THE COHEN-MACaulAY AND GORENSTEIN PROPERTIES
OF RINGS ASSOCIATED TO FILTRATIONS

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Abstract. Let \((R, \mathfrak{m})\) be a Cohen-Macaulay local ring and let \(\mathcal{F} = \{F_i\}_{i \in \mathbb{Z}}\) be an \(F_1\)-good filtration of ideals in \(R\). If \(F_1\) is \(\mathfrak{m}\)-primary we obtain sufficient conditions in order that the associated graded ring \(G(\mathcal{F})\) be Cohen-Macaulay. In the case where \(R\) is Gorenstein, we use the Cohen-Macaulay result to establish necessary and sufficient conditions for \(G(\mathcal{F})\) to be Gorenstein. We apply this result to the integral closure filtration \(\mathcal{F}\) associated to a monomial parameter ideal of a polynomial ring to give necessary and sufficient conditions for \(G(\mathcal{F})\) to be Gorenstein. Let \((R, \mathfrak{m})\) be a Gorenstein local ring and let \(F_1\) be an ideal with \(\text{ht}(F_1) = g > 0\). If there exists a reduction \(J\) of \(F\) with \(\ell(J) = g\) and reduction number \(u := r(J)\), we prove that the extended Rees algebra \(R_0(\mathcal{F})\) is quasi-Gorenstein with \(a\)-invariant \(b\) if and only if \(J^n : F_u = F_{u+b-1+g-1}\) for every \(n \in \mathbb{Z}\). Furthermore, if \(G(\mathcal{F})\) is Cohen-Macaulay, then the maximal degree of a homogeneous minimal generator of the canonical module \(\omega_{G(\mathcal{F})}\) is at most \(g\) and that of the canonical module \(\omega_{R_0(\mathcal{F})}\) is at most \(g - 1\); moreover, \(R_0(\mathcal{F})\) is Gorenstein if and only if \(J^u : F_u = F_u\). We illustrate with various examples cases where \(G(\mathcal{F})\) is or is not Gorenstein.

1. Introduction

All rings we consider are assumed to be commutative with an identity element.

A filtration \(\mathcal{F} = \{F_i\}_{i \in \mathbb{N}}\) on a ring \(R\) is a descending chain \(R = F_0 \supset F_1 \supset F_2 \supset \cdots\) of ideals such that \(F_i F_j \subseteq F_{i+j}\) for all \(i, j \in \mathbb{N}\). It is sometimes convenient to extend the filtration by defining \(F_i = R\) for all integers \(i \leq 0\).

Let \(t\) be an indeterminate over \(R\). Then for each filtration \(\mathcal{F}\) of ideals in \(R\), several graded rings naturally associated to \(\mathcal{F}\) are:

1. The Rees algebra \(R(\mathcal{F}) = \bigoplus_{i \geq 0} F_i t^i \subseteq R[t]\),
2. The extended Rees algebra \(R'(\mathcal{F}) = \bigoplus_{i \in \mathbb{Z}} F_i t^i \subseteq R[t, t^{-1}]\),
3. The associated graded ring \(G(\mathcal{F}) = \frac{R'(\mathcal{F})}{(t^{-1})R'(\mathcal{F})} = \bigoplus_{i \geq 0} \frac{F_i}{F_{i+1}}\).

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If $\mathcal{F}$ is an $I$-adic filtration, that is, $\mathcal{F} = \{I^i\}_{i \in \mathbb{Z}}$ for some ideal $I$ in $R$, we denote $R(\mathcal{F}), R'(\mathcal{F})$, and $G(\mathcal{F})$ by $R(I), R'(I)$, and $G(I)$, respectively.

In this paper we examine the Cohen-Macaulay and Gorenstein properties of graded rings associated to filtrations $\mathcal{F}$ of ideals. We establish

1. sufficient conditions for $G(\mathcal{F})$ to be Cohen-Macaulay,
2. necessary and sufficient conditions for $G(\mathcal{F})$ to be Gorenstein, and
3. necessary and sufficient conditions for $R'(\mathcal{F})$ to be quasi-Gorenstein.

These results extend those given in [HKU] in the case where $\mathcal{F}$ is an ideal-adic filtration.

Let $(R, \mathfrak{m})$ be a $d$-dimensional Cohen-Macaulay local ring and let $\mathcal{F} = \{F_i\}_{i \in \mathbb{Z}}$ be an $F_1$-good filtration, where $F_1$ is $\mathfrak{m}$-primary. Assume that $J$ is a reduction of $\mathcal{F}$ with $\mu(J) = d$ and let $u := r_J(\mathcal{F})$ denote the reduction number of $\mathcal{F}$ with respect to $J$. In Theorem 3.12, we prove that $G(\mathcal{F})$ is Cohen-Macaulay, if $J : F_{u-i} = J + F_{i+1}$ for all $i$ with $0 \leq i \leq u - 1$. If $R$ is Gorenstein, we prove in Theorem 4.3 that $G(\mathcal{F})$ is Gorenstein if and only if $J : F_{u-i} = J + F_{i+1}$ for $0 \leq i \leq \lfloor \frac{u-1}{2} \rfloor$. If $R$ is regular with $d \geq 2$ and $G(\mathcal{F})$ is Cohen-Macaulay, we prove in Theorem 4.7 that $G(\mathcal{F}/J)$ has a nonzero socle element of degree $\leq d - 2$. We deduce in Corollary 4.9 that if $G(\mathcal{F})$ is Gorenstein and $F_{i+1} \subseteq \mathfrak{m} F_i$ for all $i \geq d - 1$, then $r_J(\mathcal{F}) \leq d - 2$.

