PRIME IDEAL STRUCTURE OF BIRATIONAL EXTENSIONS OF POLYNOMIAL RINGS

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1. Introduction. Our work in this paper concerns the general question: Which partially ordered sets, or equivalently topological spaces, can occur as the prime spectra of certain types of Noetherian rings? Even when the rings are quite simple, their prime spectra can be surprising. For example, let R be a one-dimensional countable semilocal domain, and let x be an indeterminate over R. In [HW], a characterization is given of the partially ordered sets that occur as $\operatorname{Spec}(R[x])$: If R is local, there are exactly two possibilities for $\operatorname{Spec}(R[x])$, one of which occurs when R is Henselian and the other when R is not Henselian. If R has more than one maximal ideal, then the spectrum of $R[x]$ is uniquely determined up to isomorphism by the number of maximal ideals of R. (In this latter case, R cannot be Henselian.)

In [HLW], our main emphasis was to demonstrate that the situation for certain spectra related to $\operatorname{Spec}(R[x])$, such as the projective line $\operatorname{Proj}(R[s,t])$, is similar to that of $\operatorname{Spec}(R[x])$, in that again just two cases occur for $\operatorname{Proj}(R[s,t])$. In addition, we included some axioms satisfied by the partially ordered set, $\operatorname{Spec}(B)$, where B is a certain type of finitely generated birational extension of $R[x]$. Since then we have become more intrigued by $\operatorname{Spec}(B)$, for we have discovered there exist infinitely many possibilities for it, although there still are two distinct basic situations corresponding to whether or not R is Henselian.

In this article, we consider the prime spectra of integral domains B between the polynomial ring $R[x]$ and $R[x, 1/f]$, for R a one-dimensional semilocal domain with maximal ideals $\mathbf{m}_1, \mathbf{m}_2, ...\mathbf{m}_n$, and $f \in R[x] - \bigcup_{i=1}^{n} \mathbf{m}_i[x]$. We are particularly interested in the case $B = R[x, g/f]$, where g and f are an $R[x]$-sequence.

We would like to thank Roger Wiegand for helpful conversations regarding this material.

This is the final version of this paper, except for suggestions the referee may make.

For convenience we repeat some conventions, notation, definitions and theorems from [HLW] and [HW]. All rings we consider are commutative and contain a multiplicative identity. The terms "local" and "semilocal" include "Noetherian."

1.1 Notation. For U a partially ordered set of finite dimension, elements u, v of U, and T a finite subset of U, we set

$$\begin{aligned} \mathrm{G}(u) &= \{w \in U \mid w > u\} \qquad \text{and} \\ \mathrm{L_e}(T) &= \{w \in U \mid w < t \iff t \in T\} \\ &= \{w \in U \mid \mathrm{G}(w) = T\} \quad . \end{aligned}$$

Note that the set called $\mathrm{L}(T)$ in [HW] is denoted $\mathrm{L_e}(T)$ here. (The notation is chosen to suggest the "exactly-less-than" set.)

Let $\mathcal{M}(U)$ denote the set of maximal elements of U of maximal height and let $\mathcal{M}_i(U)$ be the maximal elements of height i, for each i.

We will refer to two particular two-dimensional partially ordered sets from [HW], and we describe them again here for convenience.

In [HW], it was shown that if $S = \mathbf{Z} - \bigcup_{i=1}^{n} \mathbf{p}_i$, where $\{\mathbf{p}_i \mid 1 \le i \le n\}$ is a finite set of primes, and $R = \mathbf{Z}_S$, then $\mathrm{Spec}(R[x])$ is of the following type:

1.2 Definition. *A partially ordered set U is called countable n-localized integer polynomial or $C\mathbf{Z}(n)P$ if and only if*

(P0) U is countable.

(P1) U has a unique minimal element u_0.

(P2) U has dimension 2.

(P3) There exist infinitely many height-one maximal elements.

(P4) There exist n height-one nonmaximal special elements, which we denote $u_1, u_2, \dots u_n$, satisfying:

(i) $\mathrm{G}(u_1) \cup \cdots \cup \mathrm{G}(u_n) = \mathcal{M}(U)$,

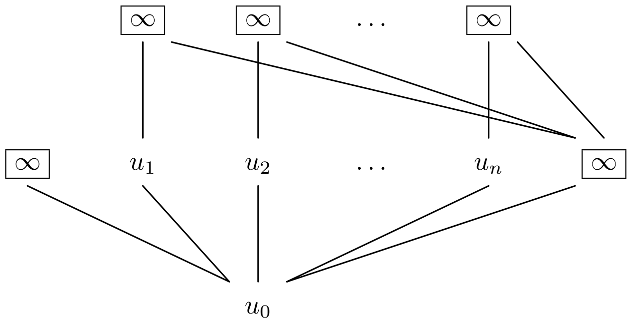
(ii) $\mathrm{G}(u_i) \cap \mathrm{G}(u_j) = \emptyset$ *for* $i \neq j$, *and*

(iii) $\mathrm{G}(u_i)$ *is infinite for each* i, $1 \le i \le n$.

(P5) For each height-one nonspecial element u, $\mathrm{G}(u)$ *is finite.*

(P6) For each nonempty finite subset T of $\{$height-2 elements of $U\}$, $\mathrm{L_e}(T)$ *is infinite.*

Pictorially, a $C\mathbf{Z}(n)P$ partially ordered set looks like this:



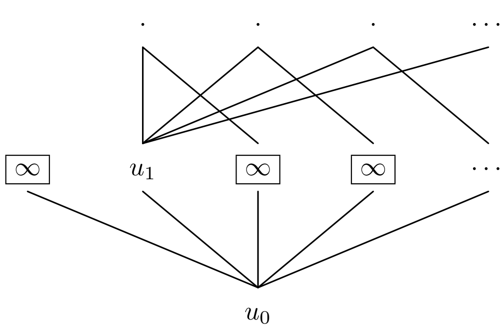
(The relationships of the lower right boxed section, determined by (P5) and (P6), are too complicated to display.)

The second partially ordered set is the type of $\operatorname{Spec}(V[x])$, where V is a countable Henselian discrete rank-one valuation domain.

1.3 Definition. *A partially ordered set U is called countable Henselian polynomial or CHP provided the same (P0)-(P5) as in $U_{C\mathbf{Z}(1)P}$ above hold and:*

(P6) For each finite subset T of $\{$height-2 elements of $U\}$ of cardinality greater than one, $\mathrm{L_e}(T)$ is empty. For each singleton $t \in \{$height-2 elements of $U\}$, $\mathrm{L_e}(\{t\})$ is infinite.

