# BUILDING NOETHERIAN AND NON-NOETHERIAN INTEGRAL DOMAINS USING POWER SERIES

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Dedicated to Jim Huckaba on the occasion of his retirement

In this mainly expository article we describe a technique, dating back at least to the 1930s, which uses power series, homomorphic images and intersections involving a Noetherian integral domain $R$ and a homomorphic image $S$ of a power series ring extension of $R$ to obtain a new integral domain $A$. Here $A$ has the form $A := L \cap S$, where $L$ is a field between the fraction field of $R$ and the total quotient ring of $S$. We give in certain circumstances necessary and sufficient conditions for $A$ to be computable as a nested union of subrings of a specific form. We also prove that the Noetherian property for the associated nested union is equivalent to a flatness condition. We present several examples where this flatness condition holds, and other examples where it fails to hold. In the first case this produces a Noetherian integral domain and in the second case a non-Noetherian domain.

## 1. Introduction.

Over the past sixty years, important examples of Noetherian integral domains have been constructed using power series, homomorphic images and intersections. The basic idea is to start with a typical Noetherian integral domain $R$ such as a polynomial ring in several indeterminates over a field and to look for unusual Noetherian and non-Noetherian extension rings inside a homomorphic image $S$ of an ideal-adic completion $R^*$ of $R$. The constructed ring $A$ has the form $A := L \cap S$, where $L$ is a field between the fraction field of $R$ and the total quotient ring of $S$. (The elements of $R^*$ are power series with coefficients in $R$.)

Several of our objectives are:

(1) To construct new examples of nontrivial Noetherian and non-Noetherian integral domains; this continues a tradition going back to Akizuki in the 1930s and Nagata in the 1950s,

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(2) To study birational extensions (i.e., extensions inside the field of fractions) of a Noetherian local domain $R$; this is related to the work of Zariski going back to the 1930s and 1940s on the problem of local uniformization along a valuation domain birationally dominating $R$,

(3) To consider the generic fiber[1] of the map $R \to R^*$, where $R^*$ is an ideal-adic completion of the Noetherian domain $R$, and investigate connections between this fiber and birational extensions of $R$.

These objectives form a complete circle, since (3) is used to accomplish (1).

The development of this technique to create new rings from well-known ones goes back to the work of Akizuki in [A] and Nagata in [N1] and has been continued by Ferrand-Raynaud[FR], Rotthaus [R1],[R2], Ogoma [O1], [O2], Brodmann-Rotthaus [BR1], [BR2], Heitmann [H1], Weston [W], and the authors [HRW1], ..., [HRW5] to produce a wide variety of Noetherian rings. In the work of Akizuki, Nagata and Rotthaus (and indeed in most of the papers cited above) the description of the constructed ring $A$ as an intersection is not explicitly stated. Instead $A$ is defined as a direct limit or nested union of subrings.

The fact that in certain circumstances the intersection domain $A$ is computable as a nested union is important to the development of this technique; if this holds one may view $A$ from two different perspectives. In general there is a natural direct limit domain $B$ associated with $A$. We examine conditions for $A$ to be equal to $B$. This motivates our formulation of limit-intersecting properties in [HRW2, (2.5)] and [HRW3, (5.1)](see Section 5.2).

A primary goal of our study is to determine for a given $R$, $S$ and $L$ whether $A := L \cap S$ is Noetherian. An important observation related to this goal is that the Noetherian property for the associated direct limit ring $B$ is equivalent to a flatness condition [HRW2, Theorem 2.12], [HRW5, Theorem 3.2] (see Section 4.5).

It took about a page for Nagata [N2, page 210] to establish the Noetherian property of Example 3.1, and the original proof of the Noetherian property of the example of Rotthaus described in Example 3.3 is about 7 pages [R1, pages 112-118]. As noted in [HRW2, Remark 3.4], the Noetherian results of [HRW2] and [HRW5] described in Section 4.5 gives the Noetherian property in these examples

[1]The *generic fiber* is the fiber over the prime ideal (0) of $R$. Thus if $U = R - (0)$, then the generic fiber of the map $R \to R^*$ is the ring $U^{-1}R^*$.

more quickly.

## 2. Elementary examples.

We begin by illustrating the construction with several examples. In these examples $R$ is a polynomial ring over the field $\mathbb{Q}$ of rational numbers.

In the one-dimensional case the situation is fairly well understood:

**Example 2.1.** Let $y$ be a variable over $\mathbb{Q}$, let $R := \mathbb{Q}[y]$, and let $L$ be a subfield of the field of fractions $\mathcal{Q}(\mathbb{Q}[[y]])$ of $\mathbb{Q}[[y]]$ such that $\mathbb{Q}(y) \subseteq L$. Then the intersection domain $A = L \cap \mathbb{Q}[[y]]$ is a rank-one discrete valuation domain (DVR) with $y$-adic completion $A^* = \mathbb{Q}[[y]]$. For example, if we work with our favorite transcendental function and put $L = \mathbb{Q}(y, e^y)$, then $A$ is a DVR having residue field $\mathbb{Q}$ and field of fractions $L$ of transcendence degree 2 over $\mathbb{Q}$.

The integral domain $A$ of Example 2.1 is perhaps the simplest example of a local Noetherian domain on an algebraic function field $L/\mathbb{Q}$ of two variables that is not essentially finitely generated over $\mathbb{Q}$, i.e., $A$ is not the localization of a finitely generated $\mathbb{Q}$-algebra. However $A$ does have a nice description as an infinite nested union of localized polynomial rings in 2 variables over $\mathbb{Q}$. Thus in a certain sense there is a good description of the elements of the intersection domain $A$ in this case.

