

Fixed Point Indices and Manifolds with Collars

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Abstract. This paper concerns a formula which relates the Lefschetz number $L(f)$ for a map $f : M \rightarrow M'$ to the fixed point index $I(f)$ summed with the fixed point index of a derived map on part of the boundary of ∂M . Here M is a compact manifold and M' is M with a collar attached.

CHAPTER I

INTRODUCTION

This Paper represents the first third of a Ph. D thesis [16] written by the first author under the direction of the second author at Purdue University in 1990. The Thesis is entitled "Fixed Point Indices and Transfers, and Path Fields" and it contains, in addition to the contents of this manuscript, an analogous formula relating to Dold's fixed point transfers and a study of path fields of differential manifolds in order to relate the formula in this manuscript with an analogous formula involving indices of vector fields.

Let M be a compact differentiable manifold with or without boundary ∂M . Assume V is a vector field on M with only isolated zeros. If M is with boundary ∂M and V points outward at all boundary points, then the index of the vector field V equals Euler characteristic of the manifold M . This is the classical Poincaré-Hopf Index Theorem. (A 2-dimensional version of this theorem was proven by Poincaré in 1885; in full generality the theorem was proven by Hopf [11] in 1926). In particular, the index is a topological invariant of M ; it does not depend on the particular choice of a vector field on M .

Marston Morse [13] extended this result to vector fields under more general boundary conditions, namely, to any vector field without zeros on the boundary ∂M ; he discovered the following formula:

$$(1.1) \quad \text{Ind}(V) + \text{Ind}(\partial V) = \chi(M),$$

where $\chi(M)$ denotes the Euler characteristic of M and ∂V is defined as follows: Let $\partial_- M$ be the open subset of the boundary ∂M containing all the points m for which the vectors $V(m)$ point inward; and let ∂V be the vector field on the boundary ∂M obtained by first restricting V to the boundary and then projecting $V|_{\partial M}$ to its component field tangent to the boundary. Then $\partial V = \partial V|_{\partial_- M}$. Furthermore, in the same paper, Morse generalized his result to indices of vector fields with nonisolated zeros. This is the formula (1.1). Now (1.1) was rediscovered by D. Gottlieb [7] and C. Pugh [15]. D. Gottlieb further found

further interesting applications in [8],[9] and [10]. Throughout this paper we shall call formula (1.1) the Morse formula for indices of vector fields.

We consider maps $f : M \rightarrow M'$ from a compact topological manifold M to M' where M' is obtained by attaching a collar $\partial M \times [0, 1]$ to M . If f has no fixed points on the boundary ∂M , we prove Theorem (2.2.1) which is the fixed point version of the Morse formula:

$$(1.2) \quad I(f) + I(r \circ f|_{\partial_- M}) = L(r \circ f),$$

where I denotes the fixed point index, r is a retraction of M' onto M which maps the collar $\partial M \times I$ onto the boundary ∂M , $\partial_- M$ is an open subset of ∂M containing all the points $x \in \partial M$ mapped outside of M under f and $L(r \circ f)$ is the Lefschetz number of the composite map $r \circ f$.

In particular, if the map $r \circ f$ is homotopic to the identity map, we have:

$$(1.3) \quad I(f) + I(r \circ f|_{\partial_- M}) = \chi(M),$$

which is similar to the Morse formula; and the map $r \circ f|_{\partial_- M}$ is analogous to the vector field $\partial_- V$.

Formula(1.3) was independently obtained by A. Dold (private letter to D. Gottlieb).

This paper is organized as follows:

In chapter II, §1, we list some properties of fixed point indices; our first main result, Theorem (2.2.1), is proven in §2.

CHAPTER II

FIXED POINT VERSION OF THE MORSE FORMULA

In this section, we use the definition of fixed point index and some well known results on fixed point index given by Dold in [3] or [5, Chap. 7] to obtain an equation for fixed point indices [Theorem 2.2.1] analogous to the Morse equation for vector field indices described in the introduction.

§1. Fixed point index and its properties.

Let X be an Euclidean neighborhood retract (ENR). Consider maps f from an open subset V of X into X whose fixed point set $F(f) = \{x \in V | f(x) = x\}$ is compact. Dold [3] defined the fixed point index $I(f)$ and proved the following properties.

(2.1.1) LOCALIZATION.

Let $f : V \rightarrow X$ be a map such that $F(f)$ is compact, then $I(f) = I(f|_W)$ for any open neighborhood W of $F(f)$ in V .