Let $J$ be a monomial parameter ideal of a polynomial ring $R = k[x_1, \ldots, x_d]$ over a field $k$. In Section 5 we consider the integral closure filtration $\mathcal{F} := \{J^n\}_{n \geq 0}$ associated to $J$. If $J = (x_1^{\alpha_1}, \ldots, x_d^{\alpha_d}) R$ and $L$ is the least common multiple of $\alpha_1, \ldots, \alpha_d$, Theorem 5.6 states that $G(\mathcal{F})$ is Gorenstein if and only if $\sum_{i=1}^{d} \frac{L}{\alpha_i} \equiv 1 \mod L$. Corollary 5.7 asserts that the following three conditions are equivalent:

1. $\sum_{i=1}^{d} \frac{L}{\alpha_i} = L + 1$,
2. $G(\mathcal{F})$ is Gorenstein and $r_J(\mathcal{F}) = d - 2$, and
3. the Rees algebra $R(\mathcal{F})$ is Gorenstein. Example 5.13 demonstrates the existence of monomial parameter ideals for which the associated integral closure filtration $\mathcal{E}$ is such that $G(\mathcal{E})$ and $R(\mathcal{E})$ are Gorenstein and $\mathcal{E}$ is not an ideal-adic filtration.

In Section 6 we consider a $d$-dimensional Gorenstein local ring $(R, \mathfrak{m})$ and an $F_1$-good filtration $\mathcal{F} = \{F_i\}_{i \in \mathbb{Z}}$ of ideals in $R$, where $\text{ht}(F_1) = g > 0$. Assume there exists a reduction $J$ of $\mathcal{F}$ with $\mu(J) = g$ and reduction number $u := r_J(\mathcal{F})$. In Theorem 6.1, we prove that the extended Rees algebra $R'(\mathcal{F})$ is quasi-Gorenstein with $\mathfrak{a}$-invariant $b$ if and only if $(J^n : F_u) = F_{n+b-u+g-1}$ for every $n \in \mathbb{Z}$. If $G(\mathcal{F})$ is Cohen-Macaulay, we prove in Theorem 6.2 that the maximal degree of a
homogeneous minimal generator of the canonical module $\omega_{G(\mathcal{F})}$ is at most $g$ and that of the canonical module $\omega_{R'(\mathcal{F})}$ is at most $g - 1$. With the same hypothesis, we prove in Theorem 6.3 that $R'(\mathcal{F})$ is Gorenstein if and only if $J^u : F_u = F_u$.

In Section 7 we present and compare properties of various filtrations.

2. Preliminaries

**Definition 2.1.** Let $\mathcal{F} = \{F_i\}_{i \in \mathbb{Z}}$ be a filtration of ideals in $R$ and let $I$ be an ideal of $R$.

1. The filtration $\mathcal{F}$ is called **Noetherian** if the Rees ring $R(\mathcal{F})$ is Noetherian.
2. The filtration $\mathcal{F}$ is called an **$I$-good filtration** if $IF_i \subseteq F_{i+1}$ for all $i \in \mathbb{Z}$ and $F_{n+1} = IF_n$ for all $n >> 0$. The filtration $\mathcal{F}$ is called a **good filtration** if it is an $I$-good filtration for some ideal $I$ in $R$.
3. A **reduction** of a filtration $\mathcal{F}$ is an ideal $J \subseteq F_1$ such that $JF_n = F_{n+1}$ for all large $n$. A **minimal reduction** of $\mathcal{F}$ is a reduction of $\mathcal{F}$ minimal with respect to inclusion.
4. If $J \subseteq F_1$ is a reduction of $\mathcal{F}$, then
   \[ r_J(\mathcal{F}) = \min \{ r \mid F_{n+1} = JF_n \text{ for all } n \geq r \} \]
   is the **reduction number** of $\mathcal{F}$ with respect to $J$.
5. If $L$ is an ideal of $R$, then $\mathcal{F}/L$ denotes the filtration $\{(F_i + L)/L\}_{i \in \mathbb{Z}}$ on $R/L$. The filtration $\mathcal{F}/L$ is Noetherian, resp. good, if $\mathcal{F}$ is Noetherian, resp. good.

**Remark 2.2.** If the filtration $\mathcal{F}$ is Noetherian, then $R$ is Noetherian and $R'(\mathcal{F})$ is finitely generated over $R$ [BH, Proposition 4.5.3]. Moreover, $\dim R'(\mathcal{F}) = \dim R + 1$ and $\dim G(\mathcal{F}) \leq \dim R$, with $\dim G(\mathcal{F}) = \dim R$ if $F_1$ is contained in all the maximal ideals of $R$ [BH, Theorem 4.5.6]. Furthermore, one has $\dim R(\mathcal{F}) = \dim R + 1$, if $F_1$ is not contained in any minimal prime ideal $p$ in $R$ with $\dim(R/p) = \dim(R)$ (cf. [Va]). Assume the ring $R$ is Noetherian, then the filtration $\mathcal{F} = \{F_i\}_{i \in \mathbb{Z}}$ is a good filtration if and only if it is an $F_1$-good filtration, and $\mathcal{F}$ is an $F_1$-good filtration if there exists an integer $k$ such that $F_n \subseteq (F_1)^{n-k}$ for all $n$ if and only if the Rees algebra $R(\mathcal{F})$ is a finite $R(F_1)$-module [B, Theorem III.3.1.1 and Corollary III.3.1.4].

If $\mathcal{F} = \{F_i\}_{i \in \mathbb{Z}}$ is a filtration on $R$, then we have
\[
R(F_1) = \bigoplus_{n \geq 0} F_1^n t^n \subseteq R(\mathcal{F}) = \bigoplus_{n \geq 0} F_n t^n \subseteq R[t].
\]
If \( R \) is Noetherian and \( \mathcal{F} = \{ F_i \}_{i \in \mathbb{Z}} \) is an \( F_1 \)-good filtration, then \( R(\mathcal{F}) \) is a finite \( R(F_1) \)-module, and hence \( R(\mathcal{F}) \) is integral over \( R(F_1) \). Thus, in this case, we have \( F_1^n \subseteq F_n \subseteq \overline{F_1^n} \), for all \( n \geq 0 \), where \( \overline{F_1^n} \) denotes the integral closure of \( F_1^n \). Notice also that if \( \mathcal{F} \) is an \( F_1 \)-good filtration, then \( J \) is a reduction of \( \mathcal{F} \iff J \) is a reduction of \( F_1 \).

The proof of Remark 2.3 is straightforward using the definition of an \( F_1 \)-good filtration.

