array}{ccccccccccc}
X \times E & \xrightarrow{g'} & j^*(A^*E_\infty) & \xrightarrow{\tilde{j}} & A^*E_\infty & \xrightarrow{h} & CX \times E & \xrightarrow{h^{-1}} & A^*E_\infty & \xrightarrow{\tilde{A}} & E_\infty \\
\Big\downarrow{\scriptstyle 1 \times p} & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow{\scriptstyle 1 \times p} & & \Big\downarrow & & \Big\downarrow{\scriptstyle p_\infty} \\
X \times B & \underset{1}{\longleftrightarrow} & X \times B & \underset{j}{\longrightarrow} & CX \times B & \underset{1}{\longleftrightarrow} & CX \times B & \underset{1}{\longleftrightarrow} & CX \times B & \underset{A}{\longrightarrow} & B_\infty
\end{array}$$

Observe that $\tilde{A} \circ h^{-1} \circ h \circ \tilde{j} \circ g' \cong \tilde{A} \circ \tilde{j} \circ g' = \tilde{\varphi} \circ g' = g$, where "$\cong$" means homotopic by a fibre preserving homotopy over φ.

Now $g \cong (\tilde{A}h^{-1}) \circ (h\tilde{j}g')$, and $\overline{\tilde{A}h^{-1}}: CX \to L(E, E_\infty; k)$, so $\bar{g}$ is quasi-homotopic to a constant map.

LEMMA 3. *Φ has the covering quasi-homotopy property for finite* CW-*complexes, by which we mean that, if $F: X \to L^*(E, E_\infty; k)$ is a quasi-continuous map from a compact* CW-*complex X, then any homotopy $h_t: X \to L(B, B_\infty; k)$ of $\Phi \circ F$ can be covered by a quasi-homotopy H_t, that is $\Phi \circ H_t = h_t$*

PROOF. This can be seen by a proof similar to Lemma 2, and the observation that any map from $X \to L(B, B_\infty; k)$ is quasi-continuous.

The import of Lemma 3 is that it gives us an exact sequence of the Q_i which is the analogue of the homotopy exact sequence for a fibration. Since $\pi_i(L(B, B_\infty; k)) = Q_i(L(B, B_\infty; k))$ by the corollary in the previous section, we have exact sequence

$$\cdots \longrightarrow Q_n(L^{**}) \xrightarrow{i_*} Q_n(L^*) \xrightarrow{\Phi_*} \pi_n(L) \xrightarrow{d} Q_{n-1}(L^{**}) \longrightarrow \cdots,$$

where L^{**} is short for $L^{**}(E, E_\infty, k)$; L^* is short for $L^*(E, E_\infty; k)$, and L is short for $L(B, B_\infty; k)$.

LEMMA 4. $\pi_i(L(B, B_\infty; k)) \cong Q_{i-1}(L^{**}(E, E))$.

PROOF. From the above exact sequence and Lemma 2, we see that d is one to one and onto. Thus $d: \pi_i(L) \to Q_{i-1}(L^{**})$ is a homomorphism for $i > 0$. In the case of $i = 1$, we shall define the group structure on $Q_0(L^{**})$ to be that induced by $\pi_1(L)$. Since $L^{**}(E, E)$ is the same as $L^{**}(E, E_\infty; k)$, we have proved the lemma.

COROLLARY. $\pi_i(L(B, B_\infty; k)) \cong \pi_{i-1}(L^{**}(E, E))$ *if E is first countable.*

THEOREM 1. $\pi_1(L(B, B_\infty; k)) \cong \mathfrak{L}(E)$.

PROOF. Observe that the elements of $Q_0(L^{**})$ correspond to the elements of $\mathfrak{L}(E)$ in a natural manner. Lemma 4 tells us that $Q_0(L^{**}) \cong \pi_1(L)$, where the group structure of $Q_0(L^{**})$ is induced by the group structure of $\pi_1(L)$. This group structure turns out to be precisely the one induced by composition of fibre homotopy equivalences; that is the group $\mathfrak{L}(E)$.