(2.1.2) ADDITIVITY.

Given a map $f : V \rightarrow X$ and V is a union of open subsets V_j , $j = 1, 2, \dots, n$ such that the fixed point sets $F^j = f(f) \cap V_j$ are mutually disjoint. Then for each j , $I(f|_{V_j})$ is defined and

$$I(f) = \sum_{j=1}^n I(f|_{V_j}).$$

(2.1.3) UNITS.

Let $f : V \rightarrow X$ be a constant map. Then

$$\begin{aligned} I(f) &= 1 && \text{if } f(V) = p \in V \quad \text{and} \\ I(f) &= 0 && \text{if } f(V) = p \notin V \end{aligned}$$

(2.1.4) NORMALIZATION.

If f is a map from a compact ENR X to itself, then $I(f) = L(f)$ where $L(f)$ is the Lefschetz number of the map f .

(2.1.5) MULTIPLICATIVITY

Let $f : V \rightarrow X$ and $f' : V' \rightarrow X'$ be maps such that the fixed point sets $F(f)$ and $F(f')$ are compact, then fixed point index of the product $f \times f' : V \times V' \rightarrow X \times X'$ is defined and

$$I(f \times f') = I(f) \cdot I(f').$$

(2.1.6) COMMUTATIVITY.

If $f : U \rightarrow X'$ and $g : U' \rightarrow X$ are maps where $U \subseteq X$ and $U' \subseteq X'$ are open subsets, then the two composites $gf : V = f^{-1}(U') \rightarrow X$ and $fg : V' = g^{-1}(U) \rightarrow X'$ have homeomorphic fixed point sets. In particular, $I(fg)$ is defined if and only if $I(gf)$ is defined, in that case,

$$I(fg) = I(gf).$$

(2.1.7) HOMOTOPY INVARIANCE

Let $H : V \times I \rightarrow X$ be a homotopy between the maps f_0 and f_1 . Assume the set $F = \{x \in V | H(x, t) = x \text{ for some } t\}$ is compact; then

$$I(f_0) = I(f_1).$$

For our purposes it is useful to reformulate the properties of additivity (2.1.2) and homotopy invariance (2.1.7) in the form of the following propositions.

(2.1.8) PROPOSITION.

Assume X is compact and V is an open subset of X . Let $f : \bar{V} \rightarrow X$ be a map without fixed points on $Bd(V)$. If $\{V_j\}$, $j = 1, 2, \dots, n$ are mutually disjoint open subsets of V and whose union $\bigcup_{j=1}^n V_j$ contains all the fixed points of f , then

$$I(f|_V) = \sum_{j=1}^n I(f|_{V_j}).$$

(2.1.9) PROPOSITION.

Assume X is compact and V is an open subset of X . Let $\bar{H} : \bar{V} \times I \rightarrow X$ be a homotopy from \bar{f}_0 and \bar{f}_1 where \bar{f}_0 and \bar{f}_1 are maps from \bar{V} , the closure of V to X . If $\bar{H}(x, t) \neq x$ for all $x \in Bd(V)$ and for all t , then

$$I(f_0) = I(f_1) \quad \text{where} \quad f_0 = \bar{f}_0|_V \quad \text{and} \quad f_1 = \bar{f}_1|_V.$$

Proof: Since $H = \overline{H}|_{V \times I}$ is a homotopy from f_0 to f_1 , it suffices to verify that the set $F = \{x \in V | H(x, t) = x \text{ for some } t\}$ is compact. Let $\{x_j\}$ be a sequence in F converging to $x \in \overline{V} = V \cup Bd(V)$. There exists a sequence $\{t_j\}$ in I such that $\overline{H}(x_j, t_j) = x_j$. Since I is compact, the sequence $\{t_j\}$ converges to a point $t \in I$. By the continuity of \overline{H} , we have $\overline{H}(x, t) = x$. On the other hand, we know that $\overline{H}(x, t) \neq x$ for all $x \in Bd(V)$; thus, $x \in V$ and $H(x, t) = x$. Consequently, $x \in F$. Therefore, F is a closed subset of a compact space, hence F is compact. This proves the proposition.

§2. The main formula.

Consider a compact topological manifold M with boundary ∂M . We attach a collar to M and call the resulting manifold $M' : M' = M \cup_{\partial M \sim \partial M \times \{0\}} \partial M \times [0, 1]$. Let $f : M \rightarrow M'$ be a map such that $f(x) \neq x$ for all $x \in \partial M$. Since M is compact, the fixed point set $F(f)$ is a compact set contained in $\overset{\circ}{M} = M \setminus \partial M$. For such $f : M \rightarrow M'$, we define the index of f , denoted by $I(f)$, to be the fixed point index of the map $f|_{\overset{\circ}{M}}$ given in §1.