**Remark 2.3.** Let \((R, \mathfrak{m})\) be a Noetherian local ring and let \( \mathcal{F} = \{ F_i \}_{i \in \mathbb{Z}} \) be an \( F_1 \)-good filtration of \( R \). Set

\[
R(\mathcal{F})_+ = \bigoplus_{i \geq 1} F_it^i,
\]

\[
R(\mathcal{F})_+(1) = \bigoplus_{i \geq 0} F_{i+1}t^i,
\]

\[
G(\mathcal{F})_+ = \bigoplus_{i \geq 1} G_i, \quad \text{where} \quad G_i = F_i/F_{i+1} \quad i \geq 0.
\]

Then we have the following:

1. \( \sqrt{F_1 \cdot R(\mathcal{F})} = \sqrt{R(\mathcal{F})_+(1)}. \)
2. \( \sqrt{F_it^i \cdot R(\mathcal{F})} = \sqrt{R(\mathcal{F})_+} \) for each \( i \geq 1. \)
3. \( \sqrt{G_i \cdot G(\mathcal{F})} = \sqrt{G(\mathcal{F})_+} \) for each \( i \geq 1. \)
4. \( (G(\mathcal{F})_+)^n \subseteq \bigoplus_{i \geq n} G_i = G_n \cdot G(\mathcal{F}) \) for all \( n >> 0. \)

We use Lemma 2.4 in Section 6.

**Lemma 2.4.** Let \((R, \mathfrak{m})\) be a Noetherian local ring and let \( \mathcal{F} = \{ F_i \}_{i \in \mathbb{Z}} \) be an \( F_1 \)-good filtration of ideals in \( R \). Let \( G := G(\mathcal{F}) = \bigoplus_{i \geq 0} F_i/F_{i+1} = \bigoplus_{i \geq 0} G_i \) and \( G_+ := \bigoplus_{i \geq 1} F_i/F_{i+1}. \) If grade\( G_+ \geq 1 \), then for each integer \( n \geq 1 \) we have:

1. \( F_{n+i} : F_i = F_n \quad \text{for all} \quad i \geq 1. \)
2. \( F_n = \cap_{j \geq 1}(F_{n+j} : F_j) = \cup_{j \geq 1}(F_{n+j} : F_j). \)

**Proof.** (1) For a fixed \( i \geq 1 \) we have \( G_i^m \subseteq G_i G \) for some \( m >> 0 \) by Remark 2.3. Therefore grade\( G_i G \geq 1. \) It is clear that \( F_n \subseteq F_{n+i} : F_i. \) Assume there exists \( b \in (F_{n+i} : F_i) \setminus F_n. \) Then \( b \in F_j \setminus F_{j+1} \) for some \( j \) with \( 0 \leq j \leq n - 1, \) and \( 0 \neq b^* = b + F_{j+1} \in F_j/F_{j+1} = G_j. \) Since \( b \in (F_{n+i} : F_i), \) we have \( b^* G_i = 0, \) and so \( b^* G_1 G = 0. \) This is a contradiction.

(2) Item (2) is immediate from item (1). \( \square \)
The $I$-adic filtration $\mathcal{F} = \{I^i\}_{i \in \mathbb{Z}}$ is an $I$-good filtration. We describe in Examples 2.5 and 2.6 other examples of good filtrations.

**Example 2.5.** Let $I$ be a proper ideal of a Noetherian ring $R$. If $I$ contains a non-zero-divisor, then Ratliff and Rush consider in [RR] the following ideal associated to $I$:

$$\tilde{I} = \bigcup_{i \geq 1} (I^{i+1} : I^i).$$

The ideal $\tilde{I}$ is now called the *Ratliff-Rush* ideal associated to $I$, or the *Ratliff-Rush closure* of $I$. It is characterized as the largest ideal having the property that $(\tilde{I})^n = I^n$ for all sufficiently large positive integers $n$. Moreover, for each positive integer $s$

$$\tilde{I}^s = \bigcup_{i \geq 1} (I^{i+s} : I^i),$$

and there exists a positive integer $n$ such that $\tilde{I}^k = I^k$ for all integers $k \geq n$ [RR, (2.3.2)]. Consequently, $\mathcal{F} = \{\tilde{I}^i\}_{i \in \mathbb{N}}$ is a Noetherian $I$-good filtration.

**Example 2.6.** Let $(R, m)$ be a Noetherian local ring with $\dim R = d$ and let $I$ be an $m$-primary ideal. The function $H_I(n) = \lambda(R/I^n)$ is called the Hilbert-Samuel function of $I$. For sufficiently large values of $n$, $\lambda(R/I^n)$ is a polynomial $P_I(n)$ in $n$ of degree $d$, the Hilbert-Samuel polynomial of $I$. We write this polynomial in terms of binomial coefficients:

$$P_I(n) = e_0(I) \binom{n+d-1}{d} - e_1(I) \binom{n+d-2}{d-1} + \cdots + (-1)^d e_d(I).$$

The coefficients $e_i(I)$ are integers and are called the Hilbert coefficients of $I$. In particular, the leading coefficient $e_0(I)$ is a positive integer called the multiplicity of $I$.

As was first shown by Shah in [Sh], if $(R, m)$ is formally equidimensional of dimension $d > 0$ with $|R/m| = \infty$, then for each integer $k$ in \{0, 1, \ldots, d\} there exists a unique largest ideal $I_{\{k\}}$ containing $I$ and contained in the integral closure $\tilde{I}$ such that

$$e_i(I_{\{k\}}) = e_i(I) \quad \text{for} \quad i = 0, 1, \ldots, k.$$
The ideal \( I_{(k)} \) is called the \( k \)-th coefficient ideal of \( I \), or the \( e_k \)-ideal associated to \( I \).

The ideal \( I_{(0)} \) is the integral closure \( \overline{I} \) of \( I \), and if \( I \) contains a regular element, then \( I_{(d)} \) is the Ratliff-Rush closure of \( I \).

