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THE TRACE OF AN ACTION AND THE DEGREE OF A MAP

BY

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ABSTRACT. Two integer invariants of a fibration are defined: the *degree*, which generalizes the usual notion, and the *trace*. These numbers represent the smallest transfers for integral homology which can be constructed for the fibrations. Since every action gives rise to a fibration, we have the *trace of an action*. A list of properties of this trace is developed. This list immediately gives, in a mechanical way, new proofs and generalizations of theorems of Borsuk-Ulam, P. A. Smith, Conner and Floyd, Bredon, W. Browder, and G. Carlsson.

1. Introduction. Let X be any topological space, and let (G, X) denote the action of a group G, or, more generally, a monoid of homotopy equivalences, on X. We define a nonnegative integer $\operatorname{tr}(G, X)$ called the *trace* of the action. This trace expresses the relationships of related actions by means of divisibility properties of integers. It *divides* almost every interesting integer associated to (G, X): for example, the order of G (if G is finite) or the Euler-Poincaré number of X if $H_*(X; \mathbf{Z})$ is finitely generated. It *characterizes* free actions of finite groups on compact manifolds as well as interesting classes of actions involving elementary abelian p-groups and compact Lie groups. Finally it is *invariant* under equivariant retraction, concordance, and equivariant degree one maps between oriented manifolds.

This trace can be calculated for many actions and these results and the divisibility, characterization, and invariance results alluded to above form a calculus wherein a number of famous results can be easily proved. Among these are the Borsuk-Ulam theorem; Smith's theorem that a finite group cannot act freely on $\mathbf{R}^n$; Carlsson's theorem about elementary abelian p groups acting freely on $S^n \times S^n \times \cdots \times S^n$; Conner and Floyd's theorem that if an elementary abelian p-group acts without fixed point on a closed manifold M the manifold must bound; and others, such as an elementary p-group cannot act with a single fixed point smoothly on a closed oriented manifold.

We collect here the principal properties of the trace for easy reference. Then we give some examples of the use of these properties by proving some of the above-mentioned theorems. We then discuss how the trace is defined in terms of two other integer invariants; the *degree of a map* (which generalizes the degree of maps between closed connected oriented manifolds) and the *fibre number* (G, X) of an action.

We conclude this introduction with a description of the contents of this paper.

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(1.1) THEOREM.

(A) *For any arbitrary action* (G, X) *the trace* $\mathrm{tr}(G, X)$ *is a nonnegative integer satisfying the following properties.*

(1) *If* $f\colon (G, X) \to (G, Y)$ *is an equivariant map, then* $\mathrm{tr}(G, Y) | \mathrm{tr}(G, X)$. (6.2)(a).

(2) *If* $\varphi\colon H \to G$ *is a homomorphism and* (H, X) *is induced from* (G, X) *by* φ, *then* $\mathrm{tr}(H, X) | \mathrm{tr}(G, X)$. (6.2)(b).

(3) *If* (G, X) *has a stationary point, then* $\mathrm{tr}(G, X) = 1$. (6.7).

(4) *If* $r\colon (G, X) \to (G, Y)$ *is an equivariant retraction then* $\mathrm{tr}(G, X) = \mathrm{tr}(G, Y)$. (6.3).

(B) *If* X *is homotopy equivalent to a CW complex, then*

(1) $\mathrm{tr}(G, X) = \mathrm{tr}(G, Y)$ *if* $f\colon (G, X) \to (G, Y)$ *is a quasiequivalence* (*i.e. a G-map which is also a homotopy equivalence*). (11.1).

(2) $\mathrm{tr}(G, X) | \Lambda_f$, *where* Λ_f *is the Lefschetz number of a G-map* f, *if* $H_*(X; \mathbf{Z})$ *is finitely generated*. (6.7).

(3) $\mathrm{tr}(G, X) | \chi(X)$, *the Euler-Poincaré number, if* $H_*(X, \mathbf{Z})$ *is finitely generated*. (6.7).

(C) *Let* G *be a finite group of order* $|G|$.

(1) $\mathrm{tr}(G, X) \mid |G|$. (6.9).

(2) *If* $\mathrm{tr}(G, X) = |G|$, *then* (G, X) *is free*.

(3) *Let* N *be a manifold which is dominated by a finite CW complex. Then* $\mathrm{tr}(G, N) = |G|$ *if and only if* (G, N) *is a free action*. (6.10).

(D) *Let* G *be a compact Lie group acting on a compact connected manifold* N.

(1) $\mathrm{tr}(G, N) \mid \prod_{r=1}^{\dim N} \exp(H^{r+1}(B_G; H_r(N; \mathbf{Z})))$ *if* G *acts trivially on* $H_*(N; \mathbf{Z})$ (*cf.* **[Bro]**). (9.1).

(2) *If* G *is connected then either* $\mathrm{tr}(G, N) = 0$ *or it divides the order of the Weyl group*; $\mathrm{tr}(G, N) = 0$ *if and only if no isotropy subgroup has maximal rank* (*i.e. contains the maximal torus* T). (6.11).

(3) *If* M_1 *and* M_2 *are connected closed oriented manifolds of the same dimension and* $f\colon (G, M_1) \to (G, M_2)$ *is a G-map, then*

(a) $\mathrm{tr}(G, M_2) \mid \mathrm{tr}(G, M_1) \mid (\deg f)\mathrm{tr}(G, M_2)$. (8.1).

(b) *In particular if* $\deg f = 1$, *then* $\mathrm{tr}(G, M_1) = \mathrm{tr}(G, M_2)$. (8.2).

(c) *If* G *preserves orientation and* $g\colon M_1 \to M_2$ *is another G-map, then* $\mathrm{tr}(G, M_1) | \Lambda_{f,g}$, *the coincidence number*. (6.7).

(E) *If a group* G *acts smoothly on a closed connected oriented manifold* M, *then* $\mathrm{tr}(G, M) | P_\omega$ *where* P_ω *is any Pontrjagin number*. (6.7). *Similarly if* G *acts holomorphically on a closed oriented complex manifold, then* $\mathrm{tr}(G, M)$ *divides any Chern number*.

(F) (Browder) *If* $G = (\mathbf{Z}_p)^r = \mathbf{Z}_p \times \cdots \times \mathbf{Z}_p$ *for* p *a prime, and if* M *is a smooth action, then* $\mathrm{tr}(G, M) =$ (*the number of points in the smallest orbit*). (7.4).

Now we shall consider some immediate applications of the trace. First we note that Smith's theorem (no finite group can act freely on $\mathbf{R}^n$) and the Borsuk-Ulam theorem (essentially that any self-map $f\colon S^n \to S^n$ of the unit sphere into itself satisfying $f(-x) = -f(x)$ must have odd degree) follow immediately from (C)(3) and (B)(2) which proves the following generalization [**G5**, Theorem 3].

(1.2) COROLLARY. *If a finite group acts freely on a manifold N dominated by a finite complex and if f is an equivariant self-map, then* $|G|\,|\Lambda_f$.

Carlsson's theorem [**Ca**], ($(\mathbf{Z}_p)^r$ can act freely on $(S^n)^k$ and trivially on $H_*((S^n)^k; \mathbf{Z})$ only if $r \leqslant k$) can be generalized as follows in the smooth case.

(1.3) COROLLARY (BROWDER). *If G is an elementary abelian p-group which acts trivially on $H_*((S^n)^k; \mathbf{Z})$, then the order of the smallest orbit cannot exceed p^k.*

PROOF. Let p^l be the order of the smallest orbit. Then combining (F) with (D)(1) gives us

$$p^l = \operatorname{tr}\big(G, (S^n)^k\big) \Big| \prod_{n=1}^{kn} \exp\Big(H^{n+1}\big(G; H_n\big((S^n)^k, \mathbf{Z}\big)\big)\Big) = p^k.$$

Conner and Floyd in [**CF1**, (43.7)] prove that if (G, M) is a smooth action of an elementary abelian p-group on a closed connected oriented manifold, then $p|P_\omega$, any Pontrjagin number of M, if (G, M) has no stationary point. This follows immediately from (E) and (F) and in fact we have the following generalization.

(1.4) COROLLARY. *If the minimal orbit of G has order p^l, then $p^l|P_\omega$.*

Similarly if a torus T acts smoothly without a stationary point on a closed connected oriented manifold, then all the Pontrjagin numbers vanish. This follows immediately from (D)(2) and (E). Compare with [**Os**].

More applications will appear in the body of the paper. Also some of the hypotheses in Theorem (1.1) are relaxed in later statements.

Now we describe the definition of $\operatorname{tr}(G, X)$. In fact we shall define several integer invariants for an action (G, X). In each case, the invariant is actually defined for fibrations or for maps. Then the invariant for the universal fibration $X \to X_G \xrightarrow{p_G} B_G$ is the invariant of the action (G, X).

For example, suppose that $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration so that F is connected and $H^i(F; \mathbf{Z}) = 0$ for $i > n$ and $H^n(F; \mathbf{Z}) \cong \mathbf{Z}$. Then the image of i^*: $H^n(E; \mathbf{Z}) \to H^n(F; \mathbf{Z}) \cong \mathbf{Z}$ is a subgroup of the integers. Define the *fibre number* $\Phi(p)$ to be the nonnegative integer which generates this subgroup. If (G, X) is an action and $H^i(X; \mathbf{Z}) = 0$ for $i > n$ and $H^n(X; \mathbf{Z}) \cong \mathbf{Z}$, then the fibre number of the action $\Phi(G, X) \equiv \Phi(p_G)$.

Variations of this number are obtained by changing the coefficients of the cohomology groups.

One of the most important properties of the trace can now be stated.

(1.5) THEOREM. *Let G be a group so that B_G has finite type. Let M be a closed connected orientable manifold. If G acts on M preserving orientation, then* $\operatorname{tr}(G, M) = \Phi(G, M)$. (6.4).

In general, however, $\Phi(G, X) \neq \operatorname{tr}(G, X)$.

The *degree of a map* f: $X \to Y$ is defined by looking at all the transfers to f_*: $H_*(X; \mathbf{Z}) \to H_*(Y; \mathbf{Z})$ and taking the integer attached to a "minimal transfer". This generalizes the usual definition of degrees of maps between closed oriented manifolds of the same dimension. Again one can vary the coefficients or use cohomology instead of homology to define variations of deg f.

The trace of a fibration, tr(p), is then defined to be the least common multiple of the degrees of the projections of all the pullbacks of p. Then $\mathrm{tr}(G, X) = \mathrm{tr}(p_G)$. Again there are variations based on the coefficient groups. These variant traces still have some of the properties of the trace. The version associated with $\mathbf{Z}_2$ coefficients can be used to prove Conner and Floyd's theorems [**CF1**, (30.1), (31.1)] that if $(\mathbf{Z}_2)^r$ acts smoothly on a closed manifold M then

(a) there cannot exist only one stationary point and

(b) if there is no stationary point then M bounds.

In §2 two integers are associated with the homomorphism h: $A \to B$ between abelian groups: the degree of h, and the codegree of h. Elementary properties are established.

In §3 the degree and codegree of an arbitrary map are defined. They are both shown to generalize the usual concept of degree from closed oriented manifolds to arbitrary spaces. An example is given showing that the degree and codegree are in fact different invariants. The degree depends upon transfers, and in (3.4) the latest results involving transfers are collected and phrased in terms of degree.

In §4 we study the fibre number, and in (4.2) we establish the central fact that for manifold bundles the degree equals the fibre number.

In §5 we define the trace of a fibration and show that all the invariants of degree, fibre number, and trace coincide for manifold bundles. Also we show that every fibration with fibre and base finite CW complexes is a "retract" of a manifold bundle.

In §6 we establish many of the properties mentioned in (1.1), including the characterization of free actions. We end the section by characterizing trace = 0 actions for compact connected Lie groups. Along the way we obtain as a corollary the fact that orientation reversing involutions kill all Pontrjagin numbers (6.8).

In §7 we study the trace for elementary p-group actions. Here we show that for compact manifolds the trace characterizes actions with stationary points.

In §8 we show that $\mathrm{tr}(G, M)$ is an invariant of degree 1 maps. We apply this to prove that elementary abelian p-groups cannot act on oriented closed manifolds with only one stationary point. We discuss Stong's recent solution to the problem of which finite G can act on M with only one stationary point, and we use it to show that the statement "$\mathrm{tr}(G, M) = 1$ if and only if (G, M) has a stationary point" holds only for G an elementary abelian p-group.

In §9 we calculate some examples of the trace using the Serre spectral sequence.

In §10 we show that if (S^1, M) has no fixed point and M^n is a closed oriented n-manifold, then every oriented G-vector bundle of dimension n has a nonequivariant cross section.

In §11 we show that trace is a G-quasi-homotopy type invariant. As an example of

what this implies, consider a manifold M which is dominated by a finite complex. Let C be any contractible space, and let B_G have finite type. If G acts on $M \times C$ leaving a slice $M \times c_0$ invariant, then the action is free if and only if the action on $M \times c_0$ is free.

In §12 we study the orbit map of an action $\omega\colon G \to X$. We show that it frequently induces the trivial homomorphism $0 = \omega_*\colon \tilde{H}_*(G, \mathbf{Z}) \to \tilde{H}_*(X; \mathbf{Z})$ when $\operatorname{tr}(G; X) \neq 0$.

Finally, in §13 we use the concept of fibre number to produce two simply connected smooth manifolds which are rationally equivalent but whose diffeomorphism groups are not rationally equivalent.

One result not proved here but worth mentioning is Murad Ozaydin's theorem [**Oz**], for G a compact Lie group acting on a G-complex, that the trace of the action is equal to the trace of the action restricted to the singular set. This is remarkable, given the global character of the definition of trace.

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2. The degree of a homomorphism. In topology one studies a class of maps between spaces by studying the induced homomorphisms between homology groups or homotopy groups. It is reasonable to expect that the topological properties of the class of maps should result in group theoretic properties of the induced homomorphisms. How can we describe these properties?