To see this, we regard

$$L^{**}(E, E) \longrightarrow L^*(E, E_\infty; k) \xrightarrow{\Phi} L(B, B_\infty; k)$$

as a principal fibration with $L^{**}(E, E)$ acting on L^* by composition. Let α and β be closed paths representing $[\alpha]$ and $[\beta]$ in $\pi_1(L)$. Then $\alpha\cdot\beta$ represents $[\alpha]\cdot[\beta]$. Let a and $b$represent paths lifting α and β in L^* with $a(1) = b(1) = 1_E \in L^{**}(E, E)$ which we regard as contained in $L^*(E, E_\infty; k)$. Then suppose $a(0) = f$ and $b(0) = g$ where f and $g \in L^{**}(E, E)$. Thus $[\alpha]$ and $[\beta]$ correspond to $\{f\}$ and $\{g\}$ respectively, that is the path components containing f and g respectively, under the homomorphism $d: \pi_1(L) \to \pi_0(L^{**})$. Now $[\alpha]\cdot[\beta]$ is represented by $\alpha\cdot\beta$, which is lifted to L^* by the path $(a \circ g)\cdot b$, where $(a \circ g)(t) = a(t) \circ g$. Now $(a \circ g)\cdot b$ evaluated at 0 is $a \circ g$ evaluated at 0 which is $f \circ g$. Thus $[\alpha]\cdot[\beta]$ corresponds to $\{f \circ g\} = \{f\}\cdot\{g\}$.

I wish to thank A. Dold for suggesting the technique which we used above to prove Theorem 1, resulting in a great improvement over the author's original proof. Please see *Appendix* at end of paper.

Let $p: \tilde{X} \to X$ be a covering map, with $\tilde{X}$ a covering space of X, and let X be a CW-complex. Then p is a Hurewicz fibration, and, as such, we have theoretically computed $\mathfrak{L}(\tilde{X})$ by Theorem 1. However, $\mathfrak{L}(\tilde{X})$ is just the group of covering transformations of $\tilde{X}$ and, so, well known. In fact $\mathfrak{L}(\tilde{X}) \cong N(p_*\pi_1(\tilde{X}))/p_*\pi_1(\tilde{X})$, where $N(p_*\pi_1(\tilde{X}))$ is the normalizer of $p_*\pi_1(\tilde{X})$ in $\pi_1(X)$. See [9]. It is possible to derive the above result from Theorem 1, but we shall not do it here.

4. The evaluation subgroup of the fibre

We shall consider the map $\Phi: L^*(E, E_\infty; k) \to L(B, B_\infty; k)$ for the case where $B = *$, the space consisting of one point, and $k: * \to B_\infty$ sends $*$ into the base point of B. For convenience, assume F is a locally compact CW-complex. Under these conditions, $L^{**}(F, E_\infty; k)$ is just the space of homotopy equivalences of F into itself, denoted by F^F. $L(*, B_\infty; k)$ can be regarded as B_∞, since B_∞ is connected. We write L^*_* for $L^*(F, E_\infty; k)$ for brevity. Thus L^*_* is the space of maps of F into E_∞ which are homotopy equivalences from F to any fibre of E_∞. Thus we have the Serre fibration $F^F \to L^*_* \xrightarrow{\Phi} B$. This is the associated principal fibration of $p_\infty: E_\infty \longrightarrow B_\infty$. Since $\pi_i(L^*_*) \cong 0$ for all i by Lemma 2, we see that $\pi_i(B_\infty) \cong \pi_{i-1}(F^F)$. These facts are well known, see [1] or [11].

Let ω stand for the evaluation map from either $F^F \to F$ or $L^*_* \to E_\infty$. Then we obtain the commutative diagram

$$\begin{array}{ccc} F^F & \xrightarrow{\omega} & F \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\ L^*_* & \xrightarrow{\omega} & E_\infty \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle p_\infty} \\ B_\infty & \underset{1}{\longleftrightarrow} & B_\infty \end{array}$$