Associated to \( I \) and the chain of coefficient ideals given in (1), we have a chain of filtrations

\[
(2) \quad \mathcal{F}_{d+1} \subseteq \mathcal{F}_d \subseteq \cdots \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_0,
\]

where the filtration \( \mathcal{F}_k := \{(I^n)_{n \in \mathbb{Z}}\} \), for each \( k \) such that \( 0 \leq k \leq d + 1 \). In particular, \( \mathcal{F}_{d+1} = \{I^n\}_{n \in \mathbb{Z}} \) is the \( I \)-adic filtration, and \( \mathcal{F}_0 = \{\overline{I^n}\}_{n \in \mathbb{Z}} \) is the filtration given by the integral closures of the powers of \( I \). If \( I \) contains a non-zero-divisor, then \( \mathcal{F}_d = \{\overline{I^n}\}_{n \in \mathbb{Z}} \) is the filtration given by the Ratliff-Rush ideals associated to the powers of \( I \). The filtration \( \mathcal{F}_1 = \{(I^n)_{(1)}\}_{n \in \mathbb{Z}} \) is called the \( e_1 \)-closure filtration.

In this connection, see also [C1], [C2] and [CPV]. If \( R \) is also assumed to be analytically unramified, then each of the filtrations \( \mathcal{F}_k := \{(I^n)_{n \in \mathbb{Z}}\} \) is an \( I \)-good filtration. This follows because the integral closure of the Rees ring \( R(I) = R[I] \) in the polynomial ring \( R[t] \) is the graded ring \( \bigoplus_{n \geq 0} \overline{I^n}t^n \), and a well-known result of Rees [R], [SH, Theorem 9.1.2] implies that \( \bigoplus_{n \geq 0} \overline{I^n}t^n \) is a finite \( R(I) \)-module. Thus \( \{\overline{I^n}\}_{n \in \mathbb{Z}} \) is a Noetherian \( I \)-good filtration. Moreover, if \( R \) is analytically unramified and contains a field and if \( (I^n)^* \) denotes the tight closure of \( I^n \), then \( \mathcal{F} = \{(I^n)^*\}_{n \in \mathbb{Z}} \) is an \( I \)-good filtration.

3. The Cohen-Macaulay property for \( G(\mathcal{F}) \)

Let \( (R, \mathfrak{m}) \) be a Noetherian local ring and let \( \mathcal{F} = \{F_i\}_{i \in \mathbb{Z}} \) be a Noetherian filtration on \( R \). For an element \( x \in F_1 \), let \( x^* \) denote the image of \( x \) in \( G(\mathcal{F})_1 = F_1/F_2 \). The element \( x \) is called superficial for \( \mathcal{F} \) if there exists a positive integer \( c \) such that \( (F_{n+1} : x) \cap F_c = F_n \) for all \( n \geq c \). In terms of the associated graded ring \( G(\mathcal{F}) \), the element \( x \) is superficial for \( \mathcal{F} \) if and only if the \( n \)-th homogeneous component \( 0 :_{G(\mathcal{F})} x^* \) of the annihilator of \( x^* \) in \( G(\mathcal{F}) \) is zero for all \( n >> 0 \). If \( \text{grade} \ F_1 \geq 1 \) and \( x \) is superficial for \( \mathcal{F} \), then \( x \) is a regular element of \( R \). For if \( u \in R \) and \( ux = 0 \), then \( (F_1)^c u \subseteq \bigcap_n (F_{n+1} : x) \cap F_c = \bigcap_n F_n = 0 \). Since \( \mathcal{F} \) is a Noetherian filtration, it follows that \( u = 0 \). A sequence \( x_1, \ldots, x_k \) of elements of \( F_1 \) is called a superficial sequence for \( \mathcal{F} \) if \( x_1 \) is superficial for \( \mathcal{F} \), and \( x_i \) is superficial for \( \mathcal{F}/(x_1, \ldots, x_{i-1}) \) for \( 2 \leq i \leq k \).

The following well-known fact is useful in working with filtrations.
Fact 3.1. If $x^*$ is a regular element of $G(\mathcal{F})$, then $x$ is a regular element of $R$ and $G(\frac{x}{x^*}) \cong G(\mathcal{F})/(x^*)$.

We record in Proposition 3.2 a result of Huckaba and Marley that involves what is now called Sally’s machine, cf. [RV, Lemma 1.8].

Proposition 3.2. (HM, Lemma 2.1 and Lemma 2.2) Let $(R, m)$ be a Noetherian local ring, let $\mathcal{F} = \{F_i\}_{i \in \mathbb{Z}}$ be a Noetherian filtration on $R$, and let $x_1, \ldots, x_k$ be a superficial sequence for $F$. Then the following assertions are true:

1. If grade $(G(\mathcal{F})_+) \geq k$, then $x_1^*, \ldots, x_k^*$ is a $G(\mathcal{F})$-regular sequence.
2. If grade $(G(\frac{\mathcal{F}}{x_1, \ldots, x_k})_+) \geq 1$, then grade $(G(\mathcal{F})_+) \geq k + 1$.

The following result of Huckaba and Marley generalizes to filtrations a result of Valabrega and Valla [VV, Corollary 2.7].

Proposition 3.3. (HM, Proposition 3.5) Let $(R, m)$ be a Noetherian local ring, let $\mathcal{F} = \{F_i\}_{i \in \mathbb{Z}}$ be a Noetherian filtration on $R$, and let $x_1, \cdots, x_k$ be elements of $F_1$. The following two conditions are equivalent:

1. $x_1^*, \ldots, x_k^*$ is a $G(\mathcal{F})$-regular sequence.
2. (i) $x_1, \ldots, x_k$ is an $R$-regular sequence, and
   (ii) $(x_1, \ldots, x_k)R \cap F_i = (x_1, \ldots, x_k)F_{i-1}$ for all $i \geq 1$.

Remark 3.4. Let $(R, m)$ be a Noetherian local ring and let $\mathcal{F} = \{F_i\}_{i \in \mathbb{Z}}$ be a filtration on $R$. If there exists a reduction $J$ of $\mathcal{F}$ such that $JF_n = F_{n+1}$ for all $n \geq 1$, then $F_n = F_1^n$ for all $n$, that is, $\mathcal{F}$ is the $F_1$-adic filtration.