For example, consider the set of *compact maps*, where $f\colon X \to Y$ is said to be *compact* if Y and the homotopy theoretic fibre of f are homotopy equivalent to finite CW complexes. The induced homomorphisms $f_*\colon H_*(X; \mathbf{Z}) \to H_*(Y; \mathbf{Z})$ may satisfy algebraic properties. How can we describe them? The abelian groups $A = H_*(X; \mathbf{Z})$ and $B = H_*(Y; \mathbf{Z})$ could be very general. In this case the only *a priori* conditions are that A and B should be finitely generated, so any common properties that this class of homomorphisms could have must be expressed in terms that are independent of the exact forms of A and B.

Now let $h\colon A \to B$ be a homomorphism between abelian groups. We might say that h is the trivial homomorphism, or that h is onto, or h is one-to-one, or h is an isomorphism, or h splits. Each of these properties of h does not depend on the precise form of the groups A and B. The purpose of this section is to introduce three properties of h, the *exponent* of h, the *degree* of h, and the *codegree* of h. In each case these will be nonnegative integers associated to h and independent of the form of A and B.

Of all the possible homomorphisms $h\colon A \to B$ between abelian groups, the only one we know always exists is the trivial homomorphism $0\colon A \to B$. For self-homomorphisms $h\colon A \to A$ we know a little more. Every integer $N \in \mathbf{Z}$ induces an endomorphism $N\colon A \to A$ by sending $a \to N \cdot a = a + \cdots + a$ (N times). Different integers need not induce different endomorphisms. If N_1 and N_2 induce the

same endomorphisms, then $N_1 - N_2 = 0$ as endomorphisms. The smallest positive integer which induces the trivial endomorphism on A is called the *exponent of* A, denoted $\exp A$. Then $\exp A$ divides $N_1 - N_2$, denoted $(\exp A) | (N_1 - N_2)$. If there is no positive integer which annihilates A, then $\exp A = 0$. In that case every integer induces a unique homomorphism.

Now if $h\colon A \to B$ is a homomorphism between abelian groups and N is an integer, we have $N \circ h = h \circ N$. We abbreviate $N \circ h$ by Nh.

The smallest positive integer N such that $Nh = 0$, the trivial homomorphism, will be called the *exponent of* h, denoted $(\exp h)$. If there is no such positive integer then $(\exp h) = 0$. It is clear that if $Nh = 0$ then $(\exp h) | N$. Also

$$(\exp h) | \text{g.c.d.}(\exp A, \exp B).$$

Note that $(\exp h) = 1$ if and only if $h = 0$.

In order to define the degree and codegree of $h\colon A \to B$, we must first introduce the notions of transfer and cotransfer for h. Let $\tau\colon B \to A$ be a homomorphism. If $h \circ \tau = N$, where N is the endomorphism on B induced by the integer N, then τ is a *transfer* homomorphism for h and the integer N is *associated* to τ. Of course, there may be more than one integer associated with τ if $\exp B \neq 0$. Similarly, if $\tau \circ h = N$, then τ is a *cotransfer* homomorphism for h. Here N is the endomorphism on A induced by the integer N *associated* with the cotransfer.

Note that the trivial homomorphism $0\colon B \to A$ is both a transfer and a cotransfer to $h\colon A \to B$, and the integer 0 is associated to $\tau = 0$. Thus the set of transfers, the set of cotransfers for h, and the sets of integers associated with them are always nonempty.

(2.1) DEFINITION. *The* (*co*)*degree of* h, *denoted* (co)deg h, *is the smallest positive integer associated with a* (*co*)*transfer. If no such positive integer exists, then* (co)deg $h = 0$.

(2.2) PROPOSITION. *If* N *is an integer associated with a* (*co*)*transfer, then* (co)deg $h | N$.

PROOF. Let N_1 and N_2 be integers associated with transfer τ_1 and τ_2 respectively. Then the linear combination $aN_1 + bN_2$ is an integer associated witht the transfer $a\tau_1 + b\tau_2$, where a and b are any integers. This follows since

$$h \circ (a\tau_1 + b\tau_2) = h \circ a\tau_1 + h \circ b\tau_2 = ah \circ \tau_1 + bh \circ \tau_2 = aN_1 + bN_2.$$

Thus the set of integers associated to a transfer forms an ideal, deg h is its generator. This proves the proposition for transfers. The proof for cotransfers is similar.

As an example let $h\colon \mathbf{Z} \to \mathbf{Z}$ be a homomorphism. Then $\deg h = \operatorname{codeg} h = |h(1)|$.

Now $\deg h | \exp B$ and $\operatorname{codeg} h | \exp A$. Degree and codegree characterize isomorphisms. That is, $\deg h = \operatorname{codeg} h = 1$ if and only if h is an isomorphism. Now $\deg h = 1$ if and only if h is onto and B is a direct summand of A. Also $\operatorname{codeg} h = 1$ if and only if h is one-to-one and A is a direct summand of B. However $\deg h$ and $\operatorname{codeg} h$ do not characterize surjectivity or injectivity as $\mathbf{Z} \twoheadrightarrow \mathbf{Z}/n\mathbf{Z}$ and $\mathbf{Z} \overset{n}{\rightarrowtail} \mathbf{Z}$ show.

(2.3) PROPOSITION. $\deg h_2 | \deg(h_2 \circ h_1) | (\deg h_1) \cdot (\deg h_2)$.

PROOF. By Proposition (2.2) we need only construct a transfer and then the degree will divide the associated number. So let τ_1, τ_2, and τ_3 be transfers associated with the degrees of h_1, h_2, and $h_3 = h_2 \circ h_1$ respectively. Then $h_1 \circ \tau_3$ is a transfer for h_2 with associated number

$$h_2 \circ (h_1 \circ \tau_3) = (h_2 \circ h_1) \circ \tau_3 = \deg(h_2 \circ h_1).$$

Also $\tau_1 \circ \tau_2$ is a transfer for h_3 with associated number

$$\begin{aligned}(h_2 \circ h_1) \circ (\tau_1 \circ \tau_2) &= h_2 \circ (h_1 \circ \tau_1) \circ \tau_2 = h_2 \circ (\deg h_1) \circ \tau_2 \\ &= (\deg h_1) h_2 \circ \tau_2 = (\deg h_1)(\deg h_2).\end{aligned}$$

(2.4) PROPOSITION.

$$\operatorname{codeg} h_1 | \operatorname{codeg}(h_2 \circ h_1) | (\operatorname{codeg} h_1)(\operatorname{codeg} h_2).$$

PROOF. Note that the term on the left involves h_1, whereas in the analogous Proposition (2.3) for degree, the term on the left involves h_2. The proof proceeds as in the above proposition. Here $\tau_3 \circ h_2$ is a cotransfer for h_1 associated with $\operatorname{codeg}(h_2 \circ h_1)$, and $\tau_1 \circ \tau_2$ is a cotransfer for $h_2 \circ h_1$ associated with

$$(\operatorname{codeg} h_1) \cdot (\operatorname{codeg} h_2).$$

Let ΣA_i and ΣB_i be direct sums of abelian groups. We say that a homomorphism $h: \Sigma A_i \to \Sigma B_i$ is *graded* if $h = \Sigma h_i$ such that $h_i: A_i \to B_i$ are homomorphisms.

(2.5) PROPOSITION. *Let $h: \Sigma A_i \to \Sigma B_i$ be a graded homomorphism. Then*

$$\deg h = \text{l.c.m.}\{\deg h_i\}, \quad \operatorname{codeg} h = \text{l.c.m.}\{\operatorname{codeg} h_i\}.$$

PROOF. Let $\tau: \Sigma B_i \to \Sigma A_i$ be a transfer which realizes $\deg h$. That is, $h \circ \tau = \deg h$. Let

$$\tau_i: B_i \to \Sigma B_i \xrightarrow{\tau} \Sigma A_i \xrightarrow{\text{proj}} A_i.$$

Then $\Sigma \tau_i$ is also a transfer associated to $\deg h$. This follows since $\tau(b_i) = \tau_i(b_i) + x$, where $b_i \in B_i$ and $x \in \Sigma_{j \neq i} A_j$, the complement to A_i, so

$$(\deg h) b_i = h\tau(b_i) = h(\tau_i(b_i) + x) = h_i \tau_i(b_i) + h(x).$$

Now $h(x)$ is in the complement of B_i since h is graded. Hence $(\deg h)(b_i) = h_i \tau_i(b_i)$ and $h(x) = 0$. So we have shown that $h \circ (\Sigma \tau_i) = (\deg h)$. Now it is easy to see that the least common multiple of the degrees of the h_i's has an associated graded transfer $\Sigma \bar{\tau}_i$, where $\bar{\tau}_i$ is a transfer associated to $\deg(h_i)$ multiplied by the quotient of the least common multiple and $\deg h_i$. Hence $\deg h | \text{l.c.m.}\{\deg h_i\}$. But also since $h_i \tau_i = \deg h$ for all i, we see that $\text{l.c.m.}\{\deg h_i\} | \deg h$. Therefore $\deg h = \text{l.c.m.}\{\deg h_i\}$, as was to be proved. Similarly for codegree.

The situation is more complicated for ungraded maps. If A and B are finitely generated, then $A \cong F \oplus T$ and $B \cong F' \oplus T'$, where F and F' are free abelian groups and T and T' are torsion subgroups. An ordinary homomorphism $h: A \to B$ does not respect the direct sum structure even through $h|T = h_T: T \to T'$.

The following example is used in §3 to show that degree $\neq$ codegree.

(2.6) EXAMPLE. Let $A = \mathbf{Z} \oplus \mathbf{Z}_8$. Let $\underline{a}$ denote the generator of $\mathbf{Z}$, and let α denote the generator of $\mathbf{Z}_8$. Let h: $A \to A$ be defined by $h(a) = 2a + \alpha$ and $h(\alpha) = 4\alpha$. An examination of h shows that $\deg h = 8$. On the other hand, $h_T = h|\mathbf{Z}_8$: $\mathbf{Z}_8 \to \mathbf{Z}_8$ is given by $h_T(\alpha) = 4\alpha$. Hence $\deg(h_T) = 4$. Also

$$h_F\colon \mathbf{Z} \to \mathbf{Z} \oplus \mathbf{Z}_8 \xrightarrow{\text{proj}} \mathbf{Z}$$

is given by $h_F(a) = 2a$. Hence $\deg h_F = 2$. So

$$\text{l.c.m.}\{\deg h_T, \deg h_F\} = 4 \neq 8 = \deg h.$$

3. The degree of a map. Let f: $X \to Y$ be a map. We define the *degree* of f by $\deg f = \deg f_*$, where f_*: $H_*(X, \mathbf{Z}) \to H_*(Y; \mathbf{Z})$. The *codegree* of f, denoted $\operatorname{cdg} f$, is defined by $\operatorname{cdg} f = \operatorname{codeg} f^*$, where f^*: $H^*(Y; \mathbf{Z}) \to H^*(X; \mathbf{Z})$.

While our main concern will be $\deg f$ and $\operatorname{cdg} f$, it is convenient to extend the definition for any abelian group of coefficients A. Thus $\deg_A f = \deg f_*$, where f_*: $H_*(X, A) \to H_*(Y; A)$, and $\operatorname{cdg}_A f = \operatorname{codeg} f^*$, where f^*: $H^*(Y, A) \to H^*(X; A)$.

Now $\deg_{A\oplus B} f = \text{l.c.m.}\{\deg_A f, \deg_B f\}$. This follows from Proposition (2.5) and the fact that

$$\begin{aligned} f_*\colon H_*(X; A \oplus B) &\cong H_*(X; A) \oplus H_*(X; B) \\ &\to H_*(Y; A \oplus B) \cong H_*(Y; A) \oplus H_*(Y; B) \end{aligned}$$

is a graded homomorphism. Similarly

$$\operatorname{cdg}_{A\oplus B} f = \text{l.c.m.}\{\operatorname{cdg}_A f, \operatorname{cdg}_B f\}.$$

This fact tells us that the situation when $A \cong \mathbf{Z}_n$ is particularly important. We use the notation $\deg_n f$ and $\operatorname{cdg}_n f$ when $A = \mathbf{Z}_n$. In the event that n is a prime p we see that $\deg_p f$ equals either 1 or p. Note that $\deg_p f = 1$ if and only if f_*: $H_*(X; \mathbf{Z}_p) \to H_*(Y; \mathbf{Z}_p)$ is onto. Similarly $\operatorname{cdg}_p f = 1$ or p and $\operatorname{cdg}_p f = 1$ if and only if f^*: $H^*(Y; \mathbf{Z}_p) \to H^*(X; \mathbf{Z}_p)$ is one to one. Thus for finite complexes X and Y

$$\deg_p f = \operatorname{cdg}_p f.$$

In general, $\deg f \neq \operatorname{cdg} f$. This may be seen from the following example.

(3.1) EXAMPLE. We construct a free chain complex C consisting of a minimal number of chains so that $H_n(C) \cong \mathbf{Z} \oplus \mathbf{Z}_8$ and $H_i(C) = 0$ for $i \neq n$. Now construct a chain map f: $C \to C$ which induces h: $H_n(C) \to H_n(C)$, where h is the homomorphism of Example (2.6). For cohomology, $H^{n+1}(C) \cong \mathbf{Z}_8$, and $H^n(C) \cong \mathbf{Z}$, and f induces h_T: $H^{n+1}(C) \to H^{n+1}(C)$, and f induces h_F: $H^n(C) \to H^n(C)$. Now we can realize C by a Moore space $M(\mathbf{Z} \oplus \mathbf{Z}_8, n)$ and the chain map f by a topological map f: $M \to M$. Hence from Example (2.6) we see that

$$\deg f = 8 \neq 4 = \operatorname{cdg} f.$$

Now let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be continuous maps on topological spaces.

(3.2) PROPOSITION.

$$\deg g | \deg(g \circ f) | (\deg f)(\deg g)$$

and

$$\operatorname{cdg} g | \operatorname{cdg}(g \circ f) | (\operatorname{cdg} f)(\operatorname{cdg} g).$$

PROOF. The proof for degree follows directly from Proposition (2.3). The proof for codegree follows from Proposition (2.4), since $\operatorname{cdg}(g \circ f) = \operatorname{codeg}((g \circ f)^*) = \operatorname{codeg}(f^* \circ g^*)$. The variation between Propositions (3.2) and (2.4) has prompted us to distinguish between "cdg" for a map and "codeg" for a homomorphism.