Pictorially, a CHP partially ordered set looks like this:



It was shown in [HW] that these axiom systems are categorical. Also,

1.4 Theorem. [HW, Theorem 2.7] *For any one-dimensional semilocal countable domain R with exactly n maximal ideals,* $\operatorname{Spec}(R[x])$ *is either $C\mathbf{Z}(n)P$ or CHP. Furthermore* $\operatorname{Spec}(R[x])$ *is CHP if and only if R is Henselian and $n = 1$.*

1.5 Proposition. [HLW, Proposition 3.1] *Let $(R, \mathbf{m}_1, \dots, \mathbf{m}_n)$ be a one-dimensional semilocal domain, x an indeterminate, $A = R[x]$, $f \in A - \bigcup_{i=1}^{n} \mathbf{m}_i[x]$, and let B be a finitely generated A-algebra strictly between A and $A[1/f]$. Then* $\operatorname{Spec}(B)$ *satisfies the following axioms from $C\mathbf{Z}(n)P$ or CHP (Definitions 1.2 and 1.3):*

(i) (P0) holds if R is countable.

(ii) (P1)–(P3) hold without additional hypotheses.

(iii) There are only finitely many height-one elements Q of $\operatorname{Spec}(B)$ *for which* $\mathrm{G}(Q)$ *is infinite. Moreover, each of these height-one prime ideals contains a maximal ideal of R.*

(iv) If fA has prime radical, then the number of these height-one prime ideals Q is greater than the number n of maximal ideals of R, and the Q's need not be comaximal.

2. Construction of extra height-one j-primes.

The term *j-prime* refers to a prime ideal which is an intersection of maximal ideals. Clearly a nonmaximal j-prime must be the intersection of *infinitely* many maximal ideals. In a Noetherian domain of dimension 2, a height-one nonmaximal prime P contained in infinitely many maximal ideals is a j-prime, and it is the intersection of every infinite set of maximal ideals that contain it (since if $c \notin P$, then there exist only finitely many primes minimal over the ideal (c, P)).

When R is a one-dimensional semilocal Noetherian domain, it is easily seen that the non-maximal height-one j-primes of $\operatorname{Spec}(R[x])$ are in one-to-one correspondence with the maximal ideals of R, via $P \mapsto P \cap R$. The spectra of birational extensions B of $R[x]$ may differ from $\operatorname{Spec}(R[x])$ in that we may have extra height-one j-primes of $\operatorname{Spec}(B)$ which contract to the same maximal ideal of R.

Let j-$\operatorname{Spec}(B) = \{j\text{-primes of } B\}$; and for each prime ideal, $Q \subset B$, define

$$j\text{-}\operatorname{rad}(Q) = \bigcap \{ M \text{ maximal} \mid M \supseteq Q \}.$$

We see that the essential thrust of the axioms in Definitions 1.2 and 1.3, for the two cases of $\operatorname{Spec}(R[x])$, is the description of the j-spectrum (axioms (P4) and (P3)) and the specification of j-radicals. (For example, axiom (P6) says that for each finite set T of height-two maximals, there are infinitely many height-one prime ideals P having $j\text{-rad}(P) = \cap T$.) In what follows we shall attempt to do the same thing for $\operatorname{Spec}(B)$. In this section we consider $j\text{-}\operatorname{Spec}(B)$.

Remark. Suppose that B is an integral domain between $R[x]$ and $R[x, 1/f]$, where the coefficients of f generate the unit ideal in R. Let P be a non-maximal height-one j- prime of B. Then

(1) Each maximal ideal of B of height two must contain a maximal ideal of R.
(2) P contains a maximal ideal of R.

Item (1) is clear by the dimension inequality [M, Theorem 15.5, p. 118]. To see item (2), let $\mathbf{m}_1, \mathbf{m}_2, ...\mathbf{m}_n$ be the maximal ideals of R and put $P = \bigcap M_\alpha$, where $\{M_\alpha\}$ is a set of height-two maximal ideals in B. Then $P \cap R = \bigcap M_\alpha \cap R \supseteq \bigcap_{i=1}^{n} \mathbf{m}_i \neq 0$.

For most of the rest of this section, we suppose that $(R, \mathbf{m})$ is a local one-dimensional Noetherian domain, K its field of fractions, $k = R/\mathbf{m}$, $f, g \in R[x]$ is an $R[x]$-sequence, and

$$A = R[x] \subseteq B = R[x][g/f] \subseteq R[x, 1/f].$$

We will also consider B as $R[x, y]/(fy - g)$, where y is another indeterminate.

Before we proceed, we illustrate our construction by re-examining two examples constructed in [HLW]. In these examples, R is a discrete rank-one valuation domain with maximal ideal $\mathbf{m} = aR$.

Example 1. Let $f = x^2 + a^3$ and $g = x$, so that $B = R[x][x/(x^2 + a^3)]$. Let $P_1 = aB[1/f] \cap B = (a, a^3/(x^2 + a^3))B$ and $P_2 = (a, x)B$. Then P_1 and P_2 are both height-one j-primes, and they are comaximal since

$$1 = \frac{x^2}{x^2 + a^3} + \frac{a^3}{x^2 + a^3} \in P_2 + P_1 \quad .$$

This becomes even clearer using the two-variable description of B:

$$\frac{B}{aB} \cong \frac{k[x, y]}{(x^2 y - x)} = \frac{k[x, y]}{(x(xy - 1))} \ .$$

Thus B/aB has two minimal primes, generated by the images of x and $xy - 1$, obviously comaximal and corresponding to two height-one primes of B which contain

aB. Note that $fR[x] = fK[x] \cap R[x]$ is a prime ideal, because a generates $\mathbf{m}$ and so no fractional root of a can be in K.

Pictorially, j-Spec(B) looks like this for Example 1:

$$\begin{tikzcd}
& \boxed{\infty} & \boxed{\infty} \\
\boxed{\infty} & P_1 = (a, xy-1)B \arrow[u, dash] & P_2 = (a, x)B \arrow[u, dash] \\
& (0) \arrow[ul, dash] \arrow[u, dash] \arrow[ur, dash] &
\end{tikzcd}$$

Example 2. Let $f = x$ and $g = a$, so that $B = A[a/x]$. Let $P_1 = (a/x)B = aB[1/f] \cap B$ and $P_2 = xB$. Then P_1 and P_2 are both height-one j-primes, and they are not comaximal since $(x, a/x)B$ contains both of them. In the other description of B,

$$\frac{B}{aB} \cong \frac{k[x,y]}{(xy)} \quad .$$

Obviously the ideals generated by the images of x and y are not comaximal since they are both contained in the image of (x, y).

Pictorially, j-Spec(B) looks like this for Example 2:

$$\begin{tikzcd}
& \boxed{\infty} & (x, a/x)B & \boxed{\infty} \\
\boxed{\infty} & P_1 = (a/x)B \arrow[u, dash] \arrow[ur, dash] & & P_2 = xB \arrow[u, dash] \arrow[ul, dash] \\
& (0) \arrow[ul, dash] \arrow[u, dash] \arrow[urr, dash] & &
\end{tikzcd}$$

In both of these examples, only one of the nonmaximal height-one j-primes "survives" in $B[1/f]$, namely $P_1 = \mathbf{m}B[1/f] \cap B$.