Proof. For every $n \geq 2$ we have $F_n = JF_{n-1} = J^2F_{n-2} = \cdots = J^{n-1}F_1 \subseteq F_1^n$. □

Corollary 3.5. Let $(R, m)$ be a Cohen-Macaulay local ring and let $\mathcal{F} = \{F_i\}_{i \in \mathbb{Z}}$ be an $F_1$-good filtration on $R$, where $F_1$ is $m$-primary. If there exists a reduction $J$ of $\mathcal{F}$ with $\mu(J) = \dim R$ and $JF_n = F_{n+1}$ for all $n \geq 1$, then the associated graded ring $G(\mathcal{F})$ is Cohen-Macaulay.

Proof. Remark 3.4 implies that $\mathcal{F}$ is the $F_1$-adic filtration. Hence $G(\mathcal{F})$ is Cohen-Macaulay by [S1, Theorem 2.2] or [VV, Proposition 3.1]. □

Proposition 3.6 is a result proved by D.Q. Viet([Vi, Corollary 2.1]). It generalizes to filtrations a result of Trung and Ikeda ([TI, Theorem 1.1]), and is in the nature of the well-known result of Goto-Shimoda ([GS]).
Let \( \alpha(G(\mathcal{F})) = \max\{n \mid [\mathcal{H}^n_m(G(\mathcal{F}))]_n \neq 0\} \) denote the \( \alpha \)-invariant of \( G(\mathcal{F}) \) ([GW, (3.1.4)]), where \( m \) is the maximal homogeneous ideal of \( R(\mathcal{F}) \) and \( \mathcal{H}^n_m(G(\mathcal{F})) \) is the \( i \)-th graded local cohomology module of \( G(\mathcal{F}) \) with respect to \( m \).

**Proposition 3.6.** ([Vi, Corollary 2.1]) Let \( (R, m) \) be a \( d \)-dimensional Cohen-Macaulay local ring and let \( \mathcal{F} = \{F_i\}_{i \in \mathbb{Z}} \) be an \( F_1 \)-good filtration on \( R \), where \( F_1 \) is \( m \)-primary. Then the following conditions are equivalent:

1. \( R(\mathcal{F}) \) is Cohen-Macaulay.
2. \( G(\mathcal{F}) \) is Cohen-Macaulay with \( \alpha(G(\mathcal{F})) < 0 \).

**Remark 3.7.** Let \( (R, m) \) be a \( d \)-dimensional Cohen-Macaulay local ring and let \( \mathcal{F} = \{F_i\}_{i \in \mathbb{Z}} \) be an \( F_1 \)-good filtration on \( R \), where \( F_1 \) is \( m \)-primary. Assume that there exists a reduction \( J \) of \( \mathcal{F} \) with \( \mu(J) = d \). If \( R(\mathcal{F}) \) is Cohen-Macaulay, then Proposition 3.6 implies that \( \alpha(G(\mathcal{F})) < 0 \). Since \( r_{\mathcal{F}}(\mathcal{F}) = r_{\mathcal{F}}(\mathcal{F}/J) = \alpha(G(\mathcal{F}/J)) - \alpha(G(\mathcal{F})) + d \), it follows that \( r_{\mathcal{F}}(\mathcal{F}) < d \).

**Proposition 3.8.** Let \( (R, m) \) be a \( d \)-dimensional regular local ring and let \( \mathcal{F} = \{F_i\}_{i \in \mathbb{Z}} \) be an \( F_1 \)-good filtration on \( R \), where \( F_1 \) is \( m \)-primary. Assume there exists a reduction \( J \) of \( \mathcal{F} \) with \( \mu(J) = d \). If \( G(\mathcal{F}) \) is Cohen-Macaulay, then \( r_{\mathcal{F}}(\mathcal{F}) < d \).

**Proof.** We have \( R(F_1) = \bigoplus_{n \geq 0} F_1^n t^n \subseteq R(\mathcal{F}) = \bigoplus_{n \geq 0} F_n t^n \subseteq R[t] \). Since \( \mathcal{F} = \{F_i\}_{i \in \mathbb{Z}} \) is an \( F_1 \)-good filtration, \( R(\mathcal{F}) \) is a finite \( R(F_1) \)-module, and thus \( R(\mathcal{F}) \) is integral over \( R(F_1) \). Hence we have \( F_1^n \subseteq F_n \subseteq F_1^n \), for all \( n \geq 0 \). Since \( J \) is a minimal reduction of \( F_1 \), it follows that \( F_1^n \subseteq J \), for every \( n \geq d \) by the Briançon-Skoda theorem ([LS, Theorem 1]). Therefore we have \( F_n = F_n \cap J \) for \( n \geq d \). Since \( G(\mathcal{F}) \) is Cohen-Macaulay, Proposition 3.3 shows that \( F_n \cap J = JF_{n-1} \). Thus \( r_{\mathcal{F}}(\mathcal{F}) < d \).

**Remark 3.9.** Let \( (R, m) \) be a 2-dimensional Cohen-Macaulay local ring and let \( \mathcal{F} = \{F_i\}_{i \in \mathbb{Z}} \) be an \( F_1 \)-good filtration on \( R \), where \( F_1 \) is \( m \)-primary.

1. If \( R(\mathcal{F}) \) is Cohen-Macaulay, then Remark 3.7 and Remark 3.4 imply that \( \mathcal{F} = \{F_i\}_{i \in \mathbb{Z}} \) is the \( F_1 \)-adic filtration.
2. If \( R \) is also regular and \( G(\mathcal{F}) \) is Cohen-Macaulay, then Proposition 3.8 and Remark 3.4 imply that \( \mathcal{F} = \{F_i\}_{i \in \mathbb{Z}} \) is the \( F_1 \)-adic filtration.

Let \( (R, m) \) be a \( d \)-dimensional Cohen-Macaulay local ring and let \( \mathcal{F} = \{F_i\}_{i \in \mathbb{Z}} \) be an \( F_1 \)-good filtration on \( R \), where \( F_1 \) is \( m \)-primary. Assume that \( J \) is a reduction of \( \mathcal{F} \) with \( \mu(J) = d \) and let \( r_{\mathcal{F}}(\mathcal{F}) = u \) denote the reduction number of \( \mathcal{F} \) with respect
to $J$. We determine sufficient conditions for $G(F)$ to be Cohen-Macaulay involving the reduction number $u$ and residuation with respect to $J$. The dimension one case plays a crucial role, so we consider this case first.