The two-dimensional (two variable) case is more interesting. The following theorem of Valabrega [V] is useful in considering this case.

**Theorem.** *Let $C$ be a DVR, let $y$ be an indeterminate over $C$, and let $L$ be a subfield of $\mathcal{Q}(C[[y]])$ such that $\mathcal{Q}(C)(y) \subseteq L$. Then the integral domain $D = L \cap C[[y]]$ is a two-dimensional regular local domain having completion[2] $\widehat{D} = \widehat{C}[[y]]$, where $\widehat{C}$ is the completion of $C$.*

Applying Valabrega's theorem, we see that the intersection domain is a two-dimensional regular local domain with the "right" completion in the following two examples:

**Example 2.2.** Let $x$ and $y$ be indeterminates over $\mathbb{Q}$ and let $C = \mathbb{Q}(x, e^x) \cap \mathbb{Q}[[x]]$. Then $A_1 := \mathbb{Q}(x, e^x, y) \cap C[[y]]$ is a two-dimensional regular local domain having completion $\mathbb{Q}[[x, y]]$.

[2]By the *completion* of a local ring, we mean the ideal-adic completion with respect to the powers of its maximal ideal.

**Example 2.3.** Let $x$ and $y$ be indeterminates over $\mathbb{Q}$ and let $C = \mathbb{Q}(x, e^x) \cap \mathbb{Q}[[x]]$ as in Example 2.2. Then $A_2 := \mathbb{Q}(x, y, e^x, e^y) \cap C[[y]]$ is a two dimensional regular local domain having completion $\mathbb{Q}[[x, y]]$.

There is a significant difference, however, between the integral domains $A_1$ of Example 2.2 and $A_2$ of Example 2.3. As is shown in [HRW3, Section 2], the 2-dimensional regular local domain $A_1$ of Example 2.2 is, in a natural way, a nested union of 3-dimensional regular local domains. It is possible therefore to describe $A_1$ rather explicitly. On the other hand, the 2-dimensional regular local domain $A_2$ of Example 2.3 contains, for example, the element $(e^x - e^y)/(x - y)$. As discussed in [HRW3, Section 2], the associated nested union domain $B$ naturally associated with $A_2$ is 3-dimensional and non-Noetherian.

**Remark 2.4.** It is shown in [HRW1, Theorem 3.9] that if we go outside the range of Valabrega's theorem, that is, if we take more general subfields $L$ of $\mathcal{Q}(\mathbb{Q}[[x, y]])$ such that $\mathbb{Q}(x, y) \subseteq L$, then the intersection domain $A = L \cap \mathbb{Q}[[x, y]]$ can be, depending on $L$, a localized polynomial ring in $n \geq 3$ variables over $\mathbb{Q}$ or even a localized polynomial ring in infinitely many variables over $\mathbb{Q}$. In particular $A = L \cap \mathbb{Q}[[x, y]]$ need not be Noetherian.

## 3. Historical examples.

A Noetherian local domain $R$ is said to be *analytically irreducible* if its completion $\widehat{R}$ is again an integral domain. Related to singularities of algebraic curves, there are classical examples of one-dimensional local Notherian domains $R$ such that $\widehat{R}$ is not an integral domain, i.e., $R$ is analytically reducible. For example, let $X$ and $Y$ be variables over $\mathbb{Q}$ and let $R = \mathbb{Q}[X, Y]_{(X,Y)}/(X^2 - Y^2 - Y^3)$. Then $R$ is a one-dimensional Noetherian local domain since the polynomial $X^2 - Y^2 - Y^3$ is irreducible in the polynomial ring $\mathbb{Q}[X, Y]$. Let $x$ and $y$ denote the images in $R$ of $X$ and $Y$, respectively. Then $\widehat{R}$ is also the $y$-adic completion of $R$ and is equal to $R^* = \mathbb{Q}[X][[Y]]/(X^2 - Y^2((1 + Y))$. Since $(1 + Y)^{1/2} \in \mathbb{Q}[[Y]]$, we see that $X^2 - Y^2(1 + Y)$ factors in $\mathbb{Q}[X][[Y]]$ as $(X - Y(1 + Y)^{1/2}) \cdot (X + Y(1 + Y)^{1/2})$. Thus $R^*$ is not an integral domain.

In this example, the integral domain $R$ is not *normal* or equivalently, *integrally closed*. That is, there are monic polynomials in the polynomial ring $R[Z]$ that have roots in the fraction field of $R$ that are not in $R$. For example, the polynomial

$Z^2 - (1+y) \in R[Z]$ has $x/y$ as a root.

If $R$ is a normal one-dimensional Noetherian local domain, then $R$ is a rank-one discrete valuation domain (DVR) and it is well-known that the completion of $R$ is again a DVR. Thus $R$ is analytically irreducible.

Zariski showed (cf. [ZS, pages 313-320]) that the normal Noetherian local domains that occur in algebraic geometry are *analytically normal*, i.e., the completion of such a domain is again a normal domain. In particular, the normal local domains occuring in algebraic geometry are analytically irreducible.

This motivated the question of whether there exists a normal Noetherian local domain for which the completion is not a domain. Nagata produced such examples. He also pinpointed sufficient conditions for a normal Noetherian local domain to be analytically irreducible [N, (37.8)].