Definition. The nonmaximal height-one j-primes P of B that *survive* in $B[1/f]$, that is, $PB[1/f] \subsetneq B[1/f]$, will be called *survivors*. Those that do not survive, that is, $PB[1/f] = B[1/f]$, will be called *transients*.

Remark. For any semilocal Noetherian one-dimensional domain R, there exists exactly one survivor j-prime of B contracting to each maximal ideal of R, since the height-one j-primes of $B[1/f] = R[x, 1/f]$ are in one-to-one correspondence with the maximals of R. (Here we are using the fact that f was chosen outside $\bigcup_{i=1}^{n} \mathbf{m}_i R[x]$.)

Questions answered and observations made in (2.2) and (2.3).

(1) For R local, can such a birational extension B be constructed so that more than one maximal ideal contains a given pair of height-one j-primes? Answer "no". (Theorem 2.2.3)

(2) For R local and n a positive integer, there exists such a birational extension B of $R[x]$ with exactly n height-one j-primes. (Corollary 2.3)

(3) At most two height-one j-primes can be contained in a given maximal ideal. If a pair of height-one j-primes is contained in a maximal ideal, one of these will be a survivor, and the other will be a transient j-prime. Every pair of transient j-primes is co-maximal. It is possible to construct a ring so that a specified number of the transient j-primes are co-maximal with a survivor j-prime and another specified number of them are not.

2.1 Notation. Let $(R, \mathbf{m}, k)$ be a local one-dimensional domain, g, f an $R[x]$-sequence and set $B = R[x, g/f] \cong R[x,y]/(fy - g)$, $B/\mathbf{m}B \cong k[x,y]/(\bar{f}y - \bar{g})$, where $\bar{f}, \bar{g}$ are the images of f, g in $k[x]$. Let π denote the natural map $B \to B/\mathbf{m}B$.

In the polynomial ring $k[x,y]$ over the field k, write the factorization of $\bar{f}y - \bar{g}$ into a product of irreducibles in the form

$$\bar{f}y - \bar{g} = p(qy - s) = h_1 \cdot h_2 \cdots h_m \cdot p_1 \cdot p_2 \cdots p_n \cdot (qy - s).$$

where (i) q and s are relatively prime elements of $k[x]$,

(ii) $h_1, h_2, \ldots, h_m, p_1, p_2, \ldots, p_n$ are powers of pairwise relatively prime irreducible elements of $k[x]$,

(iii) $h_1, h_2, \ldots, h_m$ divide a power of q, and

(iv) $p_1, p_2, \ldots, p_n$ are relatively prime to q.

2.2 Theorem. *Let $(R, \mathbf{m}, k), B, f, g, m, n$ etc. be as in 2.1. Then*

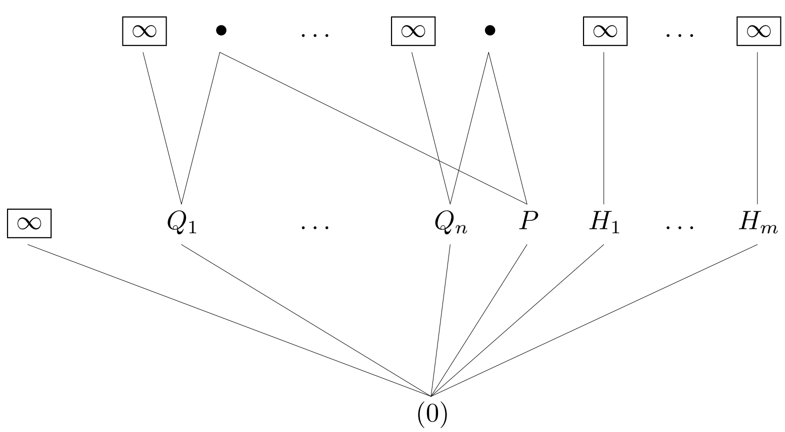
(2.2.1) B has exactly $m+n+1$ non-maximal nonzero j-primes, of which $m+n$ are transient and one is a survivor j-prime P. The transient j-primes are associated to $\pi^{-1}(h_i(x))$ and $\pi^{-1}(p_j(x))$, for $1 \le i \le m$, $1 \le j \le n$, and the survivor j-prime is associated to $\pi^{-1}(q(x)y - s(x))$.

(2.2.2) Furthermore, the transient j-primes $H_1, \ldots, H_m$ corresponding to the factors $h_1(x), \ldots, h_m(x)$ are comaximal with the survivor j- prime, but the transient j-primes $Q_1, \ldots, Q_n$ corresponding to $p_1(x), \ldots, p_n(x)$ are not comaximal with the survivor j-prime.

(2.2.3) Every pair of height-one j-primes of B is contained in at most one maximal ideal of B. If both of the height-one j-primes are transient, then they are comaximal.

(2.2.4) Every maximal ideal of B contains at most two height-one j-primes, at most one of which is transient.

Pictorially, j-Spec(B) looks like this:



Proof. By the Remark at the beginning of this section, every non-maximal height-one j-prime of B contains $\mathbf{m}$ and therefore corresponds to a nonmaximal ideal of

$$B/\mathbf{m}B \cong k[x, y]/(\bar{f}y - \bar{g}) = S.$$

Since S is a Hilbert ring, every prime ideal of S is an intersection of maximal ideals. We conclude that the height-one j-primes of B are in one-to-one correspondence with the minimal primes of S; and hence with the primes of $k[x, y]$ associated to the polynomials $h_1(x), \dots, h_m(x)$, $p_1(x), \dots, p_n(x)$ and $q(x)y - s(x)$. The prime ideal $(q(x)y - s(x))S = \mathbf{p}$ corresponds to a survivor j-prime of B since it survives in the localization $S[1/\bar{f}]$. The prime radicals of the ideals (h_i) and (p_j) correspond to transient j-primes, since they do not survive in $S[1/\bar{f}]$.