The fibre homotopy exact sequences for these fibrations then give us the following commutative diagram

$$\begin{array}{ccccccccc} \longrightarrow & \pi_i(F^F) & \xrightarrow{i_*} & \pi_i(L^*_*) & \xrightarrow{\Phi_*} & \pi_i(B_\infty) & \xrightarrow{d'_\infty} & \pi_{i-1}(F^F) & \longrightarrow \\ & \downarrow{\scriptstyle \omega_*} & & \downarrow{\scriptstyle \omega_*} & & \updownarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \omega_*} & \\ \longrightarrow & \pi_i(F) & \xrightarrow{i_*} & \pi_i(E_\infty) & \xrightarrow{p_{\infty *}} & \pi_i(B_\infty) & \xrightarrow{d_\infty} & \pi_{i-1}(F) & \longrightarrow \end{array}$$

THEOREM 2. $d_\infty(\pi_i(B_\infty)) = \omega_*(\pi_{i-1}(F^F))$. *In particular,* $d_\infty(\pi_2(B_\infty)) = G(F)$

PROOF. From the diagram $\omega_* d'_\infty = d_\infty$. Since $\pi_i(L^*_*) \cong 0$, d'_∞ is an isomorphism, hence $d_\infty\pi_i(B_\infty) = \omega_* d'_\infty \pi_i(B_\infty) = \varphi_*\pi_{i-1}(F^F)$. This proves the theorem.

Let $p: E \rightarrow X$ be a fibration, with fibre F, which corresponds to the homotopy class of a map $k: X \rightarrow B_\infty$. The we obtain the commutative diagram

$$\begin{array}{ccccccccc} \longrightarrow & \pi_i(F) & \xrightarrow{i_*} & \pi_i(E) & \xrightarrow{p_*} & \pi_i(X) & \xrightarrow{d} & \pi_{i-1}(F) & \longrightarrow \\ & \updownarrow{\scriptstyle 1_*} & & \downarrow{\scriptstyle \tilde{k}_*} & & \downarrow{\scriptstyle k_*} & & \updownarrow{\scriptstyle 1_*} & \\ \longrightarrow & \pi_i(F) & \underset{i_*}{\longrightarrow} & \pi_i(E_\infty) & \underset{p_{\infty *}}{\longrightarrow} & \pi_i(B_\infty) & \underset{d_\infty}{\longrightarrow} & \pi_{i-1}(F) & \longrightarrow \end{array}$$

where $\tilde{k}: E \rightarrow E_\infty$ covers k.

COROLLARY 1. $d(\pi_i(X)) \subseteqq \omega_*\pi_{i-1}(F^F)$. *In particular,* $d(\pi_2(X)) \subseteqq G(F)$.

PROOF. Since $d = d_\infty k_*$,

$$d(\pi_i(X)) = d_\infty k_*(\pi_i(X)) \subseteqq d_\infty(\pi_i(B_\infty)) = \omega_*(\pi_{i-1}(F^F)) .$$

It should be noted that we can prove Corollary 1 without reference to the classifying space. In fact, consider

$$\begin{array}{ccccc} B^n \times F & \xrightarrow{\tilde{\alpha}} & f^*S^n & \xrightarrow{\tilde{f}} & E \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle p} \\ B^n & \underset{\alpha}{\longrightarrow} & S^n & \underset{f}{\longrightarrow} & X \end{array}$$

where B^n is the unit n-ball, $\alpha\colon (B^n, \partial B^n) \longrightarrow (S^n, s_0)$ a map which carries $B^n - \partial B^n$ homeomorphically onto $S^n - s_0$, where s_0 is the base point of S^n. Also $f\colon S^n \longrightarrow X$ is any map and $\tilde{\alpha}$ and $\tilde{f}$ are the natural liftings of α and f. Let $c\colon B^n \longrightarrow B^n \times F$ be defined by $c(b) = (b, *)$. Then $\tilde{f}\tilde{\alpha}c$ lifts $f\alpha$. Now $\tilde{f}\tilde{\alpha}c$ restricted to $\partial B^n = S^{n-1}$ is a map from S^{n-1} to F, and in fact it represents the homotopy class of $d[f]$, where $[f]$ is the homotopy class of f. Also observe that $\tilde{f}\tilde{\alpha} \mid \partial B^n \times F$ is associated with a map $g\colon S^{n-1} \longrightarrow L(F, F; 1)$ and $\tilde{f}\tilde{\alpha}c \mid S^{n-1}$ is just the composition of g and the evaluation map. Hence Corollary 1.