**Theorem 3.10.** Let $(R, \mathfrak{m})$ be a one-dimensional Cohen-Macaulay local ring and let $\mathcal{F} = \{F_i\}_{i \in \mathbb{Z}}$ be an $F_1$-good filtration, where $F_1$ is $\mathfrak{m}$-primary. Assume there exists a reduction $J = xR$ of $\mathcal{F}$ with reduction number $r_J(\mathcal{F}) = u$ such that

$$J : F_{u-i} = J + F_{i+1} \text{ for all } i \text{ with } 0 \leq i \leq u - 1.$$ 

Then the following two assertions are true:

1. $F_u : F_{u-i} = F_i$ for $1 \leq i \leq u$, and
2. $G(\mathcal{F})$ is a Cohen-Macaulay ring.

**Proof.** Notice that $J^jF_u = F_{j+u} = F_jF_u$ for all $j \geq 0$. (*)

To establish item (1), we first prove the following claim.

**Claim 3.11.** $F_i \subseteq F_u : F_{u-i} \subseteq J + F_i$ for $1 \leq i \leq u$.

**Proof of Claim.** For $1 \leq i \leq u$, we have

$$F_i \subseteq F_u : F_{u-i} \subseteq F_uF_i : F_{u-i}F_u$$

$$= J^uF_u : J^{u-i}F_u \quad \text{by (*)}$$

$$= J^IF_u : F_u \quad \text{since } J = (x) \text{ with } x \text{ a regular element}$$

$$\subseteq J^i : F_u$$

$$= (J^{i+1} : J) : F_u \quad \text{since } J = (x) \text{ with } x \text{ regular}$$

$$= J^{i+1} : JF_u$$

$$= J^{i+1} : F_{u+1}$$

$$\subseteq J^{i+1} : J^IF_{u-(i-1)} \quad \text{since } J^IF_{u-(i-1)} \subseteq F_{u+1}$$

$$= J : F_{u-(i-1)} \quad \text{since } J = (x) \text{ with } x \text{ regular}$$

$$= J + F_i \quad \text{by assumption.}$$

This establishes Claim 3.11.
For the proof of (1), we use induction on $i$. If $i = 1$, the assertion is clear in view of Claim 3.11. Assume that $i \geq 2$. Then we have

$$F_u : F_{u-i} = (J + F_i) \cap (F_u : F_{u-i})$$

by Claim 3.11

$$= [J \cap (F_u : F_{u-i})] + [F_i \cap (F_u : F_{u-i})]$$

since $F_i \subseteq F_u : F_{u-i}$

$$= J((F_u : F_{u-i}) : J) + F_i$$

since $J = (x)$ and $F_i \subseteq F_u : F_{u-i}$

$$= J(F_u : JF_{u-i}) + F_i$$

$$\subseteq J(F_uF_u : JF_{u-i}F_u) + F_i$$

by ($*$)

$$\subseteq J(J^uF_u : J^uF_{u-(i-1)}) + F_i$$

since $J^uF_{u-(i-1)} \subseteq F_{u+u+1-i}$

$$= J(F_u : F_{u-(i-1)}) + F_i$$

since $J = (x)$

$$= JF_{i-1} + F_i$$

by the induction hypothesis

$$= F_i.$$  

This establishes item (1).

For item (2), we show that $J \cap F_i = JF_{i-1}$ for $1 \leq i \leq u$. It is clear that $J \cap F_i \supseteq JF_{i-1}$. We prove that $J \cap F_i \subseteq JF_{i-1}$. For $1 \leq i \leq u$, we have

$$J \cap F_i = J(F_i : J)$$

since $J = (x)$ with $x$ regular

$$\subseteq J(F_iF_u : JF_u)$$

$$= J(J^iF_u : JF_u)$$

by ($*$)

$$\subseteq J(J^iF_u : J^iF_{u-(i-1)})$$

since $J^iF_{u-(i-1)} \subseteq JF_u$

$$= J(F_u : F_{u-(i-1)})$$

since $J = (x)$ with $x$ regular

$$= JF_{i-1}$$

by item (1).

By Proposition 3.3, $G(\mathcal{F})$ is Cohen-Macaulay. 

Theorem 3.12 is the main result of this section.

**Theorem 3.12.** Let $(R, \mathfrak{m})$ be a $d$-dimensional Cohen-Macaulay local ring and let $\mathcal{F} = \{F_i\}_{i \in \mathbb{Z}}$ be an $F_1$-good filtration, where $F_1$ is $\mathfrak{m}$-primary. Assume that $J$ is a reduction of $\mathcal{F}$ with $\mu(J) = d$, and let $u := r_J(\mathcal{F})$ denote the reduction number of $\mathcal{F}$ with respect to $J$. If

$$J : F_{u-i} = J + F_{i+1}$$

for all $i$ with $0 \leq i \leq u - 1$,

then the associated graded ring $G(\mathcal{F})$ is Cohen-Macaulay.
Proof. We may assume that $R/m$ is infinite. There is nothing to prove if $d = 0$. If $d = 1$, then $G(F)$ is Cohen-Macaulay by Theorem 3.10. Assume that $d \geq 2$. There exists elements $x_1, \ldots, x_d$ that form a minimal generating set for $J$ and a superficial sequence for $F$. Set $\overline{F} := R/(x_1, \ldots, x_{d-1})$, $\overline{m} := m/(x_1, \ldots, x_{d-1})$, and $\overline{F} := F/(x_1, \ldots, x_{d-1}) = \{F_i\}_{i \in \mathbb{Z}}$ where $F_i = F_iR$ for all $i \in \mathbb{Z}$. Then $(\overline{F}, \overline{m})$ is a 1-dimensional Cohen-Macaulay local ring and $\overline{F} = \{F_i\}_{i \in \mathbb{Z}}$ is an $F_1$-good filtration, where $F_1$ is $m$-primary. Since $J$ is a minimal reduction of $F$ with $u := r_J(F)$, $J \cdot F_n = F_{n+1}$ for all $n \geq u$, and hence $J = (x_d)$ is a minimal reduction of $F$ and $\overline{u} := r_{\overline{J}}(\overline{F}) \leq u$. Finally, we need to check that $J : \overline{F}_{\overline{u}-i} = J + F_{i+1}$ for $0 \leq i \leq \overline{u}-1$. Since $\overline{u} \leq u$, we have

$$J : \overline{F}_{\overline{u}-i} \subseteq J : F_{u-i} \subseteq J : F_{u-i} = J + F_{i+1} = J + F_{i+1}.$$  