In Example 3.1, we present a special case of a construction of Nagata [N2, Example 7, pages 209-211] of a 2-dimensional normal Noetherian local domain that is analytically reducible.

**Example 3.1 (Nagata).** Let $x$ and $y$ be algebraically independent over $\mathbb{Q}$ and let $R$ be the localized polynomial ring $R = \mathbb{Q}[x,y]_{(x,y)}$. Then the completion of $R$ is $\widehat{R} = \mathbb{Q}[[x,y]]$. Let $\alpha \in y\mathbb{Q}[[y]]$ be an element that is transcendental over $\mathbb{Q}(x,y)$, e.g., $\alpha = e^y - 1$. Let $\rho = x + \alpha$. Now define[3] $A := \mathbb{Q}(x,y,\rho^2) \cap \mathbb{Q}[[x,y]]$. Then $A$ is Noetherian (in fact a 2-dimensional regular local domain[4]). Moreover $\rho^2$ is a prime element of $A$, so if $D := (A[z]/(z^2-\rho^2))_{(x,y,z)}$, then $D$ is an integral domain. As is shown by Nagata, $D$ is, in fact, a normal Noetherian local domain. The element $z^2$, however, factors as a square in $\widehat{D}$: $z^2 = (x+\alpha)^2$ in $\widehat{D}$. Thus $D$ has completion $\widehat{D} = \mathbb{Q}[[x,y,z]]/(z-(x+\alpha))(z+(x+\alpha))$ which is not an integral domain.

**Remark 3.2.** The two-dimensional regular local domain $A$ of Example 3.1 has a principal prime ideal $\rho^2 A$ that factors in $\widehat{A} = \mathbb{Q}[[x,y]]$ as the square of the prime element $\rho$ of $\widehat{A}$. Therefore the map $A \to \widehat{A} = \mathbb{Q}[[x,y]]$ is not a regular morphism.[5]

[3] The original definition by Nagata is in terms of a nested union of subrings. He then proves that this nested union is Noetherian with completion $\mathbb{Q}[[x,y]]$. It then follows that the nested union is an intersection as defined here.

[4] This example constructed by Nagata (historically) is the first occurence of a 2-dimensional regular local domain containing a field of characteristic zero that fails to be pseudo-geometric. As such, the example fails to satisfy one of the conditions in the definition of an excellent ring. For the definition and information on excellent rings see [M1, Chapter 13], [M2, Section 32] and [R4].

[5] A homomorphism $\phi : S \to T$ of Noetherian rings is said to be *regular* if it is flat with geometrically regular fibers [M2, page 255].

The existence of examples such as the normal Noetherian local domain $D$ of Example 3.1 naturally motivated the question: Is a *pseudo-geometric domain* (in the terminology of Grothendieck, a *universally Japanese domain*; in that of Matsumura, a *Nagata domain*) necessarily excellent? It was shown by Rotthaus in [R1] that pseudo-geometric domains need not be excellent.

In Example 3.3, we present a special case of the construction of Rotthaus [R1] of a 3-dimensional regular local domain $A$ such that the formal fibers of $A$ are geometrically reduced, but are not geometrically regular. The integral domain $A$ is pseudo-geometric but is not excellent.

**Example 3.3 (Rotthaus).** Let $x, y, z$ be algebraically independent over $\mathbb{Q}$ and let $R$ be the localized polynomial ring $R = \mathbb{Q}[x, y, z]_{(x,y,z)}$. Let $\tau_1 = \sum_{i=1}^{\infty} a_i y^i \in \mathbb{Q}[[y]]$ and $\tau_2 = \sum_{i=1}^{\infty} b_i y^i \in \mathbb{Q}[[y]]$ be power series such that $y, \tau_1, \tau_2$ are algebraically independent over $\mathbb{Q}$, for example, $\tau_1 = e^y - 1$ and $\tau_2 = e^{y^2} - 1$. Let $u := x + \tau_1$ and $v := z + \tau_2$. Define[6] $A := \mathbb{Q}(x, y, z, uv) \cap (\mathbb{Q}[x, z]_{(x,z))}[[y]])$. It is shown in [R1] that $A$ is Noetherian and that the completion of $A$ is $\widehat{A} = \mathbb{Q}[[x, y, z]]$, so $A$ is a 3-dimensional regular local domain. Since $u, v$ are part of a regular system of parameters of $\widehat{A}$, it is clear that $(u, v)\widehat{A}$ is a prime ideal of height two. It is shown in [R1], that $(u, v)\widehat{A} \cap A = uvA$. Thus $uvA$ is a prime ideal and $\widehat{A}_{(u,v)\widehat{A}}/uv\widehat{A}_{(u,v)\widehat{A}}$ is a non-regular formal fiber of $A$. Therefore $A$ is not excellent.

## 4. Constructions, pictures and Noetherian results.

Let $R$ be a Noetherian integral domain and let $a \in R$ be a nonzero nonunit. Then the $a$-adic completion of $R$ is the ring $R^* := R[[y]]/(y - a)$ [N2, (17.5)]. Thus the elements of $R^*$ are power series in $a$ with coefficients in $R$, but without the uniqueness of expression as power series that occurs in the formal power series ring $R[[y]]$.

