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HOMOLOGY TANGENT BUNDLES AND UNIVERSAL BUNDLES[1]

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ABSTRACT. We find results about the evaluation map from the group of homeomorphisms of a closed manifold M and also about fibre bundles where M is the fibre. These facts follow from the observation that the homology tangent bundle is induced from a universal bundle pair.

1. **Introduction.** The object of this paper is to prove the following theorem:

THEOREM 12. *Let G be any group of homeomorphisms acting on a closed oriented topological manifold M and let $\omega: G \to M$ be the evaluation map at the base point $*$. Then $\chi(M)\omega^*: \tilde{H}^*(M; R) \to \tilde{H}^*(G; R)$ is trivial where R is any ring of coefficients with a unit and $\chi(M)$ is the Euler-Poincaré number of M.*

Note that the theorem applies to the coset mapping $\rho: G \to G/H$.

The proof of Theorem 12 makes use of the following observation: Let G be the path-connected component of the group of homeomorphisms of any topological manifold M and let H be the isotropy subgroup at $*$. Then we have the fibre bundle $M \xrightarrow{i} B_H \to B_G$.

THEOREM 5. *The inclusion i induces the homology tangent bundle from a "universal" fibre pair over B_H.*

We use a theorem of R. F. Brown to obtain Theorem 12 from Theorem 5. Along the way we obtain a topological version of a theorem of Borel [**2**].

THEOREM 9. *Let $M \to E \xrightarrow{\pi} B$ be any fibre bundle with an orientation preserving structural group, where M is a closed, orientable, topological manifold. Let p be a prime such that $\chi(M) \not\equiv 0 \bmod p$. Then $\pi^*: H^*(B; Z_p) \to H^*(E; Z_p)$ is injective.*

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2. **Preliminaries.** Let G be a topological group and let H be a closed subgroup. Let E be a space on which G operates on the right such that the projections $E \to E/G$ and $E \to E/H$ are principal fibre bundles with fibres G and H respectively. Also, we have the fibre bundle $\rho: E/H \to E/G$ with fibre G/H.

Let G operate on the left of a space F. Then, as usual, $E \times_G F$ will denote the quotient space of $E \times F$ under the equivalence relation $(e, x) \sim (eg, g^{-1}x)$. The equivalence class of (e, x) will be denoted by $\langle e, x\rangle$. Thus $\langle e, x\rangle$ is a point of $E \times_G F$ and $\langle e, x\rangle = \langle eg, g^{-1}x\rangle$. Similarly, we shall denote points of E/H by $\langle e\rangle_H$ where $e \in E$ and points of E/G will be denoted $\langle e\rangle_G$.

The map $p: E \times_G F \to E/G$ given by $p(\langle e, x\rangle) = \langle e\rangle_G$ is the projection of a fibre bundle with fibre F and group G.

Also, we need the concept of bundles induced by a map. Let $E \to^{p} B$ be a bundle and $X \to^{f} B$ be a map. The induced bundle $f^*(X) \to^{p(f)} X$ is a bundle with the same fibre and $f^*(X)$ is the subspace of $E \times X$ given by points of the form (e, x) such that $p(e) = f(x)$. We shall denote points in $f^*(X)$ by (e, x). Then the projection map $p(f)$ is given by $(e, x) \mapsto x$.

We wish to consider the square of the bundle $E/H \to^{\rho} E/G$. This is the bundle $G/H \to \rho^*(E/H) \to E/H$. The points of $\rho^*(E/H)$ have the form $(\langle e\rangle_H, \langle e'\rangle_H)$ where $\langle e\rangle_G = \langle e'\rangle_G$. Note there is a unique $g \in G$ such that $e' \cdot g = e$.

There is another bundle with fibre G/H over E/H. This is $G/H \to E \times_H G/H \to E/H$. These two bundles are equivalent. We record some standard facts below for the reader's convenience:

LEMMA 1. *The two bundles $\rho^*(E/H) \to E/H$ and $E \times_H G/H \to E/H$ are equivalent. The bundle equivalence is given by the map $\rho^*(E/H) \to^{\alpha} E \times_H G/H$ which sends $(\langle e\rangle_H, \langle e'\rangle_H) \mapsto \langle e, g^{-1}H\rangle$ where g is defined by the equation $e' \cdot g = e$.*

LEMMA 2. *There exists a cross-section $s: E/H \to \rho^*(E/H)$ to the fibration $\rho^*(E/H) \to E/H$ given by $\langle e\rangle_H \mapsto (\langle e\rangle_H, \langle e\rangle_H)$. The fibration restricts over the fibre G/H of $E/H \to^{\rho} E/G$ to projection on the second factor $G/H \times G/H \to G/H$. Then the cross-section s restricted to the fibre G/H is the diagonal $G/H \to^{\Delta} G/H \times G/H$. We may also regard s as the cross-section $E/H \to E \times_H G/H$ given by $\langle e\rangle_H \mapsto \langle e, H\rangle$.*