The other inclusion is shown as follows:

$$(J + F_{i+1}) \cdot \overline{F}_{\overline{u}-i} = J \cdot \overline{F}_{\overline{u}-i} + F_{i+1} \cdot \overline{F}_{\overline{u}-i} \subseteq J \cdot \overline{F}_{\overline{u}-i} + F_{i+1} \subseteq J,$$

and hence $J + F_{i+1} \subseteq J : \overline{F}_{\overline{u}-i}$. By Theorem 3.10, $G(F)$ is Cohen-Macaulay. Since $\dim(G(F)) = 1$, we have grade $G(\overline{F})_+ = 1$, and thus by Proposition 3.2 (2), grade$(G(F)_+)^{\ast} = d$. Therefore $G(F)$ is Cohen-Macaulay. \hfill $\Box$

Remark 3.13. The sufficient conditions given in Theorem 3.12 in order that $G(F)$ be Cohen-Macaulay are not necessary conditions. For example, with $R = k[[t^5, t^6, t^9]]$ and $m = (t^5, t^6, t^9)R$ as in [HKU, Example 3.6], then $G(m)$ is Cohen-Macaulay and the ideal $J = t^5R$ is a minimal reduction of $m$ with reduction number $r_J(m) = 3$. However, $t^9 \in (J : m^2) \setminus J + m^2$.

4. The Gorenstein property for $G(F)$

In this section, we give a necessary and sufficient condition for $G(F)$ to be Gorenstein. We first state this in dimension zero. Among the equivalences in Theorem 4.2, the equivalence of (1) and (3) are due to Goto and Iai [GI, Proposition, 2.4]. We include elementary direct arguments in the proof. We use the floor function $\lfloor x \rfloor$ to denote the largest integer that is less than or equal to $x$.

Lemma 4.1. Let $(R, m)$ be a zero-dimensional Gorenstein local ring and let $F = \{F_i\}_{i \in \mathbb{Z}}$ be an $F$-good filtration. Assume that $F_u \neq 0$ and $F_{u+1} = 0$, that is, $u = r_{(0)}(F)$. Let $G := G(F) = \bigoplus_{i=0}^{u} F_i/F_{i+1} = \bigoplus_{i=0}^{u} G_{i}$ and let $S := \text{Soc}(G) = \bigoplus_{i=0}^{u} S_i$ denote the socle of $G$. Then the following hold:

1. $G_0 = F_0/F_1$.
2. $G_{i+1} = G_{i} / (m G_{i})$ for $0 \leq i \leq u$.
3. $\text{Soc}(G) = S$.
4. $G_{i}$ is a Cohen-Macaulay module for $0 \leq i \leq u$.
5. $G$ is a Gorenstein module.

Proof. The proof follows from the definition of $G(F)$ and the properties of $F$-good filtrations.
We may assume that $G$ denote the socle of $G_1$. For a Gorenstein local ring, we have $S$ contains $F$.

Let $R=\mathbb{F}_p$ for some prime $p$. Assume that $S_0=0$ and generate $R$. Theorem 4.2.

Proof. (1): We may assume that $u>0$. Let $k=R/\mathfrak{m}$ and write $\mathfrak{M}=\mathfrak{m}/F_1$ for the unique maximal homogeneous ideal of $G$. For $0\leq i \leq u$ we have $S_1=\mathfrak{m}$.

(2): $S_u=F_u \cap (0: \mathfrak{m})$, because $F_{u+i}=0$ for $i \geq 1$ and $0: \mathfrak{m} \subseteq 0: F_1 \subseteq \cdots \subseteq 0: F_u$.

(3): Since $S_u=0:R_u \mathfrak{m} \subseteq 0:R_u F_1= F_u$ and $(R, \mathfrak{m})$ is a zero-dimensional Gorenstein local ring, we have $S_u \cong k$. □

**Theorem 4.2.** Let $(R, \mathfrak{m})$ be a zero-dimensional Gorenstein local ring and let $\mathcal{F}=\{F_i\}_{i \in \mathbb{Z}}$ be an $F_1$-good filtration. Assume that $F_u \neq 0$ and $F_{u+1}=0$, that is, $u=r_{(0)}(\mathcal{F})$. Let $G:=G(\mathcal{F})=\bigoplus_{i=0}^u F_i/F_{i+1} = \bigoplus_{i=0}^u G_i$ and let $S:=\text{Soc}(G)=\bigoplus_{i=0}^u S_i$ denote the socle of $G$. The following are equivalent:

1. $G(\mathcal{F})$ is Gorenstein.
2. $S_i=0$ for $0 \leq i \leq u-1$.
3. $0:F_{u-i}=F_{i+1}$ for $0 \leq i \leq u-1$.
4. $0:F_{u-i}=F_{i+1}$ for $0 \leq i \leq \lfloor \frac{u-1}{2} \rfloor$.
5. $\lambda(G_i)=\lambda(G_{u-i})$ for $0 \leq i \leq \lfloor \frac{u-1}{2} \rfloor$.

Proof. (1) $\iff$ (2): $G(\mathcal{F})$ is Gorenstein if and only if $\dim_k S=1$ if and only if $S_i=0$ for $0 \leq i \leq u-1$, by Lemma 4.1.(3).