Let $r : M' \rightarrow M$ be the retraction from M' to M given by the formula:

$$\begin{cases} r(m) = m & \text{for } m \in M \\ r(b, t) = (b, 0) \sim b & \text{for } (b, t) \in \partial M \times [0, 1]. \end{cases}$$

Now we can formulate the main result of the section.

(2.2.1) THEOREM.

Let M be a compact manifold with boundary ∂M and $f : M \rightarrow M'$ and $r : M' \rightarrow M$ be as above. Then

$$I(f) + I(rf|_{\partial_- M}) = L(rf)$$

where $L(rf)$ denotes the Lefschetz number of $rf : M \rightarrow M$ and $\partial_- M = \{x \in \partial M | f(x) \notin M\}$.

This equation relates the fixed point index of the map f with the fixed point index of the “projection map” rf defined on a part of the boundary and it is similar to the equation of Morse for indices of the vector fields.

We need the following lemmas to prove theorem (2.2.1).

(2.2.2) LEMMA

$$I(rf|_{\overset{\circ}{M}}) = I(f|_{\overset{\circ}{M}}) = I(f).$$

Proof: From the commutativity of fixed point indices (2.1.6) and the fact that $r^{-1}(\overset{\circ}{M}) = \overset{\circ}{M}$ and $r|_{\overset{\circ}{M}}$ is the identity map, we have $I(rf|_{\overset{\circ}{M}}) = I(fr|_{r^{-1}(\overset{\circ}{M})}) = I(fr|_{\overset{\circ}{M}}) = I(f)$. This proves the lemma.

We now consider the following composite maps:

$$rf|_{\partial_M} : \partial_M \xrightarrow{f|_{\partial_M}} \partial M \times (0, 1] \xrightarrow{r} \partial M \quad \text{and}$$

$$rf|_V : V \xrightarrow{f|_V} M' \xrightarrow{r} M.$$

where V is an open subset of M such the $\partial M \subseteq V$ and $F(f) \cap \bar{V} = \emptyset$. In the next lemma, we shall prove that these two composites have the same indices.

(2.2.3) LEMMA.

$$I(r \circ f|_{\partial_M}) = I(r \circ f|_V).$$

Proof: The commutativity (2.1.6) implies that

$$I(r \circ f|_V) = I(f \circ r|_{r^{-1}(V)}).$$

It is easy to see that the fixed point set of the map $f \circ r|_{r^{-1}(V)}$ is $\{(b, t) \in \partial M \times (0, 1] | f(b) = (b, t)\}$ and the fixed point set of the map $rf|_V$ is $\{b \in \partial M | f(b) = (b, t) \text{ for some } t\}$.

We now define a homotopy G_s , $0 \leq s \leq 1$, as the composite of the following maps

$$\overline{\partial M} \times I \xrightarrow{r} \overline{\partial M} \xrightarrow{f} \partial M \times I \xrightarrow{H_s} \partial M \times I$$

where the map H_s is defined as follows:

$$H_s(b, t) = (b, s\bar{t} + (1-s)t), \quad \text{where } \bar{t} \text{ is a constant, } 0 < \bar{t} \leq 1.$$

since the map $H_0 = \text{Identity}$, we have

$$G_0(x, t) = H_0(fr(x, t)) = fr(x, t), \quad \text{and}$$

$$G_1(x, t) = H_1(fr(x, t)) = (rf \times g)(x, t)$$

where $r \circ f$ is a map from $\overline{\partial M}$ to ∂M and $g : I \rightarrow I$, $g(t) = \bar{t}$, is the constant map. Furthermore, the restriction $G_s|_{Bd(\partial_M \times I)}$ has no fixed points for any $0 \leq s \leq 1$. To see this, we look at a point $x \in Bd(\partial_M)$. We know then $f(x) \in \partial M$ and $rf(x) = f(x) \neq x$, therefore, $G_s(x, t) = H_s(fr(x, t)) = H_s(f(x)) = H_s(f(x), 0) = (f(x), s\bar{t}) \neq (x, t)$.