(3.3) COROLLARY. *If* $r\colon X \to Y$ *is a retraction, then*

$$\deg f = \deg(f \circ r) \quad \textit{and} \quad \operatorname{cdg} f = \operatorname{cdg}(f \circ r).$$

PROOF. Since r is a retraction, $\deg r = \operatorname{cdg} r = 1$. Thus $\deg f | \deg(f \circ r) | \deg f$ from the proposition. So $\deg(f \circ r) = \deg f$. Similarly for $\operatorname{cdg} f$.

Whenever a map is known to have a transfer, the degree divides the associated number. Thus there are many results concerning degree in the literature. We collect them below. We let $p\colon X \to Y$ be a map between CW complexes with homotopy theoretic fibre F.

(3.4) PROPERTIES. (a) *If* $H_*(F; \mathbf{Z})$ *is a finitely generated group, then* $\deg p | \chi(F)$. *If* $f\colon X \to X$ *satisfies* $p \circ f = p$, *then* $\deg p | \Lambda_{f'}$ *where* $\Lambda_{f'}$ *is the Lefschetz number of* $f' = f|F$. *Similarly* $\operatorname{cdg} p$ *divides* $\chi(F)$ *and* $\Lambda_{f'}$ [**G6**, *Transfer Theorem*].

(b) *If* $f, g\colon X \to X'$ *are two fibre maps and if* $p\colon X \to Y$ *and* $p'\colon X' \to Y$ *are oriented fibre bundles with* F *and* F' *oriented manifolds, then* $\deg p | \Lambda_{f',g'}$ *and* $\operatorname{cdg} p | \Lambda_{f',g'}$, *where* $\Lambda_{f',g'}$ *is the coincidence number of* f' *and* $g'\colon F \to F'$, *which are restrictions of* f *and* g [**N**].

(c) *Let* G *be a compact Lie group, acting on a space* X *of the homotopy type of a* G*-CW complex, and let* K *be a closed subgroup. Let* $p\colon X/K \to X/G$ *be the orbit map. Then* $\operatorname{cdg} p | \chi(G/K)$ [**Ol**, **LMM**].

(d) *Let* $f\colon M \to N$ *be a smooth map between smooth oriented closed manifolds. Then* $f^{-1}(x)$ *is a closed manifold for* $x \in N$ *a regular point. Then* $\deg f$ *and* $\operatorname{cdg} f$ *divide the Pontrjagin numbers of* $f^{-1}(x)$ [**G4**, *Theorem* 3].

(e) *Let* $f\colon M \to N$ *be as in* (d) *without the requirement of orientability. Then* $\deg_2 f$ *and* $\operatorname{cdg}_2 f$ *divide the Stiefel-Whitney numbers of* $f^{-1}(x)$. *In particular, if* $\deg_2 f = 2$, *then* $f^{-1}(x)$ *bounds* [**G4**].

(f) *Let* F *be a finite dimensional CW complex so that* $\pi_{2n}(F)$ *is finitely generated, where* F *is the fibre of* $q\colon X \to S^{2n+1}$. *Then* $\deg q = \operatorname{cdg} q = 1$.

(g) *Let* $H_*(F; \mathbf{Z}_p)$ *be finitely generated, where* F *is the fibre of* $X \xrightarrow{q} S^{2n}$. *Suppose* $\chi(F) \neq 0$. *Then* $\deg_p(q) = \operatorname{cdg}_p(q) = 1$.

(h) *For* F *a finite CW complex such that* $\chi(F) \neq 0$, $\deg q = 1$ *where* $X \xrightarrow{q} S^2$. (*In fact, in this situation there is a cross section.*) *In addition, if* $X \xrightarrow{q} S^{2n}$ *is a fibre bundle with a Lie group as structural group, then* $\deg q = \operatorname{cdg} q = 1$.

REMARK. Statements (f), (g), and (h) follow from facts about the evaluation subgroups $G_n(X)$. If $h\colon \pi_*(F) \to H_*(F; \mathbf{Z})$ is the Hurewicz homomorphism, then $G_{2n}(F) \subseteq \ker h$, $G_{2n+1}(F) \subseteq \ker(h \otimes \mathbf{Z}_p)$ if $\chi(X) \neq 0$, $G_1(X) = 0$ if $\chi(X) \neq 0$, and $\omega_* = 0\colon \tilde{H}_*(G; \mathbf{Z}) \to \tilde{H}_*(F; \mathbf{Z})$. See [**G3**, Theorem 1; **G2**, Theorem 4.1; **G1**, Theorem IV.1; and **BG**, Theorem 5]. Now we consider a fibration $F \xrightarrow{i} E \xrightarrow{p} S^n$ and the Serre exact sequence for it. The fact that $\omega_* = 0$ implies that p_* is onto. This gives us the results about degree.

The following theorem is the key fact about degree. A consequence shows that degree generalizes the usual definition of degree of maps between spheres of the same dimension and between manifolds of the same dimension.

(3.5) THEOREM. *Let $f\colon X \to M$ be a map from a space X to a closed oriented manifold M. (More generally, M is a Poincaré duality space.) Then $(\deg f)[M]$ generates the image of f_* in $H_m(M; \mathbf{Z})$, where $[M]$ is the fundamental class of M.*

PROOF. Let $\beta[M]$ generate the image of f_* in $H_m(M; \mathbf{Z})$.

(a) $\beta \mid \deg f$: There is a transfer τ so that $f_* \circ \tau = \deg f$ so $(\deg f)[M]$ is in the image of f_*. Hence $\beta \mid \deg f$.

(b) $\deg f \mid \beta$: Let $\Gamma \in H_m(X; \mathbf{Z})$ be an element so that $f_*(\Gamma) = \beta[M]$. We define $\tau\colon H_*(M; \mathbf{Z}) \to H_*(X; \mathbf{Z})$ by $\tau = (\cap\Gamma) \circ f^* \circ (\cap[M])^{-1}$, where $\cap\Gamma$ is cap product with Γ and of course $(\cap[M])$ is an isomorphism. Now $f_* \circ \tau = \beta$, hence $\deg f \mid \beta$. To see this suppose $x = y \cap [M]$. Then

$$f_*\tau(x) = f_*\big(f^*(y) \cap \Gamma\big) = y \cap f_*\Gamma = \beta y \cap [M] = \beta x.$$

(3.6) REMARK. *If M satisfies Poincaré duality for $\mathbf{Z}_n$ coefficients, then the same proof shows that $\deg_n f = \beta$ where $\beta[M]$ is a generator of the image of $f_*\colon H_m(X; \mathbf{Z}_n) \to H_m(M; \mathbf{Z}_n) \cong \mathbf{Z}_n$.*

(3.7) THEOREM. *Suppose $f\colon X \to M$ is a map such that M satisfies Poincaré duality for $\mathbf{Z}_n$ coefficients and $H_*(X; \mathbf{Z})$ has finite type. Then $\deg_n f = \operatorname{cdg}_n f$.*

PROOF. Define a homomorphism $\bar{\tau}\colon H^*(X; \mathbf{Z}_n) \to H^*(M; \mathbf{Z}_n)$ by letting

$$\bar{\tau}(x) \cap [M] = f_*(x \cap \Gamma),$$

where $\Gamma \in H^m(X; \mathbf{Z}_n)$ is such that $f_*(\Gamma) = (\deg h)[M]$. Then

$$\bar{\tau}\big(f_*(x)\big) \cap [M] = f_*\big(f^*(x) \cap \Gamma\big) = x \cap f_*(\Gamma) = (\deg f)x \cap [M].$$

Hence $\bar{\tau} \circ f^* = \deg f$. Hence $\operatorname{cdg}_n f \mid \deg_n f$.

We apply the universal coefficient theorem in Spanier [**Sp**, p. 248, Theorem 12]. This requires that $H_*(X; \mathbf{Z})$ has finite type.

$$\begin{array}{ccccccc}
0 \to \operatorname{Ext}\big(H^{m+1}(X; \mathbf{Z}_n); \mathbf{Z}_n\big) & \to & H_m(X; \mathbf{Z}_n) & \to & \operatorname{Hom}\big(H^m(X; \mathbf{Z}_n); \mathbf{Z}_n\big) \to 0 \\
\downarrow & & \downarrow f_* & & \downarrow f^{\#} \\
0 & \to & H_m(M; \mathbf{Z}_n) & \xrightarrow{\cong} & \operatorname{Hom}\big(H^m(M; \mathbf{Z}_n); \mathbf{Z}_n\big) \to 0 \\
 & & \wr\wr & & \wr\wr \\
 & & \mathbf{Z}_n & & \mathbf{Z}_n
\end{array}$$

Let $\bar{\tau}$: $H^m(X;\mathbf{Z}_n) \to H^m(M;\mathbf{Z}_n)$ be a cotransfer associated to $\operatorname{cdg}_n f$. Then $\bar{\tau} \in \operatorname{Hom}(H^m(X;\mathbf{Z}_n);\mathbf{Z}_n)$. Let $\tau \in H_m(X;\mathbf{Z}_n)$ be an element which maps onto $\bar{\tau}$. Then

$$f_*(\tau) = f^{\#}(\bar{\tau}) = \bar{\tau} \circ f^* = \operatorname{cdg}_n f.$$

Then Theorem 3.5 gives $\deg_n f \,|\operatorname{cdg}_n f$. Since $\operatorname{cdg}_n f \,|\deg_n f$, from the above paragraph we have $\operatorname{cdg}_n f = \deg_n f$.

4. The fibre number. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration and suppose that $H^i(F;\mathbf{Z}) \cong 0$ for $i > m$ and $H^m(F;\mathbf{Z}) \cong \mathbf{Z}$. The image of i^*: $H^m(E;\mathbf{Z}) \to H^m(F;\mathbf{Z}) \cong \mathbf{Z}$ is a subgroup of $\mathbf{Z}$ and so has a nonnegative generator Φ. We say that $\Phi = \Phi(p)$ is the *fibre* number of p. If we replace $\mathbf{Z}$ by $\mathbf{Z}_n$ in this definition, then we obtain $\Phi_n(p) \in \mathbf{Z}$, the *fibre number for* $\mathbf{Z}_n$ *coefficients*, the smallest positive integer so that $\Phi_n[\bar{F}]$ generates $\operatorname{im} i^* \subset H^m(F;\mathbf{Z}_n)$.

We say the fibration $F \to E \xrightarrow{p} B$ is *orientable* if $\pi_1(B)$ acts trivially on $H^m(F;\mathbf{Z})$, or on $H^m(F;\mathbf{Z}_n)$ as the case may be. If $\Phi \neq 0$, then it is easy to see that the fibration must be orientable. For notational convenience, we adopt the convention that $\mathbf{Z}_0 = \mathbf{Z}$.

Now suppose that $F \xrightarrow{i} E \xrightarrow{p} B$ is an m-oriented fibration. Then *integration along the fibre is defined by*

$$p^{\#}\colon H^i(E;\mathbf{Z}_n) \twoheadrightarrow E_\infty^{i-m,m} \rightarrowtail E_2^{i-m,m} \cong H^{i-m}\big(B; H^m(F;\mathbf{Z}_n)\big) \cong H^{i-m}(B;\mathbf{Z}_n)$$

and

$$p_{\#}\colon H_i(B;\mathbf{Z}_n) \cong H_i\big(B; H_m(F;\mathbf{Z}_n)\big) \cong E^2_{i,m} \twoheadrightarrow E^\infty_{i,m} \rightarrowtail H_{i+m}(E;\mathbf{Z}_n).$$

Now $p_{\#}$ and $p^{\#}$ satisfy, *up to sign*, the equations

$$p^{\#}\big(p^*x \cup y\big) = x \cup p^{\#}(y) \quad \text{and} \quad p_*\big(y \cap p_{\#}(\xi)\big) = \big(p^{\#}y\big) \cap \xi.$$

If E and B satisfy Poincaré duality, then $p_{\#} = p_!$ and $p^{\#} = p^!$, where

$$p_!(y \cap [B]) = p^*(y) \cap [E] \quad \text{and} \quad p^!(y) \cap [B] = p_*(y \cap [E]).$$

Thus $p_!$ and $p^!$ are the usual Umkehr homomorphisms for maps of Poincaré complexes.

(4.1) THEOREM. $\deg_n(p) | \Phi_n(p)$ *and* $\operatorname{cdg}_n(p) | \Phi_n(p)$.

PROOF. There exists an element $\Lambda \in H^n(E)$ such that $i^*(\Lambda) = \Phi_n[\bar{F}] \in H^n(F)$. Define homomorphisms τ: $H_i(B) \to H_i(E)$ and $\bar{\tau}$: $H^i(E) \to H^i(B)$ by $\tau(\alpha) = \Lambda \cap p_{\#}(\alpha)$ and $\bar{\tau}(x) = p^{\#}(x \cup \Lambda)$. Now $p^{\#}(\Lambda) = \Phi_n \cdot 1 \in H^0(B)$. This follows directly from the definition of $p^{\#}$. Hence

$$p_*\tau(\alpha) = p_*\big(\Lambda \cap p_{\#}(\alpha)\big) = p^{\#}(\Lambda) \cap \alpha = \Phi_n \cdot 1 \cap \alpha = \Phi_n \alpha$$

and

$$\bar{\tau}p^*(x) = p^{\#}\big(p^*x \cup \Lambda\big) = x \cup p^{\#}(\Lambda) = x \cup (\Phi_n \cdot 1) = \Phi_n x.$$

Thus τ and $\bar{\tau}$ are transfers associated to p_* and p^*, respectively. Hence $\deg p | \Phi_n$ and $\operatorname{cdg} p | \Phi_n$.

We say that a fibre bundle $F \xrightarrow{i} E \xrightarrow{p} B$ is a *manifold fibre bundle* if F, E, and B are closed connected orientable manifolds. (For $\mathbf{Z}_2$ coefficients unorientable manifolds are allowed.) The following beautiful fact [**G4**] is essential to our work:

$$i_*[F] = (p^*[\bar{B}]) \cap [E].$$

(4.2) THEOREM. *If p is a manifold fibre bundle, then* $\deg_n(p) = \Phi_n(p) = \mathrm{cdg}_n(p)$.