LEMMA 3. *There is a commutative diagram of fibre bundles*

$$\begin{array}{ccccc} G/H \times G/H & \longrightarrow & \rho^*(E/H) & \longrightarrow & E/H \\ \Delta \uparrow \downarrow p_2 & & s \uparrow \downarrow & & \downarrow p \\ G/H & \xrightarrow{\;i\;} & E/H & \xrightarrow{\;\rho\;} & E/G \end{array}$$

LEMMA 4. *There is a bundle equivalence*

$$\begin{array}{ccc} E/H & \xrightarrow{\mu} & E\times_G G/H \\ \big\downarrow \rho & & \big\downarrow p \\ E/G & \xrightarrow{1} & E/G \end{array}$$

where $\mu(\langle e\rangle_H)=\langle e, H\rangle$.

3. **The homology tangent bundle.** In this section we shall always assume that G/H is a topological manifold without boundary, which we shall also denote by M. Also, G will be a group of homeomorphisms of M onto itself, endowed with the compact open topology. Then H will be a space of homeomorphisms of M which leave the base point $* \in M$ fixed, i.e. the isotropy subgroup. We shall always take E to be contractible. Thus we have $B_G=E/G$ and $B_H=E/H$.

Now observe that H may be regarded as a group of homeomorphisms of $M-*$. Consider the universal principal bundle $H\to E\to B_H$. By regarding H as operating on M, we obtain the associated bundle $M\to E\times_H M\to B_H$. By regarding H as operating on $M-*$ we obtain the associated bundle $(M-*)\to E\times_H (M-*)\to B_H$.

Observe that $E\times_H (M-*)$ is contained as a subspace in $E\times_H M$. In fact, using the cross-section s of Lemma 2, we have $E\times_H (M-*)=(E\times_H M)-s(B_H)$. The pair $(E\times_H M, E\times_H (M-*))$ is a fibre-bundle pair in the sense of Fadell **[6]**. See also **[10**, p. 256]. The fibre is $(M, M-*)$ and the base is B_H.

By Lemma 1, we may view the fibre pair from a different point. Recall the fibre bundle $M\xrightarrow{i} B_H\xrightarrow{\rho} B_G$. Then $E\times_H M$ is just $\rho^*(B_H)$, and $E\times_H (M-*)$ is $\rho^*(B_H)-s(B_H)$. Thus it is easy to see, using Lemma 3, that the fibre bundle $M\times M\xrightarrow{p_2} M$ given by projection on the second factor is induced by $i:M\to B_H$ from the bundle $\rho^*(B_H)\to B_H$. Also, the bundle $M\times M-\Delta\xrightarrow{p_2} M$ given by projection on the second factor is induced by $i:M\to B_H$ from the bundle $\rho^*(B_H)-s(B_H)\to B_H$. We formalize this in the following theorem.

THEOREM 5. *The bundle pair* $(M\times M, M\times M-\Delta)\xrightarrow{p_2} M$ *with fibre* $(M, M-*)$ *is induced from the bundle pair* $(\rho^*(B_H), \rho^*(B_H)-s(B_H))\to B_H$ *by the inclusion map* $i: M\to B_H$.

Now let (E', E_0') be a fibre pair with fibre (F, F_0) and base space B. Then $\pi_1(B, *)$ operates on $H_*(F, F_0; Z)$. The fibre pair (E', E_0') is said to be *orientable* if $\pi_1(B, *)$ operates trivially on $H_*(F, F_0; Z)$. If $\pi_1(B, *)$ operates trivially on $H_*(F, F_0; Z_2)$, the fibre pair is called Z_2-*orientable*.

LEMMA 6. (a) *The homology tangent bundle*

$$(M, M-*) \to (M \times M, M \times M - \Delta) \xrightarrow{p_2} M$$

is Z_2-orientable. If M is orientable, then the homology tangent bundle is orientable.

(b) *Let M be orientable and let G be a group of homeomorphisms of M which preserve orientation. Then the fibre pair*

$$(M, M-*) \to (\rho^*(B_H), \rho^*(B_H) - s(B_H)) \to B_H$$

is orientable.

(c) *For any M and G, the above fibre pair is Z_2-orientable.*

PROOF. (a) was proved by Fadell in [**6**].