(2.2.2) and (2.2.3): Since any two of $h_1, \dots, h_m, p_1, \dots, p_n$ are relatively prime in $k[x]$, any two transient j-primes of B are comaximal. To see which are comaximal with the survivor j-prime, we note that a maximal ideal containing the survivor j-prime corresponds to a maximal ideal in $k[x, y]$ containing $qy - s$, i.e., the set of all elements of $k[x, y]$ that vanish at a point (x_0, y_0) in the affine plane over the algebraic closure of k for which $q(x_0)y_0 - s(x_0) = 0$. Now if $q(x_0)y_0 - s(x_0) = 0$, we cannot have $h_i(x_0) = 0$, for then $q(x_0) = 0$ and hence $s(x_0) = 0$, contradicting the hypothesis that q, s are relatively prime. Thus, h_i is not in any maximal ideal containing $qy - s$, and the corresponding transient j-prime in B is comaximal with the survivor j-prime. But if we choose x_0 so that $p_j(x_0) = 0$, then $q_j(x_0) \neq 0$, and by setting $y_0 = s(x_0)/q(x_0)$ we find a maximal ideal of $k[x, y]$ containing both p_j and $qy - s$; and hence the transient j-prime in B corresponding to p_j is not comaximal with the survivor j-prime. Finally, note that the maximal ideal of $k[x, y]$ containing p_j and $qy - s$ is the same no matter which root x_0 of p_j is used to find it; so the corresponding j-primes of B are contained in only one maximal ideal.

(2.2.4) follows from (2.2.3). $\square$

2.3 Corollary. *For every $m, n \geq 0$, there exists a birational extension B of R of the form above which has exactly $m + n$ transient j-primes and one survivor j-prime, where exactly m of the transient j-primes are comaximal with the survivor j-prime.*

Proof. This is almost obvious since enough irreducibles can always be found to satisfy the hypothesis of 2.2. The one sticky point is whether the polynomials f, g can be chosen to be an $R[x]$-sequence, to insure that $(fy - g)$ is prime. Since $R[x]$ is Cohen-Macaulay, it will suffice to find f, g with $\mathrm{ht}(f, g) = 2$. Start with $f, g \neq 0$ mod $\mathbf{m}$. Then there exist only finitely many primes in $R[x]$ minimal over (g). If $\mathrm{ht}(f, g) = 1$, then $\mathrm{ht}(f, g, a) = 2$, for some $a \in \mathbf{m}$. Now there are infinitely many elements of form $f + a, f + a^2, \dots$. Consider $I = (f + a^i, f + a^j)$, where $i < j$.

Since $a^i(a^{j-i}-1) \in I$ and $a^{j-i}-1$ is a unit, $a^i, f \in I$, so $\mathrm{ht}(I) = 2$. It follows that the height-one primes of $(f + a^i)$ are distinct for different i and so we can choose a^i to avoid the finite number of minimal primes of (g). Thus $(g, f + a^i)$ has height two. $\square$

Remark. It is not hard to see that results extending Theorem 2.2 and Corollary 2.3 can be obtained for R semilocal, with more than one maximal ideal. Since the relations would be more complicated to state, we omit them. There is some discussion of this in section 4.

3. The exactly-less-than set of a singleton is infinite.

Question. For T a finite set of height-two maximal elements of $\mathrm{Spec}(B)$, when is the exactly-less-than set infinite? (That is, when do there exist infinitely many height-one primes contained in exactly those maximals and no others?) If R is Henselian and T contains more than one element, then the exactly less than set of T is empty. For singleton sets T, we prove in this section that the exactly less than set of T is infinite. (This is true both when R is Henselian and when it is not.)

3.1 Theorem. *Suppose R is a one-dimensional semilocal domain with maximal ideals $\mathbf{m}_1, \ldots, \mathbf{m}_n$, x is an indeterminate, $f \in R[x] - \bigcup_{i=1}^n \mathbf{m}_i[x]$, $f, g \in R[x]$ is an $R[x]$-sequence and $B = R[x, g/f]$. Then for each height-two maximal ideal N of B, there exist infinitely many height-one primes P such that N is the only height-two maximal ideal containing P. That is, (P6) of $C\mathbf{Z}(n)P$ or CHP holds for singleton subsets T of the set of all height-two elements of Spec(B).*

We prove three lemmas in order to deduce the theorem. The first result is similar to Lemma 3 in [rW1] which is attributed to Wolmer Vasconcelos.

3.2 Lemma. *Let A be a one-dimensional Noetherian ring. Assume that all but at most finitely many of the height-one prime ideals P of A satisfy the two conditions: (i) P is the radical of a principal ideal, and (ii) A_P is a discrete (rank-one) valuation domain. Then every height-one prime ideal of A is the radical of a principal ideal.*

Proof. Let $Q_1, Q_2, \ldots, Q_n$ be the height-one maximals for which Q_i is not the radical of a principal ideal or A_{Q_i} is not a discrete valuation domain. Pick

$$x \in Q_1 - \left(Q_2 \cup \cdots \cup Q_n \cup \bigcup \{ \text{ height-zero primes of } A\}\right).$$

Then $\mathrm{rad}(xA) = Q_1 \cap P_1 \cap \cdots \cap P_m$, where $P_j = \mathrm{rad}(y_jA)$, for some $y_j \in A$ and A_{P_j} is a discrete valuation domain. Write $y_m = y$; since P_m does not consist of zerodivisors, y is not a zerodivisor. Since A_{P_m} is a discrete valuation domain, there are positive integers h, j such that $x^hA_{P_m} = y^jA_{P_m}$, so $z = x^h/y^j$ is a unit of A_{P_m}; thus, there exists an $s \in A - P_m$ such that $sz \in A$. Also $y^jz \in A$. Therefore $A = (s, y^j) = \{r \mid rz \in A\}$ and so $z \in A$. But since z is a unit in A_{P_m}, $z \notin P_m$. Hence $\mathrm{rad}(zA) = Q_1 \cap P_1 \cap \cdots \cap P_{m-1}$. Repeating this procedure, we get a $t \in A$ such that $\mathrm{rad}(tR) = Q_1$. $\square$

Remarks. Let A be a ring containing a field k. If k' is an algebraic field extension of k, then $A' = A \otimes_k k'$ is an integral extension ring of A that is a free A-module. Let $\{e_i\}$ be a vector space basis for A over k. For any k-automorphism ϕ of k', extend ϕ to an A-automorphism ϕ' of A' by $\phi'(\Sigma r_ie_i) = \Sigma\phi(r_i)e_i$, for $r_i \in k'$. (This is clearly a well-defined k-vector space homomorphism, since $\{e_i\}$ is a k'-basis for A'.) Let $e_je_k = \Sigma a_{jki}e_i$. Then

$$\begin{aligned}\phi'((\Sigma r_je_j)(\Sigma s_ke_k)) &= \phi'(\sum_{j,k} r_js_ke_je_k) \\ &= \phi'(\sum_{j,k} r_js_k\sum_i a_{jki}e_i) \\ &= \phi'(\sum_i(\sum_{j,k} a_{jki}r_js_k)e_i) \\ &= \sum_i(\sum_{j,k} a_{jki}\phi(r_j)\phi(s_k)e_i) \\ &= \phi'(\Sigma r_je_j)\phi'(\Sigma s_ke_k).\end{aligned}$$

Thus ϕ' is an A-automorphism of A'.