PROOF. Theorem (4.1) already shows that $\deg_n(p) | \Phi_n(p)$. Now we show that $\Phi_n(p) | \deg_n(p)$. Let $\Gamma \in H_*(E; \mathbf{Z}_n)$ be an element so that $p_*(\Gamma) = \deg_n(p)[B]$. Then Poincaré duality asserts there is a $\Lambda \in H^*(E; \mathbf{Z}_n)$ so that $\Gamma = \Lambda \cap [E]$. Then

$$\deg_n p = \langle [\bar{B}], p_*(\Gamma) \rangle = \langle p^*[\bar{B}], \Gamma \rangle = \langle i^!(1), \Gamma \rangle = \langle 1, i_!(\Gamma) \rangle.$$

The third equality follows from the above paragraph. So $i_!(\Gamma) = \deg_n(p) \cdot 1$. Now

$$\begin{aligned} \deg_n p = i_!(\Gamma) &= i_!(\Lambda \cap [E]) = (i^*\Lambda) \cap i_![E] \\ &= (i^*\Lambda) \cap [F] = k[\bar{F}] \cap [F] = k \end{aligned}$$

where k is an integer so that $i^*(\Lambda) = k[\bar{F}]$. Hence $\Phi_n(p) | k = \deg_n(p)$. Hence $\Phi_n = \deg_n$. Also $\mathrm{cdg}_n(p) = \deg_n(p)$ by Theorem (3.7).

(4.3) PROPOSITION. *Suppose p' is a pullback of p. Then* $\Phi_n(p') | \Phi_n(p)$.

PROOF. This follows immediately from the commutativity of the diagram

$$\begin{array}{ccc} H^m(F) & \overset{1}{\underset{\cong}{\leftrightarrow}} & H^m(F) \\ \uparrow i'^* & & \uparrow i^* \\ H^m(E') & \leftarrow & H^m(E) \end{array}$$

(4.4) PROPOSITION. *Suppose $H^m(F; \mathbf{Z}) \cong \mathbf{Z}$. Then* $\Phi_n(p) | \Phi(p)$.

PROOF. This follows immediately from the diagram

$$\begin{array}{ccc} \mathbf{Z} \cong H^m(F; \mathbf{Z}) & \twoheadrightarrow & H^m(F; \mathbf{Z}_n) \cong \mathbf{Z}_n \\ \uparrow i^* & & \uparrow i^* \\ H^m(E; \mathbf{Z}) & \rightarrow & H^m(E; \mathbf{Z}_n) \end{array}$$

(4.5) PROPOSITION. *Suppose F is a smooth closed manifold and p is a smooth fibre bundle with fibre F. Then* $\Phi(p) | P_\omega$, *where P_ω is any Pontrjagin number of F. Also* $\Phi_2(p) | W_\omega$, *where W_ω is any Whitney number of F.*

PROOF. The tangent bundle for F is the restriction of the bundle along the fibre for p. Characteristic numbers are natural.

5. The trace of a fibration. Let $p\colon E \to B$ be a fibration. The *trace* of p, denoted $\mathrm{tr}(p)$, is defined to be the least common multiple of all the degrees of all the pullbacks of p. If there is no least common multiple, we define $\mathrm{tr}(p) = 0$. Similarly, $\mathrm{tr}_n(p)$ is the least common multiple of $\deg_n(p')$ for all pullbacks p'.

This definition gives rise to the following property, which is analogous to Proposition (4.3).

(5.1) PROPOSITION. *If p' is a pullback of p, then* $\mathrm{tr}_n(p') | \mathrm{tr}_n(p)$.

(5.2) PROPOSITION. *Suppose* $E' \xrightarrow{p'} B$ *and* $E \xrightarrow{p} B$ *are fibrations and suppose* $r\colon E' \to E$ *is a fibre preserving map over the identity. Then* $\mathrm{tr}_n(p) | \mathrm{tr}_n(p')$.

PROOF. By Proposition (3.2) $\deg p | \deg(p \circ r) = \deg p'$. Now for any map $f\colon X \to B$ both p' and p pull back to fibrations and r induces a map over the identity between these pullbacks. Hence the degrees of the pullbacks divide in the appropriate manner, so $\mathrm{tr}(p) | \mathrm{tr}(p')$.

We may define the *cotrace* of a fibration p by setting it equal to the least common multiple of all the codegrees of all the pullbacks to p. Then Propositions (5.1) and (5.2) hold for cotrace of p also. Mostly, we shall not make use of the concept, since by the following theorem it is usually equal to $\mathrm{tr}(p)$.

(5.3) THEOREM. (a) $\mathrm{tr}_n(p) | \Phi_n(p)$ *and* $\mathrm{cotr}_n(p) | \Phi_n(p)$ *when* $\Phi_n(p)$ *is defined.* (b) *If p is a manifold bundle, then*

$$\mathrm{tr}_n(p) = \deg_n(p) = \Phi_n(p) = \mathrm{cdg}_n(p) = \mathrm{cotr}_n(p).$$

PROOF. (a) follows immediately from the definitions of $\mathrm{tr}(p)$ and $\mathrm{cotr}(p)$ and Theorem (4.1). (b) follows from the definitions and Theorem (4.2).

Thus all the concepts we have defined coalesce for manifold bundles. Now these manifold bundles are frequently encountered, and we shall show how they approximate almost every fibration we will be concerned with.

Let $F' \xrightarrow{i'} E' \xrightarrow{p'} B$ and $F \xrightarrow{i} E \xrightarrow{p} B$ be two fibrations. We say that p' *dominates* p if there exist maps $j\colon E \to E'$ and $r\colon E' \to E$ over the identity $1_B\colon B \to B$ so that $r \circ j$ is homotopic to 1_E by a homotopy over 1_B. Now consider two fibrations $F_1 \to E_1 \xrightarrow{p_1} B_1$ and $F_2 \to E_2 \xrightarrow{p_2} B_2$. We say that p_1 *governs* p_2 if B_1 dominates B_2 (that is, there exist maps $B_2 \xrightarrow{f} B_1 \xrightarrow{g} B_2$ so that $g \circ f$ is homotopic to 1_{B_2}), so that the pullback of p_2 by g is dominated by p_1. It follows that if p_3 is governed by p_2 and if p_2 is governed by p_1, then p_3 is governed by p_1.

Recall that a *compact fibration* is a fibration whose fibre and base are homotopy equivalent to finite complexes.

(5.4) PROPOSITION. *A compact fibration with connected fibre is governed by a smooth manifold bundle.*

PROOF. First the compact fibration is governed by its pullback by the homotopy equivalence of the base to a finite CW complex. This new fibration over a finite CW complex, call it $F \xrightarrow{i} E \xrightarrow{p} B$, is governed by the fibration $F \times T^d \xrightarrow{i \times 1} E \times T^d \xrightarrow{p \times x} B$, where T^d is a torus of dimension d = dimension of B. This is fibre homotopy equivalent to a smooth fibre bundle $N \to \bar{E} \to B$, where N is a compact regular neighborhood homotopy equivalent to $F \times T^d$. This follows from the closed fibre smoothing theorem [**CG**]. Now this fibration is governed by the pullback over a closed regular neighborhood of B. That one is governed by its pullback over the double of the base. So now we have a closed manifold as the base with fibre a

manifold with boundary. That makes the total space a manifold with boundary. If we double the total space we get a manifold bundle $M \to E \to B$ which governs our original fibration. If this bundle is not orientable, we can employ the trick of taking the cartesian product of the fibre and total space with $\mathbf{R}P^2$ to obtain the bundle $M \times \mathbf{R}P^2 \xrightarrow{i\times 1} E \times \mathbf{R}P^2 \to B$ which governs $M \to E \to B$. Both the total space and fibre are unoriented manifolds. If we take the oriented double covering of this new total space and then the mapping cylinder with total space, we obtain an oriented bundle of manifolds with boundary which dominates the previous bundle. Now if we double the total space we obtain the required manifold bundle.

(5.5) PROPOSITION. *Let p and p' be fibrations. If p' governs p, then* $\operatorname{tr}(p') = \operatorname{tr}(p)$.

PROOF. Let $\bar{p}$ denote the pullback of p over the base space of p' induced by the map $f\colon B' \to B$. Since $f \circ g\colon B \to B' \to B$ is homotopic to 1_B, we see that p is fibre homotopy equivalent to the pullback of $\bar{p}$ by g. By Proposition (5.1) we have $\operatorname{tr}(p)|\operatorname{tr}(\bar{p})|\operatorname{tr}(p)$. Hence $\operatorname{tr}(p) = \operatorname{tr}(\bar{p})$. Now $\bar{p}$ is dominated by p. Hence Proposition (5.2) gives us $\operatorname{tr}(\bar{p})|\operatorname{tr}(p')|\operatorname{tr}(\bar{p})$. Hence $\operatorname{tr}(p') = \operatorname{tr}(\bar{p}) = \operatorname{tr}(p)$.

The trace satisfies the same divisibility conditions as the degree in Proposition (3.4) provided the hypotheses are true under pullbacks. Thus Proposition (3.4)(a), (b) is true for trace. In particular, $\operatorname{tr}(p)|\chi(F)$ when $H_*(F; \mathbf{Z})$ is finitely generated. If p admits a cross section, then $\operatorname{tr}(p) = 1$. This follows since every pullback has a cross section, and hence all the degrees are equal to one.

(5.6) REMARK. The connectivity of the fibre in Theorem (5.4) is not essential. If $F \to E \xrightarrow{p} B$ is the compact fibration with disconnected fibre F, we may consider the fibration $F_0 \to E \xrightarrow{\tilde{p}} \tilde{B}$ given by identifying the path components of F to a point in E. Then the quotient space $\tilde{B}$ is a finite covering space of B. Now $F_0 \to E \xrightarrow{\tilde{p}} \tilde{B}$ is governed by a manifold bundle by Theorem (5.4). We can reduce to the situation where both the manifold bundle and the fibration are over $\tilde{B}$. Then composing the manifold bundle projection with $\tilde{B} \to B$ gives a manifold bundle with disconnected fibre which governs $F \to E \to B$.

6. The trace of an action. Suppose G is a group acting on a space X. We denote the action by (G, X). Now G has a universal fibre bundle $G \to E_G \to B_G$. Replacing G by the fibre X, we obtain the universal fibre bundle $X \to X_G \xrightarrow{p_G} B_G$, where $X_G = E_G \times_G X$.

(6.1) DEFINITION. *The trace of the action (G, X) is* $\operatorname{tr}(G, X) = \operatorname{tr}(p_G)$. *Similarly,* $\deg(G, X) = \deg p_G$, $\operatorname{cdg}(G, X) = \operatorname{cdg} p_G$, *and* $\Phi(G, X) = \Phi(p_G)$.

Similarly, we define $\operatorname{tr}_n(G, X)$, $\deg_n(G, X)$, $\operatorname{cdg}_n(G, X)$, and $\Phi_n(G, X)$. They are related by $\deg_n(G, X)|\operatorname{tr}_n(G, X)|\Phi_n(G, X)$. This follows from the definition of trace as the least common multiple of degrees and from Theorem (5.3)(a).

(6.2) PROPOSITION. (a) *Suppose $f\colon X \to Y$ is a G-map. Then* $\operatorname{tr}_n(G, Y)|\operatorname{tr}_n(G, X)$.

(b) *Suppose $\varphi\colon H \to G$ is a homomorphism and that the action (H, X) is induced by φ from the action (G, X). Then* $\operatorname{tr}_n(H, X)|\operatorname{tr}_n(G, X)$.

PROOF. (a) A G-map gives rise to a fibre preserving map $\bar{f}$: $X_G \to Y_G$ over the identity on B_G. Then Proposition (5.2) yields the result.

(b) The homomorphism φ: $H \to G$ induces a map $\bar{\varphi}$: $B_H \to B_G$. Then p_H can be thought of as the pullback of p_G. Applying Proposition (5.1) gives the result.

These two propositions tell us that $\mathrm{tr}(G, X)$ divides the trace of any invariant subspace and is divided by the trace of any subgroup.

An *equivariant retraction* consists of $X \xrightarrow{i} Y \xrightarrow{r} X$ where both r and i are G-maps and so that $r \circ i = 1_X$.

(6.3) COROLLARY. *If X is an equivariant retract of Y, then* $\mathrm{tr}_n(G, Y) = \mathrm{tr}_n(G, X)$.

PROOF. From Proposition (6.2)(a) we see that

$$\mathrm{tr}(G, X) \,|\, \mathrm{tr}(G, Y) \,|\, \mathrm{tr}(G, X).$$

(6.4) THEOREM. *Let M be a closed connected oriented manifold of dimension m. Suppose G acts on M preserving orientation. Suppose B_G has finite type. Then* $\mathrm{tr}_n(G, M) = \Phi_n(G, M)$. *For $n = 2$, orientation may be dropped from the hypothesis.*

PROOF. Let B^{m+1} be the $m+1$ skeleton of B_G. Let $M \to E \xrightarrow{p} B^{m+1}$ be the restriction of the universal fibration $M \to M_G \xrightarrow{p_G} B_G$ to B^{m+1}. The fibre homotopy exact sequence then shows that $\pi_i(E) \xrightarrow{\cong} \pi_i(M_G)$ for $i \leqslant m$. Hence $H^m(M_G; \mathbf{Z}_n) \xrightarrow{\cong} H^m(E; \mathbf{Z}_n)$. This implies that $\Phi_n(p) = \Phi_n(p_G) = \Phi_n(G, M)$. Now we imbed the finite complex B^{m+1} in Euclidean space and take a regular neighborhood N of B^{m+1}. This is a manifold with boundary. We double N to obtain an oriented manifold DN. Then DN dominates B^{m+1}, and the pullback of p over DN induced by the dominating map gives rise to a manifold bundle p' which governs p.