We usually reserve the notation $\widehat{R}$ for the situation where $R$ is a local ring with maximal ideal $\mathbf{m}$ and $\widehat{R}$ is the $\mathbf{m}$-adic completion of $R$. If $\mathbf{m}$ is generated by elements $a_1, \ldots, a_n$, then $\widehat{R}$ is realizable by taking the $a_1$-adic completion $R_1^*$ of $R$, then the $a_2$-adic completion $R_2^*$ of $R_1^*$, $\ldots$, and then the $a_n$-adic completion of $R_{n-1}^*$.

[6]In [R1] the example is constructed as a direct limit. The fact that it is Noetherian implies it is also this intersection.

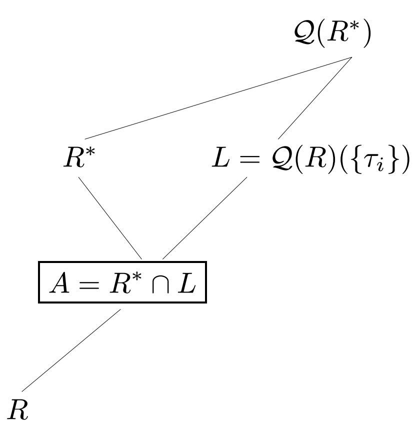
Given a Noetherian domain $R$ and a nonzero nonunit $a \in R$, there are two forms of the construction associated with the $a$-adic completion $R^*$ of $R$. Thus there are two methods for the construction.

**Method 4.1.** Suppose $\tau_1, \dots, \tau_s \in aR^*$ are algebraically independent over the fraction field $\mathcal{Q}(R)$ of $R$. Let $L = \mathcal{Q}(R)(\tau_1, \dots, \tau_s)$, and define $A := L \cap R^*$.

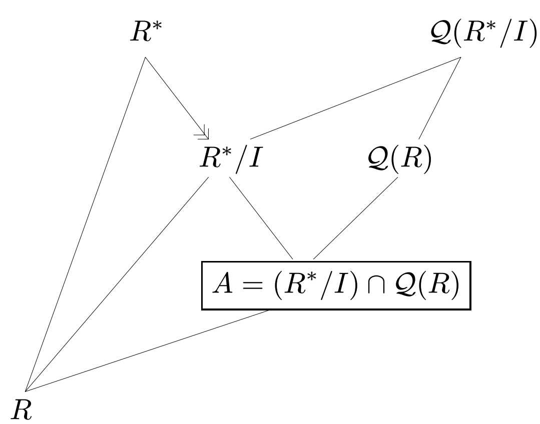
**Method 4.2.** Suppose $I$ is an ideal of $R^*$ having the property that $P \cap R = (0)$ for each $P \in \operatorname{Spec} R^*$ that is associated to $I$. Define $A := \mathcal{Q}(R) \cap (R^*/I)$.

The condition in (4.2), that $P \cap R = (0)$ for every prime ideal $P$ of $R^*$ that is associated to $I$ implies that the fraction field $\mathcal{Q}(R)$ of $R$ embeds in the total quotient ring $\mathcal{Q}(R^*/I)$ of $R^*/I$. For then $R \to R^*/I$ is an injection and regular elements of $R$ remain regular as elements of $R^*/I$.

**Pictures.** Diagrams for these constructions are drawn below.




(4.1) $A := L \cap R^*$




(4.2) $A := \mathcal{Q}(R) \cap (R^*/I)$

**Remark 4.3.** The papers [HRW2] and [HRW3] feature the construction described in (4.1) which realizes the intersection domain $A := \mathcal{Q}(R)(\tau_1, \dots, \tau_s) \cap R^*$. The construction given in (4.2) includes that given in (4.1) as a special case. To see this, let $R, a$ and $R^*$ be as in (4.1). Let $t_1, \dots, t_s$ be indeterminates over $R$, define $S := R[t_1, \dots, t_s]$, let $S^*$ be the $a$-adic completion of $S$ and let $I$ denote the ideal $(t_1 - \tau_1, \dots, t_s - \tau_s)R^*$. Consider the following diagram where $\lambda$ is the $R$-algebra isomomorphism that maps $t_i \to \tau_i$ for $i = 1, \dots, s$.

(4.4)

$$\begin{array}{ccccccc} & & S := R[t_1,\dots,t_s] & \longrightarrow & D := \mathcal{Q}(R)(t_1,\dots,t_s)\cap(S^*/I) & \longrightarrow & S^*/I \\ & & \lambda\big\downarrow & & \lambda\big\downarrow & & \lambda\big\downarrow \\ R & \longrightarrow & R[\tau_1,\dots,\tau_s] & \longrightarrow & A := \mathcal{Q}(R)(\tau_1,\dots,\tau_s)\cap R^* & \longrightarrow & R^*. \end{array}$$

Since $\lambda$ maps $D$ isomorphically onto $A$, we see that the construction given in (4.2) includes as a special case that of (4.1).

**4.5. Noetherian results.** Suppose $R$ is a Noetherian local domain and let the notation be as in (4.1). We prove in [HRW2, Theorem 2.12]:

**Theorem 4.5.1.** *The canonical map $R[\tau_1,\dots,\tau_s]\to R^*[1/a]$ is flat if and only if $A$ is Noetherian and is a nested union of localized polynomial rings in $s$ variables over $R$ as defined in Section 5.2 .*

Later, in [HRW5], we prove an analogous result in the more general setting of (4.2). With notation as in (4.2), we prove in [HRW5, Theorem 3.2]:

**Theorem 4.5.2.** *The canonical map $R\to(R^*/I)[1/a]$ is flat if and only if $A$ is simultaneously Noetherian and a localization of a subring of $R[1/a]$.*

The proof of the more general result in the setting of (4.2) is actually shorter and more direct than the earlier proof in [HRW2]. In [HRW5] and [HRW6] examples are constructed of the form (4.2) that cannot be realized by means of (4.1).