Remark. If we drop the restriction that F be locally compact in Theorem 2 and Corollary 1, then we must replace $\pi_i(F^F)$ by $Q_i(F^F)$. The proofs are completely analogous to the given proofs.

Now we apply the results of § 2 to get some interesting corollaries. Result (c), § 2, gives us

COROLLARY 2. *Let F have the homotopy type of a* CW-*complex. Then $d(\pi_2(X)) \subseteqq Z(\pi_1(F))$, the center of $\pi_1(F)$.*

COROLLARY 3. *Let F have the homotopy type of a polyhedron with a compact path-component whose Euler-Poincaré number is not zero. Then every Hurewicz fibre space over S^2 with fibre F admits a cross-section.*

PROOF. Let $*$ be the base point of F chosen in the above path-component. Then we have the fibre homotopy exact sequence

$$\longrightarrow \pi_2(E) \xrightarrow{p*} \pi_2(S^2) \xrightarrow{d} \pi_1(F, *) \longrightarrow$$

By Corollary 1, the remark above, and (f) in § 2, d is the trivial homomorphism. Thus $p\colon \pi_2(E) \longrightarrow \pi_2(S^2)$ is onto. Hence, by the covering homotopy property, the identity map $S^2 \longrightarrow S^2$ may be lifted to $S^2 \longrightarrow E$, and this lifting is the required cross-section.

From Corollaries 2 and 3, we see that the lifting problem

$$\begin{array}{ccc} & & E \\ & \nearrow & \big\downarrow p \\ S^2 & \xrightarrow[f]{} & B \end{array}$$

can be solved for all f whenever the fibre F is as in Corollary 3, or $Z(\pi_1(F, x_0)) = 0$ for some $x_0 \in F$.

5. The evaluation subgroup of B_∞

In this section we investigate $G(B_\infty)$. We begin by again considering our general situation. Suppose $p\colon E \longrightarrow X$ is a Hurewicz fibration with fibre F. Let $k\colon X \longrightarrow B_\infty$ be a classifying map for $p\colon E \longrightarrow X$. We define a map

$r\colon L^*(E, E; k) \to L^*_*$ where L^*_* is defined as in § 4, by $r(f) = f \mid F$ where $F = p^{-1}(*)$. In words, $r(f)$ is the restriction of $f\colon E \to E_\infty$ to the fibre F over the base point $* \in X$. Since f is fibre preserving, it sends F into a fibre of E_∞ and, by definition, f must be a homotopy equivalence on F, so $r\colon L^*(E, E_\infty; k) \to L^*_*$ is well defined.

Since $L^{**}(E, E_\infty; k)$ is a subspace of $L^*(E, E_\infty; k)$, we regard $r\colon L^{**}(E, E_\infty; k) \to L^*_*$ as the restriction of r to L^{**}. In fact, however, the image of $L^{**}(E, E_\infty; k)$ under r is just F^F_∞, the space of homotopy equivalences from F to $F_\infty = p_\infty^{-1}(k(*))$. We shall write F_∞ as F. So we have the commutative diagram

$$\begin{array}{ccc} L^{**}(E, E_\infty; k) & \xrightarrow{r} & F^F \\ \downarrow & & \downarrow \\ L^*(E, E_\infty; k) & \xrightarrow{r} & L^*_* \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi} \\ L(X, B_\infty; k) & \xrightarrow{\omega} & B_\infty \end{array}$$

where ω, again, is the evaluation map. The functions r may not be continuous, however they do induce homomorphisms between the appropriate quasi-homotopy groups, and the above diagram leads to an exact ladder just as if r were continuous.