Now we prove the result by noting that $\Phi_n(G, M) = \Phi_n(p) = \Phi_n(p') = \mathrm{tr}_n(p')$, where the first equality is explained in the proof above; the second equality follows from the fact that p' governs p and Proposition (4.3); the third equality follows from Theorem (5.3)(b). Then

$$\Phi_n(G, M) = \mathrm{tr}_n(p') \,|\, \mathrm{tr}_n(G, M) \,|\, \Phi_n(G, M)$$

implies $\Phi_n(G, M) = \mathrm{tr}_n(G, M)$: The first division follows since p' is a pullback of p_G by Proposition (5.1), and the second division follows from Theorem (5.3)(a).

(6.5) LEMMA. *Suppose M is a compact manifold and (G, M) is an action. Then there is a closed oriented manifold N and an orientation preserving action (G, N) so that M is an equivariant retract of N. The dimension of N is at most* $\dim M + 1$.

PROOF. (a) Let M be a manifold with boundary on which G acts in an orientation preserving manner. Then G acts on the double DM in an obvious way which preserves orientation and so that the ordinary retraction $DM \to M$ is a G-retraction.

(b) Now suppose that M is not orientable. Let $\tilde{M}$ denote the oriented double covering. G acts on $\tilde{M}$ by selecting the orientation preserving lifting $\tilde{g}$ of any $g \in G$. Thus G acts on $\tilde{M}$ preserving the orientation. The projection p: $\tilde{M} \to M$ is a

G-map. Now consider the mapping cylinder $\tilde{M}_p$. This is an oriented manifold with boundary. The action of G on $\tilde{M}$ extends in an obvious orientation preserving way to $\tilde{M}_p$, and the retraction $\tilde{M}_p \to M$ is a G-retraction. So we have (a). Since a G-retract of a G-retract is a G-retract, we can apply (a) to our new manifold and establish (b).

(c) Now suppose that M is orientable but that (G, M) does not preserve orientation. Then there is a homomorphism φ: $G \to \mathbf{Z}_2$ such that those $g \in G$ which reverse orientation are mapped to 1 and those g which preserve orientation are mapped to 0. Let $(\mathbf{Z}_2, S^1)$ be the action given by conjugation of the unit circle in the complex plane. Let (G, S^1) be the action induced by φ. Let $(G, M \times S^1)$ be the action defined by $g(m, s) = (g(m), (\varphi(g))(s))$. Then $(G, M \times S^1)$ is orientation preserving. Also i: $M \to M \times S^1$ given by $i(m) = m \times 1$ is a G-map, the projection p: $M \times S^1 \to M$ is a G-map, and $p \circ i = 1_M$. So M is a G-retract of $M \times S^1$, and we are back to (a).

(6.6) PROPOSITION. *Let (G, M) be an action on a compact manifold M, and let B_G have finite type. Then* $\operatorname{tr}(G, M) = \operatorname{tr}(p)$, *where p is the pullback of p_G by a* $(\dim M + 2)$*-homotopy equivalence $B \to B_G$. If B is an oriented manifold, then* $\deg(p) = \operatorname{tr}(G, M)$.

PROOF. First assume M is connected. There exists a closed oriented manifold N and orientation preserving action (G, N) so that (G, M) is an equivariant retract, by Lemma (6.5). Also $\dim N \leqslant \dim M + 1$. Since f: $B \to B_G$ is a $(\dim M + 2)$-equivalence, the pullback $\bar{p}$ has the same cohomology as $p_{(G,N)}$ up to dimension $\dim N$. Hence $\Phi(G, N) \equiv \Phi(p_G) = \Phi(\bar{p})$. By using the technique in (6.4) we can find a closed oriented manifold B' which maps into B by a $(\dim N + 1)$-equivalence. The pullback $\bar{p}'$ over B' is a manifold bundle, and, hence, by (5.3)(b),

$$\operatorname{tr}(\bar{p}') = \Phi(\bar{p}') = \Phi(G, N) = \operatorname{tr}(G, N) = \operatorname{tr}(G, M).$$

Also $\operatorname{tr}(\bar{p}') = \deg(\bar{p}')$ by (5.3)(b). Let p' be the pullback to B' of the fibration $p_{(G,M)}$. Then $\bar{p}' = p' \circ r$, where r is a retraction of the total space of $\bar{p}'$ onto the total space of p'. Thus, by (3.3), $\deg p' = \deg \bar{p}'$. Thus $\deg(p') = \deg(\bar{p}') = \operatorname{tr}(\bar{p}')$ $= \operatorname{tr}(G, M)$. Since p' is a pullback of p, we have $\deg(p') | \operatorname{tr}(p)$ by definition. Hence

$$\operatorname{tr}(G, M) = \deg(p') | \operatorname{tr}(p) | \operatorname{tr}(G, M).$$

Hence $\operatorname{tr}(p) = \operatorname{tr}(G, M)$ when M is connected.

In the case that M is disconnected, the fibrations involved will decompose into a union of fibrations, each with a connected total space. From this we can reduce to the case of disconnected M with G acting transitively on the components M_i. We pull back over the appropriate closed oriented manifold B' as before. If E is the total space, we have $E \xrightarrow{\tilde{p}} \tilde{B}' \xrightarrow{q} B'$, where $\tilde{B}'$ is the quotient space of E given by identifying each component M_i of the fibre to a point. Then $E \xrightarrow{\tilde{p}} \tilde{B}'$ is a bundle with fibre M_i, $\tilde{B}' \xrightarrow{q} B'$ an n-fold covering, where n is the number of components of

M. So the degree of $E \to B'$ is

$$\deg(q\tilde{p}) = (\deg q) \cdot (\deg \tilde{p}) = [G : H] \cdot \operatorname{tr}(H, M_i),$$

where H is the subgroup of G fixing M_i, and $n = [G : H]$, the index of H in G. Since every pullback can be factored as $E \to \tilde{B} \to B$ by identifying components of fibres to points, we can use the covering space transfer for $\tilde{B} \to B$ to show that $\operatorname{tr}(G, M) | [G : H] \cdot \operatorname{tr}(H, M_i)$. Hence, $\operatorname{tr}(G, M) = [G : H] \cdot \operatorname{tr}(H, M_i)$, and this is the degree over B' which we wanted to show.

The formula $\operatorname{tr}(G, M) = [G : H] \cdot \operatorname{tr}(H, M_i)$ is due to Ozaydin [**Oz**] and was independently discovered by Stong.

Note that B_G has finite type when G is finite or a compact Lie group.

(6.7) PROPOSITION. (a) *If* (G, X) *has a stationary point, then* $\operatorname{tr}(G, X) = 1$.

(b) *If* $H_*(X; \mathbf{Z})$ *is finitely generated, and* X *is homotopy equivalent to a CW complex, then* $\operatorname{tr}(G, X) | \chi(X)$. *In addition if* $f\colon X \to X$ *is a* G *map, then* $\operatorname{tr}(G, X) | \Lambda_f$.

(c) *If* $f, g\colon M \to N$ *are two* G*-maps between closed connected orientable manifolds and* G *preserves orientation, then* $\operatorname{tr}(G, M) | \Lambda_{f,g}$, *the coincidence number of* f *and* g.

(d) *If* M *is a smooth closed manifold and* G *acts smoothly, then* $\operatorname{tr}_2(G, M) = \Phi_2(G, M) | W_\omega$, *where* W_ω *is any Whitney number. In addition, if* M *is orientable,* $\operatorname{tr}(G, M) | \Phi(G, M) | P_\omega$, *where* P_ω *is any Pontrjagin number.*

PROOF. (a) is true since any stationary point gives rise to a cross section to $X_G \xrightarrow{p_G} B_G$. Hence, $\operatorname{tr}(X, G) = \operatorname{tr}(p_G) = 1$. (b) and (c) follow from Proposition (3.4)(a), (b). Proposition (4.5) implies (d).

We get the following amusing corollary from (d) of Proposition (6.7):

(6.8) COROLLARY. *Let* M *be a smooth oriented closed manifold with a smooth involution* $\alpha\colon M \to M$ *which reverses orientation. Then all its Pontrjagin numbers vanish. If* M *bounds, then it bounds orientably.*

PROOF. α gives rise to an action $(\mathbf{Z}_2, M)$ which is not orientation preserving. Thus $\Phi(\mathbf{Z}_2, M) = 0$. Hence, $0 | P_\omega$ so $P_\omega = 0$. If M bounds and all its $P_\omega = 0$, then M bounds orientably.

For some groups the trace fails to distinguish most actions. Thus if $(\mathbf{Z}, X)$ is any action of the integers on a path connected space X, then $\operatorname{tr}(\mathbf{Z}, X) = 1$. This follows since $B_{\mathbf{Z}} \approx S^1$ and any fibration over S^1 with path connected fibre must have a cross section.

On the other hand, for any group G there is a free action with trace equal to 1. For G acts freely on the contractible E_G, so $\operatorname{tr}(G, E_G) = 1$. In view of this it is remarkable that the trace characterizes free actions of finite groups on manifolds which are dominated by finite complexes.

(6.9) PROPOSITION. *If* G *is a finite group acting on* X, *then* $\operatorname{tr}(G, X) \,|\, |G|$, *where* $|G|$ *is the order of* G.

PROOF. Let $x \in X$ and let G_x be the isotropy subgroup fixing x. Then the orbit G/G_x is a G-space which includes equivariantly into X. Then

$$\operatorname{tr}(G, X)\left|\operatorname{tr}\left(G, \frac{G_x}{G}\right)\right|\chi\left(\frac{G_x}{G}\right) = \frac{|G|}{|G_x|}\Bigg| |G|.$$

(6.10) THEOREM. *Let M be a manifold dominated by a finite CW complex. Let G be a finite group. G acts freely on M if and only if* $\operatorname{tr}(G, M) = |G|$.

PROOF. First, suppose that G does not act freely. Then there is an $x \in M$ such that G_x is not trivial. Then $\operatorname{tr}(G, M)|(|G|/|G_x|) < |G|$.

Now suppose that G acts freely on M. We shall show that $\operatorname{tr}(G, M) = |G|$. First we assume that M is a closed orientable manifold on which G acts in an orientation preserving manner. Then the quotient space M/G is a closed oriented manifold, and the projection p: $M \to M/G$ is a covering map of degree $|G|$. Now there is a map q: $M_G = E_G \times_G M \to M/G$ induced by the projection $E_G \times M \to M$. Then $p = q \circ i$, and since G acts freely, q is a homotopy equivalence. Now the image of p^*: $H^n(M/G; \mathbf{Z}) \to H^n(M; \mathbf{Z})$ is generated by $|G| \cdot [M]$ and $\operatorname{im} p^* = \operatorname{im} i^*$. Hence $\Phi(G, M) = |G|$. Hence $\operatorname{tr}(G, M) = |G|$ in this case.

Now suppose that M is a compact manifold on which G acts. Then Lemma (6.5) states that there is a (G, N) so that N is closed, oriented, and G acts in an orientation preserving way, and so that M is a G-retract of N. Since G acts freely on M, it follows that G acts freely on N, so $|G| = \operatorname{tr}(G, N) = \operatorname{tr}(G, M)$. The first equality follows because of the above paragraph, and the second equality follows because M is an equivariant retract of N.

Finally, suppose M is a manifold dominated by a finite CW complex. The quotient M/G is also a manifold. It may not be homotopy equivalent to a finite complex, but by [**G5**, Lemma 1] it is dominated by a finite complex K. We may assume that K is a compact manifold with boundary. Then we get a pullback diagram

$$\begin{array}{ccccc} M & \xrightarrow{\tilde{i}} & \tilde{K} & \xrightarrow{\tilde{r}} & M \\ \downarrow p & & \downarrow p' & & \downarrow p \\ M/G & \xrightarrow{i} & K & \xrightarrow{r} & M/G \end{array}$$

of principal G-coverings. Since $r \circ i$ is homotopic to the identity, the left and right coverings are the same. Now G acts freely on $\tilde{K}$ since $\tilde{K}$ is a principal G bundle and $\tilde{K}$ a covering of K; hence $\tilde{K}$ is a compact manifold with boundary. Since $\tilde{i}$ and $\tilde{r}$ are bundle maps, they are G-maps. Then $\operatorname{tr}(G, M)|\operatorname{tr}(G, \tilde{K})|\operatorname{tr}(G, M)$. Hence $\operatorname{tr}(G, M) = \operatorname{tr}(G, \tilde{K})$ and $\operatorname{tr}(G, \tilde{K}) = |G|$ by the previous cases. This concludes the proof.

Now we can see that the concepts of $\operatorname{tr}(G, X)$, $\Phi(G, X)$ and $\deg(G, X)$ disagree. If we let $G = \prod_1^r \mathbf{Z}_p$ act freely on a torus T^r in a nonorientation preserving way, then $\operatorname{tr}(G, T) = p^r$, $\Phi(G, T) = 0$, and $\deg(G, T)$ is either 1 or p since $\exp(\tilde{H}_*(B_G; \mathbf{Z})) = p$.

As a corollary to Theorem (6.10), we find that a free action of a finite group G on a manifold M dominated by a finite complex must have $|G|\,|\chi(M)$ or $|G|\,|\Lambda_f$, where f: $M \to M$ is a G-map. This is Theorem 2 of [**G5**]. Also for closed smooth manifolds and free smooth actions we see that $|G|$ divides the Pontrjagin numbers of M.

Now let G be a compact connected Lie group acting on a compact manifold M. Then G/G_x is a closed manifold for any $x \in M$. Since any orbit G/G_x is an invariant subspace of M we see that $\mathrm{tr}(G, M)|\mathrm{tr}(G, G_x)|\chi(G/G_x)$. Now $\chi(G/G_x) \neq 0$ if and only if G_x contains a maximal torus T. If $G_x \supset T$, then $\chi(G/G_x)|\chi(G/T)$ and $\chi(G/T) =$ order of the Weyl group of G. Thus we see that for compact Lie group actions the trace either is zero or divides the order of the Weyl group. We know exactly when the trace equals zero for compact manifolds.