If k'/k is an algebraic Galois extension of fields with Galois group G and if G' is the set of automorphisms of A'/A defined by extending the elements of G, then G' is a group of automorphisms of A' having fixed ring A. This follows since an element Σr_ie_i of A' is fixed by every automorphism in G' if and only if each r_i is fixed by every automorphism in G.

Under the assumption that k'/k is Galois, let P be a prime ideal of A for which there exists a prime ideal P' of A' lying over P such that P' is the radical of a principal ideal aA' for some element a of A'; then each conjugate $\phi'(P')$ of P' is the radical of the principal ideal $\phi'(aA')$. Since a has only finitely many distinct conjugates with respect to G', the set $\{\phi'(P') : \phi' \in G'\}$ is finite (even though the group G' may be infinite).

We observe that P is the radical of the principal ideal bA, where b is the product of the distinct conjugates of a: Every minimal prime of bA' is of the form $\phi'(P')$ for some ϕ in G'. Thus we have

$$\mathrm{rad}(bA') = \bigcap\{\phi'(P') \ : \ \phi' \in G'\}.$$

Let Q be a prime in A that contains bA; then there is a prime Q' in A' lying over Q and hence containing bA'. It follows that Q' contains some $\phi'(P')$, and so $P = \phi'(P') \cap A \subseteq Q' \cap A = Q$. Therefore $\mathrm{rad}(bA) = P$.

3.3 Lemma. *Let A be a ring containing a field k. Suppose k^* is an algebraic closure of k, $A^* = A \otimes_k k^*$, and P is a prime ideal of A. If some prime ideal of A^* lying over P is the radical of a principal ideal, then P is the radical of a principal ideal of A.* (Cf. [AEH, (2.13)].)

Proof. Let k' be the separable algebraic closure of k in k^*, let G be the Galois group of k'/k, and set $A' = A \otimes_k k'$. Then A^* is purely inseparable over A'. Let P^* be a prime ideal of A^* lying over P and let $a^* \in A^*$ be such that $P^* = \mathrm{rad}(a^* A^*)$. Then $P' = P^* \cap A'$ is the radical of the principal ideal $(a^*)^{p^e} A'$, where p is the characteristic of k and e is chosen so that $a = (a^*)^{p^e} \in A'$. By the above Remark, it follows that P is the radical of the principal ideal bA, where b is the product of the distinct conjugates of a. $\square$

3.4 Remark. Let R be an arbitrary commutative ring, and let $\mathbf{n}$ be the nilradical of R. If I is a radical ideal of R such that $I/\mathbf{n}$ in $R/\mathbf{n}$ is the radical of an n-generated ideal in $R/\mathbf{n}$, then I is the radical of an n-generated ideal in R. (For example, if $I/\mathbf{n} = \mathrm{rad}((x + \mathbf{n})R/\mathbf{n})$, then $I = \mathrm{rad}(xR)$, since every prime ideal of R contains $\mathbf{n}$.) Thus when considering the condition that each height-one prime of a ring R is the radical of a principal ideal we may assume that R is reduced.

3.5 Lemma. *Let k be a field and let x, y be indeterminates over k. Assume that $p, q, s \in k[x]$ with p, q nonzero and $(q, s)k[x] = k[x]$. Then every maximal ideal in the ring $S = k[x, y]/(p(qy - s))$ is the radical of a principal ideal.*

Proof. Write the irreducible factorization of p in the form (from (2.1))

$$p = h_1 \cdot h_2 \cdots h_m \cdot p_1 \cdot p_2 \cdots p_n,$$

where $h_1, h_2, \dots, h_m, p_1, p_2, \dots, p_n$ are powers of pairwise relatively prime irreducible elements of $k[x]$, $h_1, h_2, \dots, h_m$ divide a power of q, and $p_1, p_2, \dots, p_n$ are relatively prime to q. By Lemma 3.3, we may assume k is algebraically closed (by passing from S to $S \otimes_k k^*$, where k^* is an algebraic closure of k); and then (increasing the number of p_i's and h_i's if necessary) we may assume the p_i's and h_i's are powers of distinct linear factors. Finally, by Remark 3.4, we may assume the p_i's and h_i's *are* linear and $p_i = x - c_i$, where $q(c_i) \neq 0$.

We have that

$$p(qy - s) = h_1 \cdot h_2 \cdots h_m \cdot p_1 \cdot p_2 \cdots p_n \cdot (qy - s).$$

Since $h_i, qy - s$ have no common points, the ideals they generate are comaximal. (If $h_i(a) = 0$, then $q(a) = 0$, which implies that $q(a)b - s(a) = -s(a) \neq 0$, for all points (a, b).) Therefore, by the Chinese Remainder Theorem,

$$S \cong k[x,y]/(h_1) \oplus \cdots \oplus k[x,y]/(h_m) \oplus S_1 \ ,$$
$$\text{where } S_1 = k[x,y]/(p_1 \cdots p_n)(qy - s) \ .$$

Note that if $s = 0$, then q must be a unit; consequently no h_i's occur and $S = S_1$.

Now the maximal ideals in the sum are of the form a maximal ideal in one summand and the unit ideal in the remaining summands, and the first summands, the $k[x,y]/(h_i)$, are all just isomorphic to $k[y]$. Thus once we show that every maximal ideal in S_1 is the radical of a principal ideal, the proof will be complete.

In fact, by Lemma 3.2, it suffices to show that if N is a maximal ideal of S_1 which does *not* correspond to one of the finite number of points $(c_i, s(c_i)/q(c_i))$, for which the curve $qy = s$ meets the line $x = c_i$, then N is the radical of a principal ideal. Note that $(S_1)_N$ is a discrete valuation domain, since N is a simple point of the variety V: $(qy - s)\prod_{i=1}^{n}(x - c_i)$. First take N corresponding to (a, b) on $qy = s$ but not on any $x = c_i$. Then the line $x = a$ meets V only at (a, b), so $N = \operatorname{rad}((x - a)S_1)$. Now take N corresponding to (c_i, b), where $b \neq s(c_i)/q(c_i)$, and set $h = (qy - s)\prod_{j \neq i}(x - c_j)$. Then the variety of $h(x, y) - h(c_i, b)$ meets V only at (c_i, b) (for, the curve $h(x, y) = h(c_i, b)$ meets the line $x = c_i$ only at the point (c_i, b), and for any point off the line $x = c_i$ at which $(qy - s)\prod_{j=1}^{n}(x - c_j)$ vanishes, h also vanishes, but $h(c_i, b) \neq 0$), so $N = \operatorname{rad}((h(x, y) - h(c_i, b))S_1)$. $\square$