## 5. Flatness, approximations and universality.

**5.1. Flatness.** The concept of flatness was introduced by Serre in the 1950's in an appendix to his paper [S]. Mumford writes in [Mu, page 424]: "The concept of flatness is a riddle that comes out of algebra, but which technically is the answer to many prayers." An $R$-module $M$ is *flat* over $R$ if tensoring with $M$ preserves exactness of every exact sequence of $R$-modules. Equivalently, $M$ is flat over $R$ if for every $m_1,\dots,m_n\in M$ and $a_1,\dots,a_n\in R$ such that $\sum a_im_i=0$, there exist for some integer $k$ elements $b_{ij}\in R$ and elements $m_1',\dots,m_k'\in M$ such that $m_i=\sum_{j=1}^k b_{ij}m_j'$ for each $i$ and $\sum_{i=1}^n a_ib_{ij}=0$ for each $j$. A finitely generated module over a local ring is flat if and only if it is free [M1, Proposition 3.G]. If $S$ is obtained as a localizaton of $R$, then $S$ is flat as an $R$-module [M1, (3.D)]. If $I$ is an ideal of a Noetherian ring $R$, then the $I$-adic completion of $R$ is flat over $R$ [M1, Corollary 1, page 170]. Thus flatness is a property that holds for several standard constructions for extensions of a Noetherian ring.

An important fact for the Noetherian results described in Section 4.5 is that if $\phi : C \to D$ is a flat homomorphism of rings, i.e., $D$ is a flat $C$-module, then $\phi$ satisfies the going-down theorem [M1, (5.D)]. This implies that for each $P \in \operatorname{Spec} D$ the height of $P$ in $D$ is greater than or equal to the height of $\phi^{-1}(P)$ in $C$.

**5.2. Approximations.** Let $R$ be a Noetherian integral domain with field of fractions $K$, let $a \in R$ be a nonzero nonunit, and let $R^*$ denote the $a$-adic completion of $R$. Associated with the constructions described in (4.1) and (4.2) there are subrings of the intersection domain $A$ which approximate $A$.

In (4.1), the elements $\tau_1, \dots \tau_s \in aR^*$ are algebraically independent over $\mathcal{Q}(R)$. Hence $U_0 := R[\tau_1, \dots, \tau_s] \subseteq A$ is a polynomial ring in $s$ variables over $R$. Each $\tau_i \in aR^*$ has a representation $\tau_i = \sum_{j=1}^{\infty} r_{ij}a^j$, where the $r_{ij} \in R$. For each positive integer $n$, we associate with this representation of $\tau_i$ the $n$-th *endpiece*, $\tau_{in} = \sum_{j=n+1}^{\infty} r_{ij}a^{j-n}$. Then $U_n := R[\tau_{1n}, \dots, \tau_{sn}]$ is a polynomial ring in $s$ variables over $R$, and for each $n$ we have a birational inclusion of polynomial rings $U_n \subseteq U_{n+1}$. Since $a$ is in the Jacobson radical of $R^*$ [M1, (24,B)], the localization $B_n := (1 + aU_n)^{-1}U_n$ of $U_n$ is also a subring of $A$. We define $U := \cup_{n=1}^{\infty} U_n$ and $B = \cup_{n=1}^{\infty} B_n$, and we say the construction (4.1) is *limit-intersecting* if $B = A$.

The limit-intersecting property depends on the choice of the elements $\tau_1, \dots, \tau_s \in aR^*$. For example, if $R$ is the localized polynomial ring $k[a, y]_{(a,y)}$, $s = 1$ and $U_0 = R[\tau_1]$, and if we define $U_0' := R[y\tau_1,]$, then $\mathcal{Q}(U_0) = \mathcal{Q}(U_0')$, so the intersecton domain $A = \mathcal{Q}(U_0) \cap R^* = \mathcal{Q}(U_0') \cap R^*$. However the approximation domain $B'$ associated to $U_0'$ is properly contained in the approximation domain $B$ associated to $U_0$. Therefore $B' \subsetneq B \subseteq A$ and the limit-intersecting property fails for the element $y\tau_1$.

For the construction (4.2), one no longer has an approximation of $A$ by a nested union of polynomial rings over $R$. Indeed, in (4.2) the extension $R \subseteq A$ is birational. However, there is an analogous approximation. We are given an ideal $I$ of $R^*$ with the property that $P \cap R = (0)$ for each $P \in \operatorname{Ass}(R^*/I)$. Let $I := (\sigma_1, \dots, \sigma_t)R^*$, where each $\sigma_i := \sum_{j=0}^{\infty} a_{ij}a^j$, and the $a_{ij} \in R$. We define $\sigma_{in}$, the $n^{\text{th}}$ *frontpiece* for $\sigma_i$, to be

$$\sigma_{in} := \sum_{j=0}^{n} (a_{ij}a^j)/a^n.$$

As an element of the total quotient ring of $R^*/I$, it is observed in [HRW5] that the frontpiece $\sigma_{in}$ is the negative of the $n^{\text{th}}$ *endpiece* of $\sigma_i$ as defined in [HRW2, (2.1)]; that is,

$$-\sigma_{in} = \sum_{j=n+1}^{\infty} (a_{ij}a^j)/a^n = \sum_{j=n+1}^{\infty} a_{ij}a^{j-n} \qquad (\text{mod } I).$$

It follows that $\sigma_{in} \in K \cap (R^*/I)$.