Now the properties (2.1.9), (2.1.5) and (2.1.3) imply

$$I(fr|_{\partial_M \times (0, 1]}) = I(rf|_{\partial_M}) \cdot I(g) = I(rf|_{\partial_M}) \cdot 1$$

$$I(rf|_V) = I(fr|_{r^{-1}(V)}) = I(fr|_{\partial_M \times (0, 1]}).$$

The last equality holds because $\partial M \times (0, 1]$ contains the fixed point set of $(fr|_{r^{-1}(V)})$. Thus, $I(rf|_V) = I(rf|_{\partial M})$.

Proof of Theorem (2.2.1):

Consider the composite $M \xrightarrow{f} M' \xrightarrow{r} M$. Let V be the open set as in Lemma (2.2.3), then V and $\overset{\circ}{M}$ are two open subsets of M such that $V \cup \overset{\circ}{M} = M$. Clearly, $F(rf) \cap \overset{\circ}{M}$ and $F(rf) \cap V$ are disjoint. Using additivity (2.1.2) and the normalization (2.1.4) of the fixed point indices, we have:

$$I(rf|_{\overset{\circ}{M}}) + I(rf|_V) = I(rf) = L(rf).$$

Lemma(2.2.2) and Lemma (2.2.3) then imply the equation:

$$I(f) + I(rf|_{\partial M}) = L(rf).$$

(2.2.4) COROLLARY. *If $f : M \rightarrow M'$ is map such that $f(x) \notin M$ for any $x \in \partial M$, then $I(f) = L(rf) - L(rf|_{\partial M})$.*

(2.2.5) COROLLARY. *If $f : M \rightarrow M'$ is without fixed points on the boundary ∂M and $f(\partial M) \subset M$, then $I(f) = L(rf)$.*

(2.2.6) EXAMPLE. *Consider a map $f : D^n \rightarrow \mathbb{R}^n$,*

(i) *if $f(S^{n-1}) \subset D^n$, then f has a fixed point.*

(ii) *if $f(S^{n-1}) \subset \mathbb{R}^n \setminus D^n$, then Corollary (2.2.4) implies that*

$$I(f) = L(rf) - L(rf|_{S^{n-1}}) = 1 - (1 + (-1)^{n-1}) \deg(rf|_{S^{n-1}}) = (-1)^n \deg(rf|_{S^{n-1}}).$$

(2.2.7) COROLLARY. *If $f : M \rightarrow M'$ is homotopic to the inclusion map $M \hookrightarrow M'$, then $I(f) + I(rf|_{\partial M}) = \chi(M)$ where $\chi(M)$ denotes the Euler characteristic of M .*

Proof: If $f : M \rightarrow M'$ is homotopic to the inclusion map $M \hookrightarrow M'$, then the composite map $rf : M \rightarrow M$ is homotopic to the identity map. Therefore $L(rf) = L(Id) = \chi(M)$.

Finally, we show that the results in this section are independent of the choice of a retraction from M' to M . Recall that the retraction $r : M' \rightarrow M$ in Theorem (2.2.1) was defined as follows:

$$\begin{aligned} r(m) &= m & \text{for } m \in M \\ r(b, t) &= (b, 0) \sim b & \text{for } (b, t) \in \partial M \times [0, 1]. \end{aligned}$$

Now, assume r' is any retraction from M' to M such that r' maps the collar $\partial M \times [0, 1]$ into the boundary ∂M . Then the following proposition is true:

(2.2.8) PROPOSITION.

$$I(f) + I(r'f|_{\partial_-M}) = L(r'f).$$

Furthermore,

$$\begin{aligned} L(rf) &= L(r'f) \quad \text{and} \\ I(rf|_{\partial_-M}) &= I(r'f|_{\partial_-M}) \end{aligned}$$

where r, f and ∂_-M are the same as in Theorem (2.2.1).

Proof: First, we prove the formula: $I(f) + I(r'f|_{\partial_-M}) = L(r'f)$.