Proof of 3.1. Let $\mathbf{m}_1, \ldots, \mathbf{m}_n$ be the maximal ideals of R. Then by the dimension inequality [M, Theorem 15.5, p.118], $N \cap R = \mathbf{m}_i$, for some i. For convenience, let $\mathbf{m} = \mathbf{m}_i$. Now N is the preimage of a maximal ideal in

$$\frac{B}{\mathbf{m}B} = \frac{R[x,y]}{(\mathbf{m}, fy - g)} = \frac{k[x,y]}{(\bar{f}y - \bar{g})},$$

where $k = R/\mathbf{m}$ and overbars denote images mod $\mathbf{m}$. By Lemma 3.5, the image of N is the radical of a principal ideal $(\bar{h})$. Let h be a preimage of $\bar{h}$ and let P be a minimal prime of h contained in N. Then N is the only height-two prime containing P. Moreover, P does not contain $\mathbf{m}$ (because $(\mathbf{m}, h)B$ has height two). Thus $P \cap R = 0$. However, for all $0 \neq a \in \mathbf{m}$, $h + a$ is a preimage of $\bar{h}$ and $h + a \notin P$. (Otherwise, $a \in P$.) Therefore we obtain infinitely many such P in this way. □

4. Axioms for spectra of birational extensions.

4.1 Definition. *Let m, n be nonnegative integers. A partially ordered set U is called countable birational polynomial of type (m, n), or $CB\mathbf{Z}(1)P(m, n)$ provided:*

(P0)–(P3),(P6) Same as in Definition 1.2 (for $C\mathbf{Z}(1)P$).

(P4′) There exist exactly $m + n + 1$ height-one elements u, the “j-elements”, with the property that $\mathrm{G}(u)$ is infinite: the “survivor” u_1 and the “transients”:

$h_1, \ldots, h_m$ *(comaximal to survivor), and*

$p_1, \ldots, p_n$ *(not comaximal to survivor) .*

Also there are exactly n height-two elements distinguished by the property that they are above more than one of the j-elements; denote them by $b_1, \ldots, b_n$ (special maximals).

These elements satisfy: (for all $1 \leq i \neq r \leq m, 1 \leq j \neq t \leq n$)

$$\mathrm{G}(u_1) \cap \mathrm{G}(h_i) = \mathrm{G}(h_i) \cap \mathrm{G}(h_r) = \mathrm{G}(h_i) \cap \mathrm{G}(p_j) = \mathrm{G}(p_j) \cap \mathrm{G}(p_t) = \emptyset$$

$$\mathrm{G}(u_1) \cap \mathrm{G}(p_j) = \{b_j\}\ , \qquad \text{and}$$

$$\mathcal{M}(U) = \mathrm{G}(u_1) \cup \bigcup_{i=1}^{m} \mathrm{G}(h_i) \cup \bigcup_{j=1}^{n} \mathrm{G}(p_j).$$

Note: If $n = 0$, then u_1 is indistinguishable from every h_i (in that there exists an order-isomorphism of U switching them).

4.2 Definition. *A partially ordered set U is called countable Henselian birational polynomial of type (m, n) or $CHP(m, n)$ provided:*

(P0)–(P3),(P4′) Same as in 4.1.

(P6′) For each finite subset T of {height-2 elements of U} of cardinality greater than one, $\mathrm{L_e}(T)$ *is empty. For each singleton $t \in$ {height-2 elements of U},* $\mathrm{L_e}(\{t\})$ *is infinite.*

Remark. These sets of axioms are categorical. That is, if U and V are two partially ordered sets satisfying 4.1, then U is order-isomorphic to V; and similarly for 4.2.

Proof of Remark. We first define $\phi : U \to V$ on all easily identified height-zero and height-one elements of U:

ϕ(the unique minimal element of U) = the unique minimal element of V.

ϕ(the survivor element u_1 of U) = the survivor element v_1 of V.

ϕ(the m transient elements (h_i's) comaximal to u_1 of U) = the m transient elements comaximal to v_1 of V.

ϕ(the n transient elements (p_j's) not comaximal to u_1 of U) = the n transient elements not comaximal to v_1 of V.

ϕ(the countably infinitely many height-one maximal elements of U) = the countably infinitely many height-one maximal elements of V.

Next we define ϕ on the height-two elements:

ϕ(the unique special maximal, b_j of U, for which $u_1 < b_j$ and $p_j < b_j$) = the unique special maximal, c_j, of V for which $v_1 < c_j$ and $\phi(p_j) < c_j$).

The rest of the height-two elements in U and V occur in countably infinite clusters, so we match them:

$\phi(\mathrm{G}(u_1) - \{b_1, \ldots, b_m\}) = \mathrm{G}(\phi(u_1)) - \{\phi(b_1), \ldots, \phi(b_m)\}$.

$\phi(\mathrm{G}(h_i) = \mathrm{G}(\phi(h_i))$, for each transient element h_i comaximal to u_1.

$\phi(\mathrm{G}(p_j) - \{b_1, \ldots, b_m\}) = \mathrm{G}(\phi(p_j)) - \{\phi(b_1), \ldots, \phi(b_m)\}$, for each transient element p_j not comaximal to u_1.

Finally we define $\phi(u)$, for each height-one element u that is not a j-element. The set of all such u is partitioned by the sets $\mathrm{L_e}(T)$ as T varies over all finite subsets (in the case of (4.1)) or all singleton subsets (in (4.2)) of the set of all height-two elements of U. By (P6) and (P6′), these sets $\mathrm{L_e}(T)$ are countably infinite, so we can match them. That is, we define

$\phi(\mathrm{L_e}(T)) = \mathrm{L_e}(\phi(T))$, for each such set T of height-two elements of U.

By inspection, we see that ϕ is a bijection $U \to V$. We claim that $u < w$ in $U \iff \phi(u) < \phi(w)$ in V. Let $u < w$ be elements of U. If $\mathrm{ht}(u) = 0$, then $\mathrm{ht}(\phi(u)) = 0$, so $\phi(u) < \phi(w)$. If $u = u_1$, the survivor, and $w = b_j$, a special maximal, then $\phi(u_1) = v_1 < \phi(b_j)$. If $u = u_1$, the survivor, but $w \neq b_j$, for any special maximal b_j, then $w \in \mathrm{G}(u_1) - \{b_1, \dots, b_m\}$, so $\phi(w) \in \mathrm{G}(\phi(u_1)) - \{\phi(b_1), \dots, \phi(b_m)\}$, which implies $\phi(u_1) = v_1 < \phi(w)$.

The argument for every transient j-transient u of U is similar.