We define

$$U_n := R[\sigma_{1n}, \dots, \sigma_{tn}], \qquad \text{and } B_n := (1 + aU_n)^{-1} R[\sigma_{1n}, \dots, \sigma_{tn}]_{(\sigma_{1n}, \dots, \sigma_{tn})},$$

where these rings are considered to be subrings of $R^*/I$.

Now $\sigma_{in} = -aa_{i,n+1} + a\sigma_{i,n+1}$, and so $R \subseteq U_0 \subseteq \cdots U_n \subseteq U_{n+1}$ and $B_n \subseteq B_{n+1}$. Set

$$U := \cup_{n=1}^{\infty} U_n, \qquad B := \cup_{n=1}^{\infty} B_n = (1 + aU)^{-1}U, \qquad \text{and } A := K \cap (R^*/I).$$

Again the fact that $a$ is in the Jacobson radical of $R^*$ implies that $B \subseteq A$. We say the construction (4.2) is *limit-intersecting* if $B = A$.

**Remark 5.3.** The following results about the nested union approximation of an integral domain $A$ constructed as in (4.2) are given in [HRW5]

(1) The definitions of $B$ and $U$ are independent of the choice of generators for $I$, and the representation of the generators $\sigma_i$ of $I$ as power series in $a$.
(2) $a(R^*/I) \cap A = aA, \qquad a(R^*/I) \cap U = aU, \qquad a(R^*/I) \cap B = aB.$
(3) $U/a^nU = B/a^nB = A/a^nA = R^*/((a^n) + I)$. All the rings $A$, $U$, $B$ have the same $a$-adic completion, that is, $A^* = U^* = B^* = R^*/I$.
(4) $R_a = U_a$, $U = R_a \cap B = R_a \cap A$ and the fraction fields of $R$, $U$, $B$ and $A$ are all equal to $K$.
(5) The rings $U = R_a \cap (R^*/I)$ and $B = (1 + aU)^{-1}U$ are uniquely determined by $a$ and the ideal $I$ of $R^*$.
(6) If $b \in B$ is a unit of $A$, then $b$ is already a unit of $B$.
(7) We have the following diagram displaying the relationships among the rings. Recall that $B = (1 + aU)^{-1}U$.

$$\begin{array}{ccccccccc}
\mathcal{Q}(R) & = & \mathcal{Q}(U) & = & \mathcal{Q}(B) & = & \mathcal{Q}(A) & \xrightarrow{\subseteq} & \mathcal{Q}(R^*/I) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
R[1/a] & = & U[1/a] & \xrightarrow{\subseteq} & B[1/a] & \xrightarrow{\subseteq} & A[1/a] & \xrightarrow{\subseteq} & (R^*/I)[1/a] \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
R & \xrightarrow{\subseteq} & U = \cup U_n & \xrightarrow{\subseteq} & B & \xrightarrow{\subseteq} & A & \xrightarrow{\subseteq} & R^*/I.
\end{array}$$

In connection with the flatness property, if $U_0 := R[\tau_1, \dots, \tau_s] \hookrightarrow R^*[1/a]$ is flat, then for each $P \in \operatorname{Spec} R^*[1/a]$ one has that $\operatorname{ht} P \geq \operatorname{ht}(P \cap U_0)$. It is shown in [HRW6, Theorem 2.2] that conversely this height inequality in certain contexts implies flatness.

**5.4. Universality.** Let $k$ be a field and let $L/k$ be a finitely generated field extension. A general question one may ask with regard to $L/k$ is how to describe those Noetherian local integral domains $(A, \mathbf{n})$ such that $k \subseteq A \subseteq L$, $A$ has fraction field $L$ and $k$ is a coefficient field for $A$, i.e., the canonical map of $A \to A/\mathbf{n}$ maps $k$ isomorphically onto $A/\mathbf{n}$ ? Given such a Noetherian local domain $(A, \mathbf{n})$, it is easy to find a Noetherian local domain $(R, \mathbf{m})$ such that

(1) $R$ contains $k$ and has fraction field $L$,
(2) $R$ is contained in $A$ and $\mathbf{m}A = \mathbf{n}$,
(3) $R$ is essentially finitely generated over $k$.