Let $X = B_\infty$ and k be the identity map 1. The above diagram gives rise to the commutative square

$$\begin{array}{ccc} \pi_1 L\big((B_\infty, B_\infty; 1)\big) & \xrightarrow{\omega *} & \pi_1(B_\infty) \\ \downarrow{\scriptstyle d*} & & \downarrow{\scriptstyle d*} \\ Q_0\big(L^{**}(E_\infty, E_\infty; 1)\big) & \xrightarrow{r *} & \pi_0(F^F) \end{array}$$

Now the vertical homomorphisms, both denoted by d_* are isomorphisms by Lemma 2, § 3. Thus we see that $\omega_* \pi_1\big(L(B_\infty, B_\infty; 1)\big) \cong r_* Q_0\big(L^{**}(E_\infty, E_\infty; 1)\big)$. The group on the left is simply $G(B_\infty)$. The group on the right is the subgroup of the group of homotopy classes of homotopy equivalences $\pi_0(F^F)$, consisting of those classes of homotopy equivalences f, $F \to F$ which extend to fibre homotopy equivalences $\bar{f}\colon E_\infty \to E_\infty$. We shall denote this group by $\mathcal{F}(E_\infty)$. The above remarks prove the following theorem.

THEOREM 3. $G(B_\infty) \cong \mathcal{F}(E_\infty)$.

For any fibration $p\colon E \to B$, we may define $\mathcal{F}(E)$ to be the subgroup of homotopy classes of homotopy equivalences from F to F, where $F = p^{-1}(*)$, which extend to a fibre homotopy equivalence from E to E.

Now, roughly speaking, $\mathcal{F}(E_\infty) = \bigcap_N \mathcal{F}(E)$, where N is the collection of all fibrations with fibre F. This may be seen by assuming that $f: F \to F$ can be extended to a fibre homotopy equivalence $\bar{f}: E_\infty \to E_\infty$. Then $\bar{f}$ induces a fibre homotopy equivalence from E to E, for any $p: E \to B$, which restricts to f on the fibre F.

We shall use Theorem 3 to compute $G(B_\infty)$ when $F = K(\pi, n)$ and π is abelian.

COROLLARY 1. *Let $F = K(\pi, n)$, where π is abelian. Then $G(B_\infty)$ is trivial.*

PROOF. We known, [7, p. 158], for example, that, for every $n-1$ connected space X, $n > 1$, there exists an n-connected fibre space over X with a $K(\pi_n(X), n-1)$ as its fibre. Now let $\pi = \pi_n(X)$, and let E be the total space of one of the above fibre spaces. Let $\bar{f}$; $E \to E$ be a fibre homotopy equivalence, and let $f: K(\pi, n-1) \to K(\pi, n-1)$ be the restriction of $\bar{f}$ to a fibre. This gives rise to the commutative diagram

$$\begin{array}{ccccc} K(\pi, n-1) & \xrightarrow{i} & E & \xrightarrow{p} & X \\ \downarrow f & & \downarrow \bar{f} & & \downarrow 1_X \\ K(\pi, n-1) & \longrightarrow & E & \xrightarrow[p]{} & X . \end{array}$$

By looking at the homotopy exact sequence for this fibration, we arrive at the commutative square

$$\begin{array}{ccc} \pi_n(X) & \xrightarrow[d]{\cong} & \pi_{n-1}(K(\pi, n-1)) = \pi \\ \downarrow 1 & & \downarrow f \\ \pi_n(X) & \xrightarrow[d]{\cong} & \pi_{n-1}(K(\pi, n-1)) = \pi . \end{array}$$

Thus $f_*: \pi \to \pi$ is the identity isomorphism. Hence $f: K(\pi, n-1) \to K(\pi, n-1)$ is homotopic to the identity. Thus $\mathcal{F}(E)$ is trivial where E represents the above fibre space, hence by Theorem 3, and the following paragraph, $G(B_\infty)$ is trivial for $F = K(\pi, n-1)$ for π abelian, $n > 1$.

It is not true that $G(B_\infty)$ is trivial for all B_∞. In fact, we have a whole class of non-trivial examples.

Let π be a group with center $Z(\pi)$ and group of outer automorphisms $\overline{\text{Aut}}(\pi)$. $\overline{\text{Aut}}(\pi)$ is the group of automorphisms of π modulo the inner automorphisms.