If u is not a j-prime, then $\mathrm{G}(u) = T$ is a finite subset of the height-two maximals and $w \in T, u \in \mathrm{L_e}(T)$. Thus $\phi(u) \in \mathrm{L_e}(\phi(T))$, so $\phi(u) < \phi(w)$. In all cases we have shown that $u < w$ in $U \Rightarrow \phi(u) < \phi(w)$ in V. Since ϕ^{-1} is defined similarly to ϕ, we have $u < w$ in $U \Leftarrow \phi(u) < \phi(w)$ in V. $\square$

4.3 Theorem. *Let $(R, \mathbf{m})$ be a (countable) one-dimensional Henselian local domain, g, f an $R[x]$-sequence and set $B = R[x, g/f] \cong R[x, y]/(fy - g), B/\mathbf{m}B = (R/\mathbf{m})[x, y]/(\bar{f}y - \bar{g})$, where $\bar{f}, \bar{g}$ are the images of f, g in $(R/\mathbf{m})[x]$. Write the factorization of $\bar{f}y - \bar{g}$ into a product of irreducibles as in (2.1). Then* Spec(B) *is $CHP(m, n)$, where m, n are given in (2.1). (If R is not countable, all axioms except (P0) hold.)*

Proof. In 1.5, it was shown that (P0)-(P3) hold. We check (P4$'$) by inspecting Theorem 2.2 and matching the elements appropriately. First denote the survivor prime from 2.2 by u_1. By an abuse of notation, we let the h_i's refer to j-elements of Spec(B) which are comaximal with u_1 as well as to the polynomials which were associated to those prime ideals. Similarly the p_j's stand for j-elements of Spec(B) which are not comaximal with u_1 as well as the polynomials. Assign $b_j, 1 \le j \le n$, to be the maximal elements above p_j and $u_1, 1 \le j \le n$. Then (P4$'$) holds.

To see that the first part of (P6$'$) holds, let T be a finite set of height-two maximals in B for which $\mathrm{L_e}(T) \neq \emptyset$, and take P in $\mathrm{L_e}(T)$. Then P is not a j-prime, so $P \cap R = 0$, and hence B/P is a one-dimensional domain finitely generated over R. By the dimension formula [M, Theorem 15.6, p. 118], B/P is algebraic over R. Since R is a one-dimensional Henselian domain, the integral closure of R in every finite algebraic field extension is a DVR. It follows that every one-dimensional domain algebraic over R is integral over R and has a unique maximal ideal. Therefore B/P

is local, i.e., T is a singleton. The second part of (P6$'$) holds by (3.1) $\square$

We suspect that a similar result holds for non-Henselian local domains, namely that the answer to the following question is "yes":

4.4 Question. *Let R be a countable one-dimensional local non-Henselian domain with maximal ideal $\mathbf{m}$, $f \in R[x] - \mathbf{m}[x]$, g, f an $R[x]$-sequence and set $B = R[x, g/f] \cong R[x,y]/(fy - g)$, $B/\mathbf{m}B = (R/\mathbf{m})[x,y]/(\bar{f}y - \bar{g})$, where $\bar{f}, \bar{g}$ are the images of f, g in $(R/\mathbf{m})[x]$. Write the factorization of $\bar{f}y - \bar{g}$ into a product of irreducibles as in (2.1). Does $\operatorname{Spec}(B)$ satisfy $CB\mathbf{Z}(1)P(m,n)$, where m, n are given in (2.1)? (If R is not countable, do all axioms except (P0) hold?)*

"Proof". It would suffice to prove that $\mathrm{L}_e(T)$ is infinite, for T a finite set of height-two maximal ideals with more than one element. $\square$

There is some evidence that this is true and we now give some examples, making use of results of Roger Wiegand [rW1][rW2]. In particular, he has given a set of axioms characterizing $U \cong \operatorname{Spec}(\mathbf{Z}[x]) \cong \operatorname{Spec}(k[x,y])$, where k is a field contained in the algebraic closure of a finite field:

(W1) U has a unique minimal element.

(W2) U has dimension 2.

(W3) For each element x of height one there are infinitely many elements $y > x$.

(W4) For each pair x, y of distinct elements of height one, there are only finitely many elements z such that $z > x$ and $z > y$.

(W5) Let S be a finite set of height-one elements of U and T a finite set of height-two elements of U. Then there exists a height-one element $w \in U$ such that $\mathrm{G}(s) \cap \mathrm{G}(w) \subseteq T \subseteq \mathrm{G}(w)$, for all $s \in S$.

Remarks. (1) If a partially ordered set U satisfies (W3) and (W4) and U contains infinitely many height-one elements, then for every finite set S of height-one elements of U, there exist infinitely many height-two elements not in $\bigcup\{\mathrm{G}(s) \mid s \in S\}$.

(2) If a partially ordered set U satisfies (W3), (W4) and (W5), then (U contains infinitely many height-one elements and) U satisfies (W) below:

(W) Let S be a finite set of height-one elements of U and T a finite set of height-two elements of U. Then there exist infinitely many height-one elements $w \in U$ such that $\mathrm{G}(s) \cap \mathrm{G}(w) \subseteq T \subseteq \mathrm{G}(w)$, for all $s \in S$.

Proof of Remark 1. Let T be a finite set of height-two elements of U. Suppose that every height-two element of U is in the union

$$H = T \cup \bigcup \{G(s) \mid s \in S\}.$$

Choose a height-one element $u \notin S$. If $\mathrm{G}(u) \subseteq H$, then

$$\mathrm{G}(u) \subseteq (T \cap \mathrm{G}(u)) \cup \bigcup \{\mathrm{G}(s) \cap \mathrm{G}(u) \mid s \in S\} .$$

But (W4) implies that each $\mathrm{G}(u) \cap \mathrm{G}(s)$ is finite, which contradicts (W3) for u.

Proof of Remark 2. It follows from (W5) and (W3) that U has infinitely many height-one elements. (For, the w in (W5) cannot be in S — or else some $\mathrm{G}(s) \cap \mathrm{G}(w)$ is infinite — so $S \cup \{w\}$ gives a new S to which to apply (W5), and so on.) Choose w_1 from (W5), using T and S as given, and set $S_1 = S \cup \{w\}$. Now by the first remark, there exists a new height-two element $t_1 \notin T \cup (\cup \{\mathrm{G}(s) \mid s \in S_1\}$. Let $T_1 = T \cup \{t_1\}$. By (W5), there exists an element w_2 with $\mathrm{G}(s) \cap \mathrm{G}(w_2) \subseteq T_1 \subseteq \mathrm{G}(w_2)$, for all $s \in S_1$. Note that $w_2 \neq w_1$ because $w_1 \not< t_1$, but $w_2 < t_1$. However we do have $T \subseteq T_1 \subseteq \mathrm{G}(w_2)$. Also note that for each $s \in S$, since $s \not< t_1$, we have $\mathrm{G}(s) \cap \mathrm{G}(w_2) \subseteq T_1 \cap \mathrm{G}(s) \subseteq T$. Thus we have produced a second element which satisfies the condition relative to T. Continuing in this way, we can get infintely many elements. $\square$

Roger Wiegand has shown [rW2, Theorem 2] for a field k that the spectrum of a 2-dimensional domain that is finitely generated as a k-algebra satisfies the axioms (W1)-(W5) if and only if k is contained in the algebraic closure of a finite field. In particular this means that (W) is also satisfied by $\mathrm{Spec}(k[x,y][g/f])$, where k is a field contained in the algebraic closure of a finite field, and g, f are a $k[x,y]$-sequence. This provides examples to which the following theorem applies.