There is then a relationship between $R$ and $A$ that is realized by passing to completions; the inclusion map $R \hookrightarrow A$ extends to a surjective homomorphism $\widehat{\phi} : \widehat{R} \to \widehat{A}$ of the $\mathbf{m}$-adic completion $\widehat{R}$ of $R$ onto the $\mathbf{n}$-adic completion $\widehat{A}$ of $A$ [M2, Theorem 8.4, page 58]. If $I = \ker(\widehat{\phi})$, then $L$ embeds in $\mathcal{Q}(\widehat{R}/I)$, the total quotient ring of $\widehat{R}/I$ and $A = L \cap (\widehat{R}/I)$. The following commutative diagram, where the vertical maps are injections, displays the relationships among these rings:

$$\begin{array}{ccccccc}
 & & \widehat{R} & \xrightarrow{\hat{\phi}} & \widehat{R}/I \cong \widehat{A} & \longrightarrow & \mathcal{Q}(\widehat{R}/I) \\
 & & \uparrow & & \uparrow & & \uparrow \\
k & \xrightarrow{\subseteq} & R & \longrightarrow & A := L \cap (\widehat{R}/I) & \longrightarrow & L \quad .
\end{array} \tag{5.4.1}$$

**Summary 5.5.**

(1) Every Noetherian local domain $A$ whose fraction field $L$ is finitely generated over a coefficient field $k$ has the form $L \cap (\widehat{R}/I)$ for a domain $R$ which is essentially finitely generated over $k$. That is, every such $A$ is realizable as the intersection of $\mathcal{Q}(R)$ with a homomorphic image of $\widehat{R}$.

(2) More generally, if we drop the assumption that $R$ and $A$ have the same fraction field, then the above argument yields: Every Noetherian local domain $A$ having a coefficient field $k$ is realizable as $L \cap (\widehat{R}/I)$, where $R$ is a local domain essentially finitely generated over $k$ and $I$ is an ideal of its completion $\widehat{R}$ having the property that $P \cap R = (0)$ for each $P \in \operatorname{Ass}(\widehat{R}/I)$.

**Remark 5.6.** A drawback with (5.5) is that it is not true for each $R, L, I$ as in (5.5) that $L \cap (\widehat{R}/I)$ is Noetherian (e.g, see part(4) of Remark 6.4 below). To make the classification more satisfying an important goal is to identify the ideals $I$ of $\widehat{R}$ and fields $L$ such that $L \cap (\widehat{R}/I)$ is Noetherian.

In relation to Theorem 4.5.2, we display, in the following diagram, the inclusions that are always flat:

$$
\begin{array}{ccccccccc}
\mathcal{Q}(R) & = & \mathcal{Q}(U) & = & \mathcal{Q}(B) & = & \mathcal{Q}(A) & \xrightarrow{\subseteq} & \mathcal{Q}(R^*/I) \\
\text{flat}\uparrow & & \text{flat}\uparrow & & \text{flat}\uparrow & & \text{flat}\uparrow & & \text{flat}\uparrow \\
R[1/a] & = & U[1/a] & \xrightarrow{\text{flat}} & B[1/a] & \xrightarrow{\subseteq} & A[1/a] & \xrightarrow{\subseteq} & (R^*/I)[1/a] \\
\text{flat}\uparrow & & \text{flat}\uparrow & & \text{flat}\uparrow & & \text{flat}\uparrow & & \text{flat}\uparrow \\
R & \xrightarrow{\subseteq} & U = \cup U_n & \xrightarrow{\text{flat}} & B & \xrightarrow{\subseteq} & A & \xrightarrow{\subseteq} & R^*/I.
\end{array}
$$

On the other hand, (i) if $B \subsetneq A$, then $B \to A$ is not flat, and (ii) the inclusion $A \subseteq R^*/I$ is flat if and only if $A$ is Noetherian.

## 6. Explicit constructions.

Using the flatness results described in Section 4.5, we formulate two methods for the construction of explicit examples. The first of these methods uses the construction technique of (4.1) while the second uses (4.2).

**Method 6.1.** Let $k$ be a field, let $a, y_1, \dots, y_n$ be variables over $k$, let $R$ be the localized polynomial ring $R := k[a, y_1, \dots, y_n]_{(a, y_1, \dots, y_n)}$ and let $\mathbf{m}$ denote the maximal ideal of $R$. Let $\tau_1, \dots, \tau_s \in ak[[a]]$ be formal power series that are algebraically independent over $k(a)$ and let $D_0 := R[\tau_1, \dots, \tau_s]_{(\mathbf{m}, \tau_1, \dots, \tau_s)}$ be the associated localized polynomial ring over the field $k$ in $n+s+1$ variables. Observe that $D_0$ is contained in the $a$-adic completion $R^*$ of $R$. It is readily seen that the map $D_0 \to R^*[1/a]$ is flat and that $D := \mathcal{Q}(D_0) \cap R^*$ is the localized polynomial ring $V[y_1, \dots, y_n]_{(a, y_1, \dots, y_n)}$, where $V := k(a, \tau_1, \dots, \tau_s) \cap k[[a]]$ is a DVR. Thus the construction method of (4.1) gives a Noetherian limit-intersecting domain $D$.

We now investigate the construction of examples inside $D$. Let $f_1, \dots, f_r$ be elements of the maximal ideal of $D_0$ that are algebraically independent over $\mathcal{Q}(R)$, and let $B_0 := R[f_1, \dots, f_r]_{(\mathbf{m}, f_1, \dots, f_r)}$ be the associated localized polynomial ring. The inclusion map $B_0 \hookrightarrow D_0$ is an injective local $R$-algebra homomorphism.

Let $A := \mathcal{Q}(B_0) \cap R^*$ and let $B$ be the associated nested union domain. Then $B$ is Noetherian and $B = A$ if and only if the map $B_0 \to R^*[1/a]$ is flat. This map factors as $B_0 \to D_0 \to D_0[1/a] \to R^*[1/a]$ and the map $D_0 \to R^*[1/a]$ is flat. Therefore $B$ is Noetherian (and so also $B = A$) if $B_0 \to D_0[1/a]$ is flat.