(6.11) THEOREM. *Let a compact connected Lie group G act on a compact manifold M. Then* $\mathrm{tr}(G, M) = 0$ *if and only if no isotropy subgroup contains a maximal torus.*

PROOF. If $G_X \supset T$ then $\chi(G/G_x) \neq 0$. Since $\mathrm{tr}(G, M)|\mathrm{tr}(G/G_x)|\chi(G/G_x) \neq 0$, we see that $\mathrm{tr}(G, M) \neq 0$.

Conversely, we assume $\mathrm{tr}(G, M) \neq 0$. Since $\mathrm{tr}(T, M)|\mathrm{tr}(G, M) \neq 0$, we see that $\mathrm{tr}(T, M) \neq 0$. Let $t \in T$ generate T, that is, the set of powers of t is dense in T. We find a sequence t_p of elements in T which converge to t such that t_p has order p, where p is a prime. Now t_p generates $\mathbf{Z}_p$ acting on M and $\mathrm{tr}(\mathbf{Z}_p, M)|\mathrm{tr}(G, M) \neq 0$. Also $\mathrm{tr}(\mathbf{Z}_p, M) = 1$ if t_p has a fixed point and $\mathrm{tr}(\mathbf{Z}_p, M) = p$ if t_p does not have a fixed point, since then $\mathbf{Z}_p$ would be a free action. Since $p|\mathrm{tr}(G, M)$ for at most a finite set of primes, we see that t_p has a fixed point for all $p > \mathrm{tr}(G, M)$. These fixed points have limit points as $p \to \infty$. These limit points must be fixed points of t. Hence they must be stationary points of T. Hence $T \subset G_x$ for some $x \in M$.

7. Elementary abelian p-group actions. Let $G = \mathbf{Z}_p$ where p is a prime. Then $\mathrm{tr}(\mathbf{Z}_p, X)$ is either p or 1 depending on whether the action is free or has a fixed point. In general the larger the trace for a finite group G, the more free-like is the action. When the trace equals one, we can only say with certainty that every element of prime order in G has a fixed point. But for the case of elementary abelian p-groups $G = (\mathbf{Z}_p)^r$, the trace equals 1 characterizes fixed points for manifolds.

(7.1) LEMMA. *Let X be a compact G-space. Let $G = (\mathbf{Z}_p)^r$. Then the action has a stationary point if and only if* $\mathrm{cdg}_p(G, X) = 1$.

PROOF. This is Corollary 1 of W. Y. Hsiang's book [**H**, p. 45].

(7.2) THEOREM. *Let M be a compact manifold and $G = (\mathbf{Z}_p)^r$. Then* $\mathrm{tr}(G, M) = 1$ *if and only if* $\mathrm{tr}_p(G, M) = 1$ *if and only if (G, M) has a stationary point.*

PROOF. If (G, M) has a stationary point, then of course $\mathrm{tr}(G, M) = 1$ and $\mathrm{tr}_p(G, M) = 1$. Now we assume that $\mathrm{tr}(G, M) = 1$. By Lemma (6.5) M is an equivariant retract of an oriented (G, N). Hence $\mathrm{tr}(G, N) = 1$ also. But $\mathrm{tr}(G, N) = \Phi(G, N)$ by Theorem (6.4). So $\Phi(G, N) = 1$. Now $\Phi_p(G, N)|\Phi(G, N)$ by Proposition (4.4). Hence $\Phi_p(G, N) = 1$. Since $\mathrm{tr}_p(G, N) = \Phi_p(G, N)$ by Theorem (6.4) again, we see that $\mathrm{tr}_p(G, N) = 1$. Since $\deg_p(G, N)|\mathrm{tr}_p(G, N)$, we see that $\deg_p(G, N) = 1$. But $\deg_p(G, N) = \mathrm{cdg}_p(G, N)$. Hence $\mathrm{cdg}_p(G, N) = 1$, and the preceding lemma tells us that (G, N) has a stationary point. Since (G, M) is a G-retract of (G, N), (G, M) also has a stationary point.

A particular case of Theorem (7.2) is that $\mathrm{tr}_2(G, M) = 1$ if and only if (G, M) has a fixed point. If this is combined with Proposition (6.7)(d), which states that $\mathrm{tr}_2(G, M)$ divides Stiefel-Whitney numbers, we obtain the following result of Conner and Floyd [**CF**$_1$, 30.1] as a corollary.

(7.3) COROLLARY. *If $G = (\mathbf{Z}_2)^r$ acts smoothly on a closed manifold M without a stationary point, then M bounds.*

PROOF. M bounds if and only if every Stiefel-Whitney number equals zero.

Now for elementary abelian p-groups we see that $\mathrm{tr}(G, M) = 1$ if and only if there is a stationary point; that is, the smallest orbit consists of one point; and we see that $\mathrm{tr}(G, M) = |G|$ if and only if the action is free; that is, the minimum orbit consists of $|G|$ points. It is natural to ask if $\mathrm{tr}(G, M)$ is equal to the order of the minimal orbit.

William Browder can show under some conditions that $\Phi(G, M) =$ (order of the minimal orbit), where M is some space and (G, M) is orientation preserving. The conditions that this holds certainly include smooth actions on smooth closed oriented manifolds. From the methods developed so far, it immediately follows that $\mathrm{tr}(G, M)$ is the order of the minimal orbit. We present a proof quite different from Browder's, which nevertheless was inspired by his proof.

(7.4) THEOREM. *Let (G, M) be a smooth action of an elementary abelian p-group $(\mathbf{Z}_p)^r$ on a compact manifold M. Then $\mathrm{tr}(G, M)$ is equal to the order of the smallest orbit.*

As corollaries we see that the order of the minimal orbit divides Pontrjagin numbers, as was mentioned in the Introduction, Euler-Poincaré numbers, and Lefschetz numbers. In fact, Murad Ozaydin [**Oz**] can show that for finite groups, the greatest common divisor of the order of all the orbits must divide the Euler-Poincaré and Lefschetz numbers.

We begin the proof by recalling a well-known lemma:

(7.5) LEMMA. *Let $G = (\mathbf{Z}_p)^r$ and let X be a G-space so that the action (G, X) has one isotropy subgroup H. Then X_G is homotopy equivalent to $B_H \times (X/G)$.*

PROOF. Let $K \cong G/H$, so that $G \cong H \oplus K$. Now let E_H and E_K be contractible spaces so that H and K act freely on each one, respectively. Then G acts freely on the contractible space $E_K \times E_H$ by letting $(k, h)(e_1, e_2) = (ke_1, he_2)$. Then $X_G = (X \times E_K \times E_H)/G$, where G acts diagonally. But this is

$$\begin{aligned}\big((X \times E_K \times E_H)/H\big)/K &= (X \times E_K \times E_H/H)/K = (X \times E_K \times B_H)/K \\ &= \big((X \times E_K)/K\big) \times B_H = X_K \times B_K.\end{aligned}$$

Since K acts freely on X, we see that X_K is homotopy equivalent to X/K; and $X/K = (X/H)/K = X/G$ since H acts trivially on X.

Now suppose that $G \cong (\mathbf{Z}_p)^r$ acts on a compact manifold M. We can filter M into invariant subspaces $M_r \subset M_{r-1} \subset \cdots \subset M_0 = M$, where M_i is the set of all points whose isotropy subgroup has rank greater than or equal to i. Then $M_i - M_{i+1}$

consists of the disjoint union of spaces $M_{i,j}$ where each point of $M_{i,j}$ has a single isotropy subgroup H_j of rank i. This filtration gives rise to a filtration $\hat{M}_r \subset \hat{M}_{r-1} \subset \cdots \hat{M}_0 = M_G$ of M_G, and $\hat{M}_i - \hat{M}_{i+1}$ is homotopy equivalent to a disjoint union of spaces, each one homotopy equivalent to $B_{H_j} \times M_{i,j}/G$. If $E \to B$ is a pullback of $M_G \to B_G$, let E_i denote the pullback of $\hat{M}_i$. Then $E_r \subset E_{r-1} \subset \cdots \subset E_0 = E$ filters E.

PROOF OF (7.4). We may assume that (G, M) is a smooth effective and orientation preserving action on a closed oriented manifold M. For if (G, M) were not effective, we consider $(G, M \times D)$, where D is a unit disk of an effective linear representation of G on R^n and G acts diagonally on the product. Now (G, M) is an equivariant retract of $(G, M \times D)$, so a minimal orbit of (G, M) will be a minimal orbit of $(G, M \times D)$, and also $\operatorname{tr}(G, M) = \operatorname{tr}(G, M \times D)$ by (6.3). Note $(G, M \times D)$ is effective. Similarly, by Lemma (6.5) we can find a smooth oriented action which has (G, M) as a retract.

So consider $M - M_1 \neq \varnothing$. This is an open G-manifold of dimension equal to $\dim M = m$ on which G acts freely. Since the action is smooth, we see that $M - M_1$ has a compact manifold with boundary as a deformation retract. Thus by (6.6) we can find a fibre bundle $M \to E \xrightarrow{q} B$ which has the property that $\operatorname{tr}(q)$ and $\operatorname{tr}(q_1)$, where $(M - M_1) \to E \xrightarrow{q_1} B$, are both realized by the degree over B and at the same time are equal to $\operatorname{tr}(G, M)$ and $\operatorname{tr}(G, (M - M_1))$, respectively.

Now since $(G, (M - M_1))$ is a free action, we see by (6.10) that $\operatorname{tr}(G, (M - M_1)) = p^r$. Hence $\deg q_1 = p^r$. Let $\dim B = b$. Then there is a $\Gamma_1 \in H_b(E - E_1; \mathbf{Z})$ so that $q_{1*}(\Gamma_1) = p^r[B]$ by Theorem (3.5). Consider the commutative diagram of inclusions and projections:

$$
\begin{array}{ccccccccc}
j\colon (E - E_1) & \xrightarrow{j_1} & (E - E_2) & \xrightarrow{j_2} & \cdots & \to & (E - E_k) & \xrightarrow{j_k} & E \\
\downarrow q_1 & & \downarrow q_2 & & & & \downarrow q_k & & \downarrow q = q_{k+1} \\
B & \leftrightarrow & B & \to & \cdots & \to & B & \leftrightarrow & B
\end{array}
$$

where k is the maximal rank of all the isotropy subgroups. For each $E - E_i$ there is a $\Gamma_i \in H_b(E - E_i; \mathbf{Z})$ so that $(q_i)_*(\Gamma_i) = (\deg q_i)[B]$.

We claim that $p\Gamma_i \in \operatorname{Image}(j_{i-1})_*$. Hence $p^k\Gamma \in \operatorname{Image}(j_*)$ so $\deg q \mid p^{r-k}$ by commutativity of the diagram. Also $q_*(p^k\Gamma) \in \operatorname{Im}(q_* j_*) = \operatorname{Im}(q_{1*}) = p^r\mathbf{Z}$. Hence $p^r \mid p^k \deg q$. Hence $p^{r-k} \mid \deg q$. Thus $\deg q = p^{r-k} =$ order of the minimal orbit.

To prove the claim we show that $H_b(E - E_i)/\operatorname{Im}(j_{i-1*})$ has exponent p. This follows if we can show that $H_b(E - E_i, E - E_{i-1}; \mathbf{Z})$ has exponent p.

So consider $H_b(E - E_i, E - E_{i-1}; \mathbf{Z})$. Since E is a closed oriented manifold of dimension $b + m$ and since the E_i are compact, Poincaré duality states that $H_b(E - E_i, E - E_{i-1}) \cong \check{H}^m(E_{i-1}, E_i)$. Since $B \to B_G$ is an $m + 1$ homotopy equivalence, $\check{H}^m(E_{i-1}, E_i) \cong \check{H}^m(\hat{M}_{i-1}, \hat{M}_i)$.

Now since the action is smooth, each M_i has an equivariant tubular neighborhood W_i in M_{i-1} which deformation retracts equivariantly onto M_i. Hence $\check{H}^m(\hat{M}_{i-1}, \hat{M}_i) \cong H^m(\hat{M}_{i-1}, \hat{W}_i)$. Excision then shows that

$$H^m(\hat{M}_{i-1}, \hat{W}_i) \cong \bigoplus_j \left(H^m\big((M_{i-1,j})/H_j\big) \times B_{H_j}, (\partial W_j)/H_j \times B_{H_j} \right).$$

Thus it is a direct sum of groups of the form $H^m((C, D) \times B_H; \mathbf{Z})$, where (C, D) is a pair of spaces whose dimension is less than m. Applying the Kunneth formula we see that this group splits as a direct sum of groups of the form $H^i((C, D); H^j(B_H; \mathbf{Z}))$, where $i + j = m$. But $H^m((C, D); H^0(B_H; \mathbf{Z})) = 0$, since C has dimension less than m. Since $H^j(B_H; \mathbf{Z})$ has exponent p for $j > 0$, we see that $H^m((C, D); H^j(B, \mathbf{Z}))$ has exponent p. So $H^m(\hat{M}_{i-1}, \hat{M}_i)$ has exponent p, and thus $H_b(E - E_i, E - E_{i-1})$ has exponent p.

8. Equivariant degree 1 maps. In this section we consider the following situation. Let M and N be oriented closed G-manifolds of dimension m. Assume that B_G has finite type. Thus G could be a compact Lie group for example. Finally, suppose $f\colon M \to N$ is an equivariant map.

(8.1) PROPOSITION. $\operatorname{tr}(G, N) | \operatorname{tr}(G, M) | (\deg f) \cdot \operatorname{tr}(G, N)$.

PROOF. $\operatorname{tr}(G, N) | \operatorname{tr}(G, M)$ by Proposition (6.2)(a). Now since f is equivariant we have a commutative diagram

$$\begin{array}{ccc} M & \overset{f}{\to} & N \\ \downarrow i & & \downarrow i \\ M_G & \underset{\bar{f}}{\to} & N_G \end{array}$$

First we assume that G acts orientably on both M and N. Then $\operatorname{tr}(G, M) = \Phi(G, M)$ and $\operatorname{tr}(G, N) = \Phi(G, M)$ by Theorem (6.4). Now

$$(\deg f) \cdot \Phi(G, N)[M] = f^*(\Phi(G, N)[N]) = f^* i^*(\Lambda)$$

for some $\Lambda \in H^n(N_G; \mathbf{Z})$. But $f^* i^*(\Lambda) = i^* \bar{f}^*(\Lambda)$. Hence $(\deg f) \cdot \Phi(G, N)[M]$ is in the image of i^*. So $\Phi(G, M) | (\deg f) \Phi(G, N)$, and this establishes the corresponding result for trace for orientable actions.