(b) Let $\alpha \in \pi_1(B_H, *)$. We shall show that α acts trivially on $H_n(M, M-*)$. There exists an isotopy $f_t: M \to B_H$ such that:

(1) each f_t is a homeomorphism of M onto a fibre of $B_H \to^p B_G$.

(2) f_0 is the identity map of M regarded as the fibre over $p(*)$,

(3) the trace of the isotopy (i.e. the path $\sigma: t \mapsto f_t(*)$) is a closed path representing α.

Now we define an isotopy $g_t: M \to \rho^* B_H$ by setting $g_t(m) = (f_t(m), f_t(*)) \in \rho^* B_H$. Note that g_t is an isotopy of the pairs $(M, M-*) \to (\rho^*(B_H), \rho^*(B_H) - s(B_H))$.

Now g_0 is the identity on the fibre $(M, M-*)$ over $* \in B_H$. On the other hand $g_1(m) = (f_1(m), *) \in (M, M-*)$. Since f_1 preserves orientation on M by hypotheses, we see that $g_1: (M, M-*) \to (M, M-*)$ must induce the identity homomorphism on $H_n(M, M-*)$. Thus α acts trivially on $H_n(M, M-*)$, which was to be shown.

(c) is obvious since $H_n(M, M-*; Z_2) \cong Z_2$.

Let $(E', E_0') \to B$ be an orientable fibre bundle pair with fibre $(M, M-*)$. Since $H^*(M, M-*) = H^*(R^m, R^m - 0)$, there exists a "Thom isomorphism" $\phi: H^i(B) \cong H^{i+n}(E', E_0')$ which is natural with respect to mappings of fibre bundle pairs. Then $\phi(1) \in H^n(E', E_0; R)$, where R is a ring of coefficients, is the "Thom class". Characteristic classes (Fadell [**6**]) are defined from the Thom class in the obvious way. For example, $\phi^{-1}(\phi(1) \cup \phi(1))$ is the Euler class.

The characteristic classes of M are defined to be the characteristic classes of the homology tangent bundle $(M, M \times M - \Delta)$.

COROLLARY 7. *If $k \in H^i(M; R)$ is a characteristic class of M, then k is in the image of $i^*: H^i(B_H; R) \to H^i(M; R)$.*

Now we assume that M is a closed orientable n-manifold. Then R. F. Brown [**4**] says the Euler class of M is equal to $\chi(M)\mu$ where $\mu \in H^n(M; R)$

is the fundamental class of the manifold and $\chi(M)$ is the Euler-Poincaré number of M. If $\varepsilon \in H^n(B_H; R)$ is the Euler class for the fibre pair $(\rho^*(B_H), \rho^*(B_H)-s(B_H)) \to B_H$, then by Theorem 5 we have $i^*(\varepsilon)=\chi(M)\mu$.

COROLLARY 8. $i^*(\varepsilon)=\chi(M)\mu$.

4. **Proof of Theorem 9.**

THEOREM 9. *Let $M\to E \to^p B$ be a fibre bundle with orientation preserving structural group and let M be a closed, orientable topological manifold. If* $\chi(M)\not\equiv 0 \pmod p$, *then p^* is injective. In addition, the theorem is still true when $p=2$ with no orientability condition on M or on the fibre bundle.*

PROOF. We know that the Euler class of M is $\chi(M)\mu \in H^n(M; Z)$, where μ is a generator, by a theorem of R. F. Brown [**4**]. Let $\chi(M)\mu$ also denote the element in Z_p cohomology. If $\chi(M)\not\equiv 0 \pmod p$, then

$$\chi(M)\mu \neq 0 \in H^n(M; Z_p).$$

Consider the fibration $M \to^i B_H \to^p B_G$. By Corollary 8, $\chi(M)\mu$ is in the image of $i^*: H^n(B_H; Z_p) \to H^n(M; Z_p)$. (Note that in order to apply Corollary 8 we need that Lemma 6 be valid. But Lemma 6(b) requires that G be orientation preserving.) For Z_2 coefficients we apply (c). By naturality, $\chi(M)\mu$ is in the image of i^* for the fibration $M \to^i E \to^p B$.

Lemma 3.1 of [**2**] (or see Theorem 14.5 of [**3**]) says the following: Suppose $F \to^i E \to^p B$ is a fibration with an n-dimensional fibre F such that $\pi_1(B)$ acts trivially on $H^n(F; Z_p)$. If $i^*: H^n(E; Z_p) \to H^n(F; Z_p)$ is nonzero, then p^* is injective.

In the case at hand, $\pi_1(B)$ operates trivially on $H^n(M; Z_p)$ since $\pi_1(B)$ operates trivially on the image of i^* (which is onto $H^n(M; Z_p)$).

REMARK. In a forthcoming paper, we shall remove the hypothesis that M is orientable.