Since $B_0$ and $D_0$ are localized polynomial rings over a field, the nonflat locus of the inclusion map $B_0 \to D_0$ is closed [M2, Theorem 24.3] and is defined by the ideal $J := \cap\{P \in \operatorname{Spec} D_0 : B_0 \to (D_0)_P \text{ is not flat }\}$. Thus we have established the following theorem.

**Theorem 6.2.** *With the notation above, we have*

(1) *If* $JD_0[1/a] = D_0[1/a]$, *then* $B$ *is Noetherian.*
(2) $B$ *is Noetherian if and only if* $JD[1/a] = D[1/a]$ *if and only if* $JR^*[1/a] = R^*[1/a]$.

Using again the factorization of $B_0 \to R^*[1/a]$ through $D_0[1/a]$, Theorem 5.5 of [HRW3] implies the following result.

**Theorem 6.3.** *With the notation above, if* $\operatorname{ht}(JD_0[1/a]) > 1$, *then* $B = A$.

**Remark 6.4.** With the notation of this section,

(1) If $B$ is Noetherian, then $B$ is a regular local ring.
(2) Example 3.1 of Nagata may be described by taking $n = s = r = 1, y_1 = y, \tau_1 = \tau$, and $f_1 = f$. Then $R = k[a, y]_{(a,y)}$, $D_0 = k[a, y, \tau]_{(a,y,\tau)}$, $f = (y+\tau)^2$, and $B_0 = k[a, y, f]_{(a,y,f)}$. The Noetherian property of $B$ is implied by the flatness property of the map $B_0 \to D_0[1/a]$. In this case, $D_0$ is

actually a free $B_0$-module with $< 1, y + \tau >$ as a free basis.

(3) Example 3.3 of Rotthaus may be described by taking $n = s = 2$, and $r = 1$. Then $R = k[a, y_1, y_2]_{(a,y_1,y_2)}$, $D_0 = R[\tau_1, \tau_2]_{(\mathbf{m},\tau_1,\tau_2)}$, $f_1 = (y_1 + \tau_1)(y_2 + \tau_2)$ and $B_0 = R[f_1]_{(\mathbf{m},f_1)}$. Again the Noetherian property of $B$ is implied by the flatness property of the map $B_0 \to D_0[1/a]$.

(4) The following example is given in [HRW6, Section 4]. Let $n = s = r = 2$, let $f_1 = (y_1 + \tau_1)^2$ and $f_2 = (y_1 + \tau_1)(y_2 + \tau_2)$. It is shown in [HRW6] for this example that $B \subsetneq A$ and that both $A$ and $B$ are non-Noetherian.

**Method 6.5.** Let $k$ be a field, let $a, y_1, \dots, y_n$ be variables over $k$, let $R$ be the localized polynomial ring $R := k[a, y_1, \dots, y_n]_{(a,y_1,\dots,y_n)}$ and let $\mathbf{m}$ denote the maximal ideal of $R$. Let $\tau_1, \dots, \tau_s \in ak[[a]]$ be formal power series that are algebraically independent over $k(a)$ and let $D_0 := R[\tau_1, \dots, \tau_s]_{(\mathbf{m},\tau_1,\dots,\tau_s)}$ be the associated localized polynomial ring over the field $k$.

Assume that $I$ is an ideal of $D_0$ such that $P \cap R = (0)$ for each $P \in \text{Ass}(D_0/I)$. Let $A := \mathcal{Q}(R) \cap (R^*/IR^*)$; then the $a$-adic completion of $A$ is $R^*/IR^*$. Using the frontpiece approximations of a generating set for $I$, it is shown in [HRW5, Section 2] that there exists a quasilocal integral domain $B = \cup_{n=1}^{\infty} B_n \subseteq A$ birationally dominating $R$ such that the $a$-adic completion of $B$ is $R^*/IR^*$. By [HRW5, Theorem 3.2], $R \to (R^*/IR^*)[1/a]$ is flat if and only if $B$ is Noetherian.

Since the map $D_0 \to R^*[1/a]$ is flat, the map $D_0/I \to (R^*/IR^*)[1/a]$ is flat. Also the map $R \to (R^*/IR^*)[1/a]$ factors as $R \to (D_0/I)[1/a] \to (R^*/I)[1/a]$. Thus $R \to (R^*/IR^*)[1/a]$ is flat if $R \to (D_0/I)[1/a]$ is flat. Since $R$ and $D_0/I := T$ are essentially finitely generated over a field, the nonflat locus of the inclusion map $R \to T$ is closed [M2, Theorem 24.3] and is defined by the ideal $J := \cap\{P \in \operatorname{Spec} T : R \to T_P \text{ is not flat }\}$. This yields the following theorem.

**Theorem 6.6.** *With the notation above, if $J(D_0/I)[1/a] = (D_0/I)[1/a]$, then $B$ is Noetherian*

**Remark 6.7.** Let the notation be as in part(4) of Remark 6.4, so $f_1 = (y_1 + \tau_1)^2$ and $f_2 = (y_1 + \tau_1)(y_2 + \tau_2)$, and let $I := (f_1, f_2)R^*$. In [HRW6, Section 4] it is shown that $C := \mathcal{Q}(R) \cap (R^*/I)$ is Noetherian and limit-intersecting. Indeed, it is shown in [HRW6, Proposition 4.5] that $C$ is a two-dimensional Noetherian local domain for which the generic formal fiber is not Cohen-Macaulay.

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