4.5 Theorem. *(1) Let $(R, \mathbf{m})$ be a (countable) local one-dimensional domain, x an indeterminate over R, $f \in R[x] - \mathbf{m}[x]$, g, f an $R[x]$-sequence, and consider $B = R[x, g/f] \cong R[x,y]/(fy - g), B/\mathbf{m}B = (R/\mathbf{m})[x,y]/(\bar{f}y - \bar{g})$, where $\bar{f}, \bar{g}$ are* *the images of* f, g *in* $(R/\mathbf{m})[x]$*. Write the factorization of* $\bar{f}y - \bar{g}$ *into a product of irreducibles as in (2.1). Suppose there exists a one-dimensional Noetherian domain* $D \subset R$ *such that* R *is a localization of* D *and such that* $\mathrm{Spec}(D[x, g/f])$ *satisfies (W). Then* $\mathrm{Spec}(B)$ *is* $CB\mathbf{Z}(1)P(m, n)$ *, where* m, n *are given in (2.1). (If* R *is not countable, all axioms except (P0) hold.)*

(2) Let R *be a (countable) semilocal one-dimensional domain with exactly* t *maximal ideals,* x *an indeterminate over* R*,* $f \in R[x]$ *such that the coefficients of* f *generate the unit ideal of* R*,* g, f *an* $R[x]$*-sequence, and set* $B = R[x, g/f] \cong R[x, y]/(fy - g)$*. Suppose there exists a one-dimensional Noetherian domain* $D \subset R$ *such that* R *is a localization of* D *and such that* $\mathrm{Spec}(D[x, g/f])$ *satisfies (W). Then* $\mathrm{Spec}(B)$ *satisfies a generalization of* $CB\mathbf{Z}(1)P(m, n)$*, given below (4.6). (If* R *is not countable, all axioms except (P0) hold.)*

4.6 Definition. *Let* t *be a positive integer and let* $(m_i, n_i)_{i=1}^t$ *be ordered pairs of non-negative integers. A partially ordered set* U *is said to be countable birational polynomial of type* $(m_1, n_1), \ldots, (m_t, n_t)$*, or* $CB\mathbf{Z}(t)P(m_1, n_1), \ldots, (m_t, n_t)$ *provided:*

(P0)–(P3),(P6) hold, from Definition 1.2 .

(P4″) There exist exactly $t + \sum_{i=1}^{t} (m_i + n_i)$ *height-one* j*-elements* u *with the property that* $\mathrm{G}(u)$ *is infinite. Exactly* t *of these* j*-elements will be denoted* $u_1, \ldots, u_t$ *("survivors").*

For each $i, 1 \leq i \leq t$*, there are* $m_i + n_i$ *associated "transient" elements:*

$$h_{i1}, \ldots, h_{im_i} \text{ (comaximal to } u_i\text{)}, \quad \text{and}$$
$$p_{i1}, \ldots, p_{in_i} \text{ (not comaximal to } u_i\text{)}.$$

There are exactly $n_1 + \cdots + n_t$ *special height-two maximal elements, distinguished by the property that they are above more than one of the* j*-elements; these are denoted by* $b_{ij}, 1 \leq i \leq t, 1 \leq j \leq n_i$*.*

These elements satisfy:

$$\mathrm{G}(u_i) \cap \mathrm{G}(p_{ij}) = \{b_{ij}\} \ ;$$

$$\mathrm{G}(u) \cap \mathrm{G}(w) = \emptyset \textit{ for every pair } (u, w) \textit{ of distinct}$$

$$j\textit{-elements } \{u, w\} \neq \{u_i, p_{ij}\} \textit{ for all } i, j \textit{ ; and}$$

$$\mathcal{M}(U) = \bigcup_{i=1}^{t} [\mathrm{G}(u_i) \cup \bigcup_{j=1}^{m_i} \mathrm{G}(h_{ij}) \cup \bigcup_{j=1}^{n_i} \mathrm{G}(p_{ij})].$$

Proof of 4.5. Let $\mathbf{m}_1, \mathbf{m}_2, \ldots \mathbf{m}_t$ be the maximal ideals of R. If $P \in \mathrm{Spec}(B)$ is such that $G(P)$ is infinite, then $P \cap R = \mathbf{m}_i$ for some i. Using $\mathbf{m}$ for $\mathbf{m}_i$, we see that $\mathrm{Spec}(R_{\mathbf{m}}[x, g/f])$ satisfies (P4$'$) and it follows that $\mathrm{Spec}(B)$ satisfies (P4$''$), with the appropriate assignment of elements.

We check that (P6) holds. For this, let T be a finite nonempty set of height-two maximals of $\mathrm{Spec}(B)$, and let S be the (finite) set of all height-one nonmaximal j-primes of $\mathrm{Spec}(B)$. Let $C = D[x, g/f]$, and let Ψ be the localization map $\mathrm{Spec}(B) \to \mathrm{Spec}(C)$. Then $\Psi(T)$ is a finite set of height-two maximals of $\mathrm{Spec}(C)$ and $\Psi(S)$ is a finite set of height-one primes of $\mathrm{Spec}(C)$. By (W), there exist infinitely many elements $w \in \mathrm{Spec}(C)$ with $\mathrm{G}_C(s) \cap \mathrm{G}_C(w) \subseteq \Psi(T) \subseteq \mathrm{G}_C(w)$, for all $s \in \Psi(S)$. (Here $\mathrm{G}_C(u)$ means $\{v \in \mathrm{Spec}(C) \mid v > u\}$.) Now each $w = \Psi(w')$ for some $w' \in \mathrm{Spec}(B)$, since $\mathrm{G}_C(w) \cap \Psi(T) \neq \emptyset$. It follows that $\mathrm{G}_B(w') \supseteq T$. (Clearly $\Psi(\mathrm{G}_B(w')) = \mathrm{G}_C(w) \cap \Psi(\mathrm{Spec}(B))$.) Suppose that $v \in \mathrm{G}_B(w')$. Since v is a height-two maximal of $\mathrm{Spec}(B)$, using (P4$'$) or (P4$''$), we have $v \in \cup\{\mathrm{G}_B(s) | s \in S\}$. But then $v \in \mathrm{G}_B(w') \cap \mathrm{G}_B(s)$ for some s in S, and so $\Psi(v) \in \mathrm{G}_C(w) \cap \mathrm{G}_C(\Psi(s)) \subseteq \Psi(T)$. Since Ψ is one-to-one, we must have $v \in T$. □

We believe that a general semilocal version of Question(4.4) may also be true.

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