COROLLARY 2. *If $Z(\pi)$ is trivial, and $\overline{\text{Aut}}(\pi)$ has a non-trivial center, then $G(B_\infty)$ is not trivial for $F = K(\pi, 1)$.*

PROOF. Recall that $\pi_i(B_\infty) \cong \pi_{i-1}(F^F)$ for F a CW-complex. Now $\pi_0(F^F)$ is the group of homotopy classes of homotopy equivalences of F into itself. For a $K(\pi, 1)$, this is just $\overline{\text{Aut}}\,(\pi)$. See [7, p. 198]. Also, by (g), § 2, $\pi_1(F^F) = Z(\pi)$ and $\pi_i(F^F) = 0$ for $i > 1$.

Hence

$$\pi_i(B_\infty) = 0 \qquad \text{if } i > 2$$
$$\pi_2(B_\infty) = Z(\pi)$$
$$\pi_1(B_\infty) = \overline{\text{Aut}}\,(\pi)\,.$$

Now $Z(\pi)$ is trivial, hence B_∞ is a $K(\overline{\text{Aut}}\,(\pi), 1)$. Hence $G(B_\infty)$ is just the center of $\overline{\text{Aut}}\,(\pi)$, and, by hypothesis, it is not trivial. This proves the corollary.

The hypotheses for Corollary 2 are fulfilled by $F = K(S(6), 1)$, where $S(6)$ is the symmetric group on 6 letters. In this case $\overline{\text{Aut}}\,(S(6)) \cong Z_2$ and $Z(S(6)) = 1$. See [8, pp. 132–3].

Thus any fibration with fibre $K(S(6), 1)$ has a self fibre homotopy equivalence which is not fibre homotopic, by fibre homotopy equivalences, to the identity.

It is interesting to remark that, showing $\mathfrak{L}(E_\infty)$ is not trivial (for fibre F), does not necessarily imply that $\mathfrak{L}(E)$ is non-trivial for all fibrations $p: E \to X$ with fibre F. However, $\mathcal{F}(E_\infty)$ non-trivial implies $\mathcal{F}(E)$ is non-trivial for all $p: E \to X$ with fibre F; hence $\mathfrak{L}(E)$ is non-trivial for all E with fibre F.

Combining Theorems 2 and 3 we get

COROLLARY. *Every Hurewicz fibre space over S^2 with fibre B_∞ has a cross-section if and only if every fibre homotopy equivalence $\tilde{f}: E_\infty \to E_\infty$ restricts on F to a map homotopic to 1_F.*

6. Remarks

Suppose we are confronted with a classifying space B_* for principal fibre bundles, or for vector bundles, or for principal fibre spaces. Then we may apply the same techniques as in § 3 to show that $\pi_1(L(X, B_*); k)$ is isomorphic to the group of principal bundle equivalences, or of vector bundle equivalences, or of principal fibre homotopy equivalences (as the case may be) for the particular bundle or fibre space whose classifying map is k. This may be done by choosing the appropriate spaces of fibre preserving maps corresponding to $L^*(E, E_\infty; k)$ and $L^{**}(E, E)$, and following the proofs of Lemmas 1, 2, 3, and 4 in § 3.

If B_* is an H-space, then $L(X, B_*; k)$ has the same homotopy type as $L(X, B_*; k')$ for any two maps $k, k': X \to B_*$. See [12]. Hence $\pi_1(L(X, B_*; k')) \cong \pi_1(L(X, B_*; k))$. For example, B_U, the classifying space for the unitary group, is an H-space, so any two principal U-bundles over a space X have isomorphic groups of principal bundle equivalences.

Note that $G(B_*)$ plays the same role that $G(B_\infty)$ does. For example, for principal H-bundles, $G(B_H)$ is isomorphic to the group of those elements x of H for which $L_x: H \longrightarrow H$, where $L_x(a) = xa$, can be extended to a principal equivalence map $E_H \longrightarrow E_H$, modulo H_0, the identity component of H.

Appendix: The proof of Lemma 1, § 3, assumes that the total space E is Hausdorff. It is possible, however, to prove Lemmas 2, 3 and 4 and Theorem 1 without restricting E to be Hausdorff by observing that Φ composed with any quasi-continuous map is still quasi-continuous. Then, using this fact, the proofs of the above lemmas may be carried out with slight modification.

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