Now if $\deg f = 0$, the result is true, so we assume that $\deg f \neq 0$. Then, since $fg = gf$ for all $g \in G$, we see that g either preserves the orientation on both M and N or reverses both orientations. Thus we are left with the case where G does not preserve the orientation of M and N.

Now we let G act on $M \times S^1$ as in part (c) of the proof of Lemma (6.5). Similarly, G acts on $N \times S^1$, and in both cases G preserves the orientation. Now

$$f\colon M \times S^1 \overset{f \times 1}{\to} N \times S^1$$

is a G-map and $\deg(f \times 1) = (\deg f)$. So applying the oriented case of the proposition, we see that

$$\operatorname{tr}(G, M \times S^1) \mid \deg(f \times 1) \operatorname{tr}(G, N \times S^1).$$

Since M is a G-retract of $M \times S^1$, we see that $\operatorname{tr}(G, M) = \operatorname{tr}(G, M \times S^1)$. Similarly for N. Hence $\operatorname{tr}(G, M) | (\deg f) \operatorname{tr}(G, N)$.

(8.2) THEOREM. *If $f\colon M \to N$ is a G-map of degree 1 between two closed orientable G-manifolds of the same dimension, and if B_G has finite type, then* $\operatorname{tr}(G, M) = \operatorname{tr}(G, N)$.

PROOF. By Proposition (8.1) we see that

$$\operatorname{tr}(G, N) \mid \operatorname{tr}(G, M) \mid (\deg f) \operatorname{tr}(G, N) = \operatorname{tr}(G, N).$$

(8.3) COROLLARY. (a) *Let G be finite and suppose $f\colon M \to N$ is a G-map of degree* 1. *Then G acts freely on M if and only if G acts freely on N (cf.* [**Bre**, *Theorem* 5.1]).

(b) *If $G \cong (\mathbf{Z}_p)^r$ for p prime, then (G, M) has a stationary point if and only if (G, N) has one. (Independently discovered by Browder.)*

(8.4) COROLLARY. *Suppose $f\colon M \to N$ is a G-map and G acts freely on M and has a fixed point on N. Then $|G| \mid \deg f$. For G a compact connected Lie group,* $\deg f = 0$.

PROOF. From Proposition (8.1),

$$\operatorname{tr}(G, M) \mid (\deg f) \cdot \operatorname{tr}(G, N).$$

Since (G, M) is a free action, $\operatorname{tr}(G, M) = |G|$ (or 0 if G is a compact Lie group). Since (G, N) has a fixed point, $\operatorname{tr}(G, N) = 1$. Hence $|G| \mid \deg f$ (or $0 \mid \deg f$ in the Lie group case).

(8.5) COROLLARY. *Suppose a finite group G acts smoothly on a closed oriented manifold M with exactly one stationary point. Then* (a) *G cannot be an elementary abelian p-group, and* (b) *the action cannot be semifree.*

PROOF. Let $x \in M$ be the stationary point. We can assume that the metric of M is G-invariant.

Let B be a small invariant open ball with center x in M. Then $M - B$ is a manifold with boundary S^{n-1}. Let D be its double. Then G acts on D in the obvious way. We will define an equivariant degree one map from $D \to M$. Since (G, M) has a fixed point and (G, D) has no fixed point in (a) or is free in (b), Corollary (8.3) gives a contradiction.

So we must construct the degree 1 G-map $f\colon D \to M$. Let f be the inclusion on the first copy of $M - B$, and let f map the second copy of $M - B$ onto $\overline{B}$ by using the collar neighborhood of ∂B in M to map onto $\overline{B}$ equivariantly and then extending the map on the boundary of the collar (which is a constant map into x) to a constant map on the rest of the second copy of $M - B$. This is the required f.

(8.6) COROLLARY. *If $G = (\mathbf{Z}_2)^r$ acts smoothly on a closed manifold M, there cannot be exactly one stationary point.*

PROOF. By the same construction as in (8.5) we obtain an equivariant map $f\colon D \to M$ from a fixed point free action to an action with a fixed point. Now $f^*[M] = [D]$, so $\Phi_2(G, M) = \Phi_2(G, D)$. Hence

$$2 = \operatorname{tr}_2(G, D) = \Phi_2(G, D) = \Phi_2(G, M) = \operatorname{tr}_2(G, M) = 1.$$

Hence, the construction is impossible, so the corollary is proved.

REMARKS. The question of which groups act smoothly on closed manifolds with a single stationary point has been studied for years. It has very recently been completely solved by R. Stong after (8.5) was discovered. First, G. D. Mostow showed that $\mathbf{Z}_p$ could not act with a single stationary point. This result fascinated Conner and Floyd so much that it gave them the impetus for starting their investigations into cobordism and group actions and resulted in their book [**CF1**]. In that book they gave an example of $\mathbf{Z}_4$ acting with a single stationary point, but they conjectured that $\mathbf{Z}_{p^n}$ for odd primes p could not act with a single stationary point. Atiyah and Bott [**AB**] showed that this conjecture was true by using their Lefschetz index theorem. Also Conner and Floyd proved it in [**CF2**]. For elementary abelian 2-groups Conner has shown Corollary (8.6) [**Co**]. Suprisingly the result (8.5) for odd primes never appeared in the literature. We shall record Stong's solution below.

(8.7) STONG'S THEOREM. (a) *A finite group G acts smoothly on some closed manifold with a unique stationary point if and only if G is not an elementary abelian 2-group.*

(b) *G acts on some orientable M with a unique stationary point if and only if G is not an elementary abelian 2-group or an abelian p-group for p odd* [**EW**].

We use Stong's theorem and the construction in (8.5) to answer the following question: For which groups G does $\operatorname{tr}(G, M) = 1$ imply that (G, M) has a stationary point?

(8.8) PROPOSITION. *Suppose G is not an elementary abelian p-group. Then there exists a smooth action on a closed orientable manifold M without a stationary point so that* $\operatorname{tr}(G, M) = 1$.

PROOF. If (G, M) has a unique fixed point, consider the construction of (8.5) to find a degree 1 equivariant map $f\colon D \to M$. Since $\operatorname{tr}(G, M) = 1$ because (G, M) has a stationary point and since $\operatorname{tr}(G, D) = \operatorname{tr}(G, M)$, we see that $\operatorname{tr}(G, D) = 1$, and (G, D) does not have a stationary point. By Stong's theorem this eliminates all groups except abelian p-groups for odd p which are not elementary. In this case we have a surjective homomorphism $\varphi\colon G \to \mathbf{Z}_{p^2}$. Let $\mathbf{Z}_p$ act freely on an odd-dimensional sphere S^n. Let $\rho\colon \mathbf{Z}_{p^2} \to \mathbf{Z}_p$ be the obvious surjection. We show in (9.3) that this action induced by ρ has $\operatorname{tr}(\mathbf{Z}_{p^2}, S^n) = 1$ and has no fixed points. So then the action (G, S^n) induced by φ has no fixed points and $\operatorname{tr}(G, S^n) | \operatorname{tr}(\mathbf{Z}_{p^2}, S^n) = 1$.

9. Trace and the Serre spectral sequence. The trace can be calculated or estimated by use of the Serre spectral sequence. Let (G, M) be an action on a compact manifold M and let B_G have finite type. Then by Proposition (6.6), $\operatorname{tr}(G, M) = \operatorname{tr}(p)$, where p is the restriction of p_G to the $(\dim M + 2)$-skeleton of B_G. If we embed this skeleton in a high-dimensional Euclidean space and take a regular neighborhood and double it, we get a closed oriented manifold B^N whose first $(\dim M + 1)$-homology and cohomology groups agree with those of B_G. In the resulting fibration $M \to E \xrightarrow{p} B^N$, the trace $\operatorname{tr}(G, M)$ will be the generator of $\operatorname{Im} p_* \subset H_N(B^N; \mathbf{Z}) \cong \mathbf{Z}$. Now this number is precisely the product of the orders of the images of the generators under $d^r\colon E^r_{N,0} \to E^r_{N-r,r-1}$. Since $E^r_{p,q}$ is a subquotient of $E^2_{p,q} \equiv H_p(B^N; \{H_q(M; \mathbf{Z})\})$, the orders of the images must divide the exponents of

$H_{N-r-1}(B^N; \{H_r(M; \mathbf{Z})\})$ for $r = 1, \ldots, \dim M$. So

$$\operatorname{tr}(G, M)| \prod_{r=1}^{\dim M} \exp H_{N-r-1}\big(B^N; \{H_r(M; \mathbf{Z})\}\big)$$

$$= \prod_{r=1}^{\dim M} \exp H^{r+1}\big(B^N; \{H_r(M; \mathbf{Z})\}\big)$$

$$= \prod_{r=1}^{\dim M} \exp H^{r+1}\big(B_G; \{H_r(M; \mathbf{Z})\}\big).$$

(by Poincaré duality [**W**, pp. 22, 23]).

Similarly, for cohomology,

$$\operatorname{tr}(G, M)| \prod_{r=1}^{\dim M} \exp H_r\big(B_G; \{H^r(M; \mathbf{Z})\}\big),$$

since here the trace must divide the product of the orders of $E_r^{N-r,r}$, $r > 0$, assuming that $E_\infty^{N,0} \cong \mathbf{Z}$. So we have

(9.1) THEOREM. *If (G, M) is an action on a connected compact manifold such that B_G is of finite type, then*

(a) $$\operatorname{tr}(G, M)| \prod_{r=1}^{\dim M} \exp\big(H^{r+1}\big(B_G; \{H_r(M; \mathbf{Z})\}\big)\big),$$

(b) $$\operatorname{tr}(G, M)| \prod_{r=1}^{\dim M} \exp\big(H_r\big(B_G; \{H^r(M; \mathbf{Z})\}\big)\big) \quad \textit{if } \operatorname{tr}(G, M) \neq 0.$$

In [**Bro**], W. Browder studied free actions on a chain complex. He found that for finite groups acting freely that the order of the group divides the number in (a). Since $\operatorname{tr}(G, M) = |G|$ for free actions, we generalize his result for M a compact manifold. We also generalize his result for the number in (b), which unfortunately, he states incorrectly in his paper.

One example of an application is the generalization of Carlsson's theorem [**Ca**], which was done in the Introduction (1.3).

Another example using connected Lie groups: Let $G = S^3$ and let M be a compact manifold so that $H_r(M; \mathbf{Q}) = 0$ for $r \equiv 3 \bmod 4$. Then

$$\prod_{r=1}^{\dim M} \exp\big(H^{r+1}\big(B(S^3); H_r(M; \mathbf{Z})\big)\big) \neq 0.$$

Hence $\operatorname{tr}(S^3, M) \neq 0$. Hence, by (6.11),

(9.2) COROLLARY. *With M as above, S^3 cannot act on M with only finite isotropy subgroups.*

Another example of the use of spectral sequences is due to W. Browder.

(9.3) PROPOSITION. *Let $\mathbf{Z}_p$ act freely on S^n. Let $(\mathbf{Z}_{p^2}, S^n)$ be induced by $\varphi\colon \mathbf{Z}_{p^2} \twoheadrightarrow \mathbf{Z}_p$. Then if $n > 1$ is odd, $\operatorname{tr}(\mathbf{Z}_{p^2}, S^n) = 1$.*

PROOF. $\tilde{H}^*((B\mathbf{Z}_p); \mathbf{Z})$ is generated by the powers of an element α of dimension two and order p. Similarly, $\tilde{H}^*(B\mathbf{Z}_{p^2}; \mathbf{Z})$ is generated by the powers of an element β of dimension 2 and order p^2. The homomorphism $\varphi\colon \mathbf{Z}_{p^2} \to \mathbf{Z}_p$ induces a map $\bar{\varphi}\colon B\mathbf{Z}_{p^2} \to B\mathbf{Z}_p$. Now $\bar{\varphi}^*(\alpha) = p\beta$. Since $(\mathbf{Z}_p, S^N)$ is a free action, $\Phi(\mathbf{Z}_p, S^n) = p$ and so in the Serre spectral sequence for $S^n \to (S^n)_{\mathbf{Z}_p} \to B\mathbf{Z}_p$, $d^n([S^n]) = \alpha^k \in E_n^{0,k}$, where $k = \frac{1}{2}(n+1)$. This spectral sequence maps into that for the action $(\mathbf{Z}_{p^2}, S^n)$. But α^k maps into $(p\beta)^k = p^k\beta^k = 0$. Thus $d^n([S^n]) = 0$ in the $(\mathbf{Z}_{p^2}, S^n)$ spectral sequence. Hence $\Phi(\mathbf{Z}_{p^2}, S^n) = 1$. Hence $\operatorname{tr}(\mathbf{Z}_{p^2}, S^n) = 1$.

It is worth noting that for $n = 1$, $\operatorname{tr}(\mathbf{Z}_{p^2}, S^1) = p$. This is easily seen, since if $f\colon S^1 \to S^1$ is given by $f(e^{i\theta}) = e^{ip\theta}$, then $f\colon S^1 \to S^1$ is equivariant from the free action $(\mathbf{Z}_{p^2}, S^1)'$ to $(\mathbf{Z}_{p^2}, S^1)$. Now f has degree p and $\operatorname{tr}(\mathbf{Z}_{p^2}, S^1)' = p^2$, and $\operatorname{tr}(\mathbf{Z}_{p^2}, S^1) \neq p^2$ since the action is not free. By (8.1) $p^2 | p \cdot \operatorname{tr}(\mathbf{Z}_{p^2}, S^1)$. So $\operatorname{tr}(\mathbf{Z}_{p^2}, S^1) = p$.