5. **Actions and characteristic classes.**

In this section we shall prove that a characteristic class of a topological manifold M is "inert" under the action $\hat{\omega}: G \times M \to M$. We say $k \in H^*(M; R)$ is *inert* under $\hat{\omega}$ if $\hat{\omega}^*(k)=1\times k$, where R is any ring of coefficients. One consequence of the inertness of characteristic classes is that $\chi(M)\omega^*$ is trivial where $\omega: G \to M$ is evaluation at a base point $*$.

We begin by studying the action $\hat{\omega}: G \times (G/H) \to G/H$ given by $\hat{\omega}(g, g'H)=gg'H$. This action fits inside the following commutative diagram.

$$(*)\qquad \begin{array}{ccc} G \times (G/H) & \xrightarrow{\hat{\omega}} & G/H \\ \Big\downarrow{\scriptstyle i \times 1} & & \Big\downarrow{\scriptstyle i} \\ E \times (G/H) & \xrightarrow{\phi} & B_H = E/H \\ \Big\downarrow{\scriptstyle p \times *} & & \Big\downarrow{\scriptstyle p} \\ B_G & \xrightarrow{1} & B_G \end{array}$$

where ϕ is defined by $\phi(e, gH)=\langle eg\rangle_H$ and $i:G\to E$ is given by $i(g)=g$ and $i:G/H\to E/H$ is given by $i(gH)=\langle g\rangle_H$. Observe that ϕ is well defined and continuous and that (*) in fact commutes.

Since E is contractible, the commutativity of (*) tells us that $i\circ\hat{\omega}$ is homotopic to $G\times(G/H)\to^{\text{proj}}G/H\to^{\iota}B_H$. This proves the following theorem.

THEOREM 10. *Any k which is in the image of $i^*:H^*(B_H; R)\to H^*(G/H; R)$ is inert under the action $\hat{\omega}$. In particular, characteristic classes of topological manifolds are inert under $\hat{\omega}$.*

REMARK. The inertness of characteristic classes of differentiable manifolds under differentiable actions will be shown in **[9]** (i.e. Stiefel-Whitney classes and Pontrjagin class are inert under differentiable actions).

The usefulness of the concept of inertness comes from the following lemma. Let $\omega:G\to M$ be given by $\omega=\hat{\omega}(\cdot, *)$ for base point $*\in M$.

LEMMA 11. *Suppose that u and v are positive dimensional cohomology elements of a space M, and suppose that v is inert under some action $\hat{\omega}$. Then $u\cup v=0$ implies $\omega^*(u)\times v=0$.*

PROOF. We have

$$\begin{aligned} 0 &= \hat{\omega}^*(u \cup v) = \hat{\omega}^*(u) \cup \hat{\omega}^*(v) \\ &= \left(\omega^*(u) \times 1 + \sum a_\iota \times b_\iota + \sum c_i * d_i\right) \cup 1 \times v \\ &= \omega^*(u) \times k + \sum a_\iota \times (b_\iota \cup k) + \sum (c_i * d_\iota) \cup (1 \times v). \end{aligned}$$

Here the a_ι and $c_i\in H^*(M; R)$ are positive dimensional and b_i and $d_\iota\in H^*(G; R)$ and $\times$ is the cohomology cross product and $*$ is the torsion product coming from the Künneth formula. It is easy to see that no term in the expansion has the right dimensions to cancel $\omega^*(u)\times k$. Hence $\omega^*(u)\times k=0$.

The above lemma, with the aid of Corollary 8 gives us the main result.

THEOREM 12. *Let M be a closed orientable topological manifold and let G be a group of homeomorphisms acting on M by the action $\hat{\omega}:G\times M\to M$. Let $\omega:G\to M$ be the evaluation map at the base point $*$. Then $\chi(M)\omega^*$: $\tilde{H}^*(M; R)\to\tilde{H}^*(G; R)$ is trivial where R is any coefficient ring with unit.*

PROOF. By Corollary 8, $i^*(\varepsilon)=\chi(M)\mu$. So $\chi(M)\mu$ is inert by Theorem 10. (We let $\chi(M)\mu$ also stand for the image of $\chi(M)\mu$ in cohomology with coefficients in R.) Since μ is the top dimensional class, $\mu\cup v=0$ for any $v\in\tilde{H}^*(M; R)$. Thus, by Lemma 11,

$$0 = \omega^*(v) \times \chi(M)\mu = \chi(M)\omega^*(v) \times \mu.$$

So $\chi(M)\omega^*(v)$ must equal zero, thus proving the theorem.

REMARK. If $R=Z_2$, the orientability requirement may be dropped.

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