Let $V_1 = \{x \in M | f(x) \in \overset{\circ}{M}\}$ and $V_2 = \{x \in M | f(x) \in M' \setminus M\}$, then V_1 and V_2 are disjoint open subsets of the manifold M and $V_1 \cup V_2$ contains all the fixed points of the map $r'f$. Indeed, if $x \notin (V_1 \cup V_2)$, then $f(x) \in \partial M$ and hence $r'f(x) = f(x) \neq x$. Proposition (2.1.8) implies the equation:

$$I(r'f) = I(r'f|_{V_1}) + I(r'f|_{V_2}).$$

Since $r'f$ is a self map from M to M , so

$$I(r'f) + L(r'f).$$

We have:

$$(i) \quad L(r'f) = I(r'f|_{V_1}) + I(r'f|_{V_2}).$$

Now, since $r'f|_{V_1} = f|_{V_1}$ and $F(f) \subseteq V_1$, then

$$(ii) \quad I(r'f|_{V_1}) = I(f|_{V_1}) = I(f).$$

Let us decompose the map $r'f|_{V_2}$:

$$r'f|_{V_2} : V_2 \xrightarrow{f|_{V_2}} \partial M \times [0, 1] \xrightarrow{r'} \partial M \xrightarrow{i} M.$$

The commutativity (2.1.6) implies that

$$(iii) \quad I(r'f|_{V_2}) = I(ir'f|_{V_2}) = I(r'f|_{i^{-1}(V_2)}) = I(r'f|_{\partial_-M}).$$

Combining equations (i), (ii) and (iii), we obtain

$$(iv) \quad I(f) + I(r'f|_{\partial_-M}) = L(r'f).$$

Next, we show that $L(r'f) = L(rf)$. It is quite easy to see that r and r' are homotopic with a homotopy $H_t : M' \rightarrow M$, $0 \leq t \leq 1$, defined as follows:

$$\begin{aligned} H_t(m) &= m && \text{for } m \in M, \\ H_t(b, s) &= r'(b, st) && \text{for } (b, s) \in \partial M \times [0, 1]. \end{aligned}$$

Clearly, $H_0 = r$ and $H_1 = r'$. So, rf and $r'f$ are homotopic and, consequently,

$$(v) \quad L(rf) = L(r'f).$$

Equations (iv) and (v) and Theorem (2.2.1) imply that

$$I(rf|_{\partial_-M}) = I(r'f|_{\partial_-M}).$$

This concludes the proof.

BIBLIOGRAPHY

- [1] J. C. BECKER AND D. H. GOTTLIEB, *Vector fields and transfers*, Manuscripta Mathematica, 72 (1991), 111–130.
- [2] R. F. BROWN, *Path fields on manifolds*, Trans. Amer. Math. Soc., 118 (1965), 180–191.
- [3] A. DOLD, *Fixed point index and fixed point theorem for Euclidean neighborhood retracts*, Topology, 4 (1965), 1–8.
- [4] ———, *The fixed point transfers of fiber-preserving maps*, Math. Z., 148 (1976), 215–224.
- [5] ———, *“Lecture on Algebraic Topology”*, Springer, Berlin - Heidelberg - New York, 1972.
- [6] E. FADELL, *Generalized normal bundles for locally-flat imbeddings*, Trans. Amer. Math. Soc., 114 (1965), 488–513.
- [7] D. H. GOTTLIEB, *A de Moivre formula for fixed point theory*, ATAS de 5^o Encontro Brasileiro de Topologia, 53 (1988), 59–67, Universidade de Sao Paulo, Sao Carlos S. P., Brasil.
- [8] ———, *A de Moivre like formula for fixed point theory*, Proceedings of the fixed point theory seminar at the 1986 International Congress of Mathematics, Robert F. Brown editor, Contemporary Math. Series of the American Math. Soc., 72, 99–106.
- [9] ———, *On the index of pullback vectorfields*, Proceedings of the 2nd Siegen Topology Symposium, August 1987, Ulrich Koschorke editor. Lecture Notes of Mathematics. Vol. **1350**, 163–170; Springer Verlag, New York City, N.Y.
- [10] ———, *Zeros of pullback vector fields and fixed point theory for bodies*, Contemporary Mathematics, 96 (1991), 163–180.
- [11] H. HOPF, *Abbildungsklassen n -dimensionaler Mannigfaltigkeiten*, Mathematische Annalen, 96 (1926), 209–224.
- [12] S. T. HU, *Fiberings of enveloping space*, Proc. London Math. Soc., (3)11 (1961), 691–707.
- [13] M. MORSE, *Singular points of vector fields under general boundary conditions*, American Journal of Mathematics, 51 (1929), 165–178.
- [14] J. NASH, *A path space and Stiefel–Whitney classes*, Proc. Nat. Sci. U. S. A., 41 (1955), 320–321.
- [15] C. PUGH, *A generalized Poincaré index formula*, Topology, 7 (1968), 217–226.
- [16] CHEN-FARNG BENJAMIN, *Fixed Point Indices, Transfers and Path Fields*, Ph. D Thesis (1990 Purdue University).