For $\mathbf{Z}_2$ acting on even spheres, $\operatorname{tr}(\mathbf{Z}_4, S^n) = 1$ for $n > 2$ since the even spheres have odd spheres imbedded equivariantly. For $n = 2$ this argument breaks down. However, Ozaydin can show that $\operatorname{tr}(\mathbf{Z}_4, S^2) = 1$. He also shows that the trace for finite G on one-dimensional or zero-dimensional G-complexes is just the g.c.d. of the orders of the orbits. This is not true for 2-complexes, as $\operatorname{tr}(\mathbf{Z}_4, S^2) = 1$ shows [**Oz**].

Finally, we give one more corollary of the combination of Theorem (9.1) and (6.11).

(9.4) COROLLARY. *Let (G, M) be a smooth action of $G = (\mathbf{Z}_p)^r$ on a compact manifold M which acts trivially on integral homology. Then*

$$\log_p(\textit{order of minimal orbit}) \leqslant \dim M.$$

10. Traces and cross sections to G-bundles.

(10.1) THEOREM. *Suppose (G, M) is an action of a compact connected Lie group on a closed oriented manifold of dimension n. If $\operatorname{tr}(G, M) = 0$, then all n-dimensional oriented G-vector bundles over M admit nonzero (nonequivariant) cross sections.*

PROOF. The obstruction to finding a nonzero cross section to ξ lies in $H^n(M; \pi_{n-1}(S^{n-1})) \cong H^n(M; \mathbf{Z})$. Since ξ is an oriented G-bundle, the Borel construction gives an oriented vector bundle $\xi_G \to M_G$. If ξ had no nonzero cross section, then $\xi_G \to M_G$ would have none. So there is an obstruction C_G in $H^n(M_G, \mathbf{Z})$ to finding a nonzero cross section. Then $i^*(C_G)$ is the obstruction in M, and since there is no cross section, $i^*(C_G) \neq 0$. Hence $\operatorname{tr}(G, M) = \Phi(G, M) \neq 0$, which is a contradiction. Note that the theorem is still true if we consider G-fibrations over M with fibre S^{n-1}.

(10.2) COROLLARY. *If (T, M^n) has no stationary point, then every oriented n dimensional T-vector bundle over M admits a nonzero cross section. Here T is a torus.*

11. Trace and concordance.

(11.1) THEOREM. *Suppose X and Y are G-spaces homotopy equivalent to CW complexes. If $f\colon X \to Y$ is a G-map which is a homotopy equivalence, then* $\operatorname{tr}(G, X) = \operatorname{tr}(G, Y)$.

PROOF. Since $f\colon X \to Y$ is a homotopy equivalence, the Borel construction gives a fibre preserving homotopy equivalence $\bar{f}\colon X_G \to Y_G$. This is a fibre homotopy equivalence by a theorem of Dold. Clearly the trace defined for fibrations is a fibre homotopy equivalence invariant.

Quinn has defined a *concordance* between two actions. We say (G, X_1) and (G, X_2) are *concordant* if there exists a diagram $X_1 \xrightarrow{f} Y \xleftarrow{g} X_2$ of G-maps and a G-space Y so that f and g are homotopy equivalences.

(11.2) COROLLARY. *Trace is preserved by concordances. In particular, if X_1 is a manifold dominated by a finite complex and G is a finite group, then any concordant action must be free.*

REMARK. (a) If we remove the condition of finite dimensionality from X_1, then there exist nonfree actions concordant to (G, X_1). The G-cone over E_G shows that the free action (G, E_G) is concordant to (G, point).

12. Trace and the orbit map. Let (G, X) be an action of a connected group on X. An orbit map $\omega\colon G \to X$ is given by sending $g \mapsto g(x_0)$ for some choice of $x_0 \in X$. We show that $\exp \tilde{\omega}^*$ and $\exp \tilde{\omega}_*$ divide $\operatorname{tr}(G, X)$ when X is a compact manifold, where $\tilde{\omega}^*\colon \tilde{H}^*(X; \Gamma) \to \tilde{H}^*(G; \Gamma)$ for any untwisted coefficients Γ.

(12.1) THEOREM. $\operatorname{tr}(G, M)\tilde{\omega}^* = 0$ *for M a compact manifold.*

(12.2) COROLLARY. *If G is a connected Lie group and if* $\operatorname{tr}(G, M) \neq 0$, *then $\tilde{\omega}_*\colon \tilde{H}_*(G; \mathbf{Z}) \to \tilde{H}_*(M; \mathbf{Z})$ is trivial.*

PROOF OF THEOREM. First we consider the case where M is closed and oriented. Let $\hat{\omega}\colon G \times M \to M$ denote the action. We get a commutative diagram

$$\begin{array}{ccc} G \times M & \xrightarrow{\hat{\omega}} & M \\ \downarrow j \times 1 & & \downarrow i \\ E_G \times M & \xrightarrow{\phi} & M_G \\ \downarrow & & \downarrow \\ B_G & \xrightarrow[1]{} & B_G \end{array}$$

From this we see that $\hat{\omega}^* i^*(k) = 1 \times i^*(k) \in H^*(G \times M; \mathbf{Z})$. Then $\hat{\omega}^*(\operatorname{tr}(G, M)[M]) = 1 \times \operatorname{tr}(G, M)[M]$. Now any element $x \in \tilde{H}^i(M; \Gamma)$ has the form

$$\hat{\omega}^*(x) = \omega^*(x) \times 1 + \Sigma a_j \times b_j + \Sigma c_k * d_k.$$

So

$$\begin{aligned} 0 &= \hat{\omega}^*(x \cup \operatorname{tr}(G, M)[M]) \\ &= \left(\tilde{\omega}^*(x) \times 1 + \Sigma a_j \times b_j + \Sigma c_j * d_j\right) \cup (1 \times \operatorname{tr}(G, M)[M]) \\ &= \tilde{\omega}^*(x) \times \operatorname{tr}(G, M)[M] + (\text{other terms of different filtration}). \end{aligned}$$

So $\tilde{\omega}^*(x) \times \operatorname{tr}(G, M)[M] = 0$. So $\operatorname{tr}(G, M)\tilde{\omega}^*(x) = 0$.

Now suppose that M has a boundary and is orientable. Then the double of M has an obvious G action, and $\omega\colon G \to M \overset{j}{\subset} DM$. Since j^* splits and

$$\operatorname{tr}(G, DM)\tilde{\omega}^* j^* = 0$$

by the previous case, we see that $\operatorname{tr}(G, DM)\tilde{\omega}^* = 0$. But $\operatorname{tr}(G, DM) = \operatorname{tr}(G, M)$.

Now suppose M is not orientable. Then the mapping cylinder of $p\colon \tilde{M} \to M$, where $\tilde{M}$ is the oriented double covering, is an oriented manifold with boundary on which G acts. Then the above case proves the result.

PROOF OF COROLLARY. If $\operatorname{tr}(G, M) \neq 0$, then the maximal torus is contained in an isotropy subgroup G_x. Hence, ω factors as $\omega\colon G \overset{\rho}{\to} G/T \to G/G_* \hookrightarrow M$. Now $\operatorname{tr}(G, G/T)\tilde{\rho}^* = 0$ by the theorem, and $\operatorname{tr}(G, G/T)|\chi(G/T) \neq 0$. So $\tilde{\rho}^*\colon \tilde{H}^*(G/T; \mathbf{Q}) \to \tilde{H}^*(G; \mathbf{Q})$ is trivial. Hence $\tilde{\rho}_*\colon H_*(G; \mathbf{Q}) \to H_*(G/T; \mathbf{Q})$ is trivial by duality. But $H_*(G/T; \mathbf{Z})$ is torsion free, so $\tilde{\rho}_* = 0$ for integral coefficients. Hence $\tilde{\omega}_* = 0$ for integral coefficients. See [**G6**] for other results on ω_*.

(12.3) THEOREM. $\operatorname{tr}(G, M)\tilde{\omega}_* = 0$ *for M a compact manifold.*

PROOF. Suppose that M is a closed oriented manifold. Let $x \in H_i(G; \mathbf{Z})$ for $i > 0$. Then $\hat{\omega}_*(x \times [M]) = 0$ for dimensional reasons. Thus if $\Phi = \operatorname{tr}(G, M)$ we have

$$\hat{\omega}_*\big(\hat{\omega}^*(\Phi \cdot [\overline{M}]) \cap (x \times [M])\big) = \Phi \cdot [\overline{M}] \cap \hat{\omega}_*(x \times [M]) = 0.$$

Also

$$\begin{aligned}\hat{\omega}_*\big(\hat{\omega}^*(\Phi \cdot [\overline{M}]) \cap (x \times [M])\big) &= \hat{\omega}_*\big((1 \times \Phi \cdot [\overline{M}]) \cap (x \times [M])\big) \\ &= \pm\Phi\hat{\omega}_*(x \times 1) = \pm\Phi \cdot \omega_*(x).\end{aligned}$$

Hence, $\Phi \cdot \omega_*(x) = 0$ for closed oriented manifolds. The extension to compact manifolds follows as in Theorem (12.1).

13. An example for $(\mathrm{Diff})_{\mathbf{Q}}$. Two CW complexes, X and Y, are *rationally homotopy equivalent* if there exists a sequence of maps $X \to X_1 \leftarrow X_2 \to \cdots \leftarrow Y$ such that each map induces an isomorphism on homology with rational coefficients.

Let M and N be rationally equivalent closed oriented manifolds. Let $\mathscr{H}_0(M)$ and $\mathscr{H}_0(N)$ be the monoids of self-homotopy equivalences homotopic to the respective identities.

Now there exists a functor which takes a nilpotent CW complex X to a rational space $X_{\mathbf{Q}}$ and gives a map $f\colon X \to X_{\mathbf{Q}}$ which induces an isomorphism of rational homology groups. This functor has the pleasant property that $\mathscr{H}_0(X_{\mathbf{Q}})$ is homotopy equivalent to $(\mathscr{H}_0(X))_{\mathbf{Q}}$. This property extends to classifying spaces, so $B(\mathscr{H}_0(X_{\mathbf{Q}}))$ is homotopy equivalent to $(B\mathscr{H}_0(x))_{\mathbf{Q}}$. Thus we have a commutative diagram

$$\begin{array}{ccc} X & \to & X_{\mathbf{Q}} \\ \downarrow & & \downarrow \\ E_\infty & \to & (E_\infty)_{\mathbf{Q}} \\ \downarrow & & \downarrow \\ B_\infty & \to & (B_\infty)_{\mathbf{Q}} \end{array}$$

where $E_\infty \to B_\infty$ is the universal oriented fibration and each horizontal map induces an isomorphism on rational homology. Thus if X and Y are rationally homotopy equivalent, there exists a sequence of fibrations and fibre maps between them preserving the rational homology. See [**HMR**, Theorems 3.11, 3.12].

(13.1) THEOREM. $\Phi(\mathscr{H}_0, M) = 0$ *if and only if* $\Phi(\mathscr{H}_0, N) = 0$.

PROOF. The sequence of commutative diagrams from $M \xrightarrow{i_1} E_1 \to B\mathscr{H}_0(M)$ and $N \xrightarrow{i_1} E_2 \to B\mathscr{H}_0(N)$ alluded to in the above paragraph results in two fibrations having the same rational cohomology, and the fibre inclusions i_1 and i_2 must induce the same homomorphism on rational cohomology.

(13.2) EXAMPLE. Now we consider an example of a smooth manifold M such that $\Phi(\mathscr{H}_0, M) = 0$ but $\Phi(\mathrm{Diff}_0 M, M) = 1$. This M is rationally homotopy equivalent to $S^3 \times S^4 \times S^5$ and

$$\Phi\big(\mathrm{Diff}_0(S^3 \times S^4 \times S^5), S^3 \times S^4 \times S^5\big) = 0.$$

This results in the fact that $(\mathrm{Diff}_0 S^3 \times S^4 \times S^5)_{\mathbf{Q}}$ is not equivalent to $(\mathrm{Diff}_0 M)_{\mathbf{Q}}$, for otherwise the fibre numbers Φ would agree.

By a theorem of Sullivan [**Su**, Theorem 13.2], for every closed oriented simply connected manifold X and for certain elements $[V_i] \in \oplus_i H^{4i}(X; \mathbf{Q})$ there is a smooth oriented closed simply connected manifold M and a map $f\colon M \to X$ so that f is a rational equivalence, the degree of f equals 1, and the Pontrjagin classes $p_i \in H^{4i}(M; \mathbf{Z})$ satisfy $\rho(p_i) = f^*(V_i)$, where $\rho\colon H^*(M; \mathbf{Z}) \to H^*(M; \mathbf{Q})$. The elements $[V_i]$ can be arbitrarily chosen if $\dim X \not\equiv 0 \pmod 4$ and must satisfy the Hirzebruch index formula for $\dim X = 4k$. That is, $\sigma(X) = \langle L_k(V_i), [X] \rangle$.

Now let $X = S^3 \times S^4 \times S^5$. Choose $V_1 = [S^4]$, $V_2 = 62[S^3 \times S^5]$, and $V_3 = 13[S^3 \times S^4 \times S^5]$. Since $\dim X = 12 = 4 \times 3$, the appropriate L genus is $L_3 = (1/945)(62V_3 - 13V_1V_2 + 2V_1^3)$, also $\sigma(X) = 0$, since $H^6(X; \mathbf{Z}) = 0$. Hence the Pontrjagin numbers of M are 0, 62, and 13. Now g.c.d.$(13, 62) = 1$, so $\Phi(\mathrm{Diff}_0, M) = 1$.

On the other hand $\Phi(\mathscr{H}_0, S^3 \times S^4 \times S^5) = 0$ since the fibration $S^3 \times (S^4 \times S^5) \to E_{S^3} \times (S^4 \times S^5) \xrightarrow{p} B_{S^3}$ has $\Phi(p) = 0$. Since $S^3 \times S^4 \times S^5$ is rationally equivalent to M, we see that $\Phi(\mathscr{H}_0, M) = 0$ by the theorem. Also $(\mathrm{Diff}_0, S^3 \times S^4 \times S^5) = 0$ since $\Phi(p) = 0$.

REMARK. If we knew that the classifying spaces involved were of finite type, we could have used trace instead of Φ in the above argument.

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