(2) $\implies$ (3): Suppose that $S_i=0$ for $0 \leq i \leq u-1$. Then $S=S_u \cong k$, by Lemma 4.1.(3). Hence there exists $0 \neq s^* \in S_u$ such that $S=s^*k$. Let $0 \leq i \leq u-1$. The containment ”$\supseteq$" is clear, because $F_{u+1}=0$. To see the other containment, we assume that $0:F_{u-j} \not\subseteq F_{j+1}$ for some $j$ with $0 \leq j \leq u-1$. In this case there exists an element $\beta \in 0:F_{u-j}$, but $\beta \not\in F_{j+1}$, and hence we can choose an integer $v$ with $0 \leq v \leq j$ such that $\beta \in F_v \setminus F_{v+1}$. Hence $0 \neq \beta^* = \beta + F_{v+1} \in F_v/F_{v+1}$. Since the graded ring $G$ is an essential extension of $\text{Soc}(G)$, we have $\beta^*G \cap \text{Soc}(G) \neq 0$. Then there exists a non-zero element $\xi$ such that $\xi \in \beta^*G \cap \text{Soc}(G)$. Since $S=S_u=s^*k$, we can express $s^* = \beta^* \omega^* = \beta \omega + F_{u+1}$, for some $\omega \in F_{u-v}$. Then $\beta \omega \neq 0$, because $s^* \neq 0$. This is impossible, because $\beta \in 0:F_{u-j}$ and $\omega \in F_{u-v} \subseteq F_{u-j}$, as $v \leq j$. 

(3) $\implies$ (4): This is clear.

(4) $\implies$ (5): For $0 \leq i \leq \left\lfloor \frac{u-1}{2} \right\rfloor$, we have
\[
\lambda(G_{u-i}) = \lambda(F_{u-i}/F_{u-i+1})
\]
\[
= \lambda(R/F_{u-i+1}) - \lambda(R/F_{u-i})
\]
\[
= \lambda(0 : F_{u-i+1}) - \lambda(0 : F_{u-i}) \quad \text{by [BH, Proposition 3.2.12]}
\]
\[
= \lambda(F_{i}) - \lambda(F_{i+1}) \quad \text{by condition (4)}
\]
\[
= \lambda(F_{i}/F_{i+1}) = \lambda(G_{i}).
\]

(5) $\implies$ (3): For $0 \leq i \leq u - 1$, we have
\[
\lambda(F_{i+1}) = \lambda(F_{i+1}/F_{u+1}) \quad \text{since } F_{u+1} = 0
\]
\[
= \lambda(G_{i+1}) + \lambda(G_{i+2}) + \cdots + \lambda(G_{u})
\]
\[
= \lambda(G_{u-(i+1)}) + \lambda(G_{u-(i+2)}) + \cdots + \lambda(G_{u-u}) \quad \text{by condition (5)}
\]
\[
= \lambda(R/F_{u-i}) = \lambda(0 : F_{u-i}) \quad \text{by [BH, Proposition 3.2.12]}
\]

Since $F_{u+1} = 0$, we have $F_{i+1} \subseteq 0 : F_{u-i}$ for $0 \leq i \leq u - 1$. We conclude that $F_{i+1} = 0 : F_{u-i}$, because these two ideals have the same length.

(3) $\implies$ (2): Let $0 \leq i \leq u - 1$. By Lemma 4.1.(1), we have
\[
S_i = \frac{F_i \cap (F_{i+1} : m) \cap (F_{i+2} : F_1) \cap \cdots \cap (F_u : F_{u-(i+1)}) \cap (F_{u+1} : F_{u-i}) \cap \cdots \cap (F_{i+u+1} : F_u)}{F_{i+1}}
\]
\[
\subseteq \frac{F_{u+1} : F_{u-i}}{F_{i+1}}
\]
\[
= \frac{0 : F_{u-i}}{F_{i+1}} \quad \text{since } F_{u+1} = 0
\]
\[
= \frac{F_{i+1}}{F_{i+1}} \quad \text{by condition (3)}.
\]

Hence $S_i = 0$ for $0 \leq i \leq u - 1$. \(\square\)

**Theorem 4.3.** Let $(R, m)$ be a $d$-dimensional Gorenstein local ring and let $\mathcal{F} = \{F_i\}_{i \in \mathbb{Z}}$ be an $F_1$-good filtration, where $F_1$ is $m$-primary. Assume there exists a minimal reduction $J$ of $\mathcal{F}$ such that $\mu(J) = d$, and let $u := r_f(\mathcal{F})$ denote the reduction number of $\mathcal{F}$ with respect to $J$. The following are equivalent:

(1) $G(\mathcal{F})$ is Gorenstein.

(2) $J : F_{u-i} = J + F_{i+1}$ for $0 \leq i \leq u - 1$.

(3) $J : F_{u-i} = J + F_{i+1}$ for $0 \leq i \leq \left\lfloor \frac{u-1}{2} \right\rfloor$.

**Proof.** The equivalence of items (2) and (3) follows from the double annihilator property in the zero-dimensional Gorenstein local ring $R/J$, see, for example [BH,
To prove the equivalence of (1) and (2), by Theorem 3.12, we may assume that \( G(\mathcal{F}) \) is Cohen-Macaulay. Choose \( x_1, \ldots, x_d \) in \( F_1 \) such that \( J = (x_1, \ldots, x_d)R \) and \( x_1, \ldots, x_d \) is a superficial sequence for \( \mathcal{F} \). Since \( G(\mathcal{F}) \) is Cohen-Macaulay, the leading forms \( x_1^*, \ldots, x_d^* \) in \( F_1/F_2 \) are a \( G(\mathcal{F}) \)-regular sequence by Proposition 3.2, and hence we have the isomorphism

\[
G(\mathcal{F})/(x_1^*, \ldots, x_d^*) \cong G(\mathcal{F}/J)
\]
as graded \( R \)-algebras. Set \( \bar{R} := R/J, \bar{m} := m/J, \) and \( \bar{\mathcal{F}} := \mathcal{F}/J = \{\bar{F}_i\}_{i \in \mathbb{Z}} \), where \( \bar{F}_i = F_iR \) for all \( i \in \mathbb{Z} \). Then \( (\bar{R}, \bar{m}) \) is a zero-dimensional Gorenstein local ring and \( \bar{\mathcal{F}} \) is a \( \bar{F}_1 \)-good filtration with \( \bar{F}_{u+1} = 0 \) and \( \bar{F}_u \neq 0 \). To show the last equality suppose that \( \bar{\mathcal{F}}_u = 0 \). In this case \( F_u \subseteq J \), and hence \( F_u = F_u \cap J = JF_{u-1} \), as \( G(\mathcal{F}) \) is Cohen-Macaulay. This is impossible since \( u := r_J(\mathcal{F}) \). Now we have

\[
G(\mathcal{F}) \text{ is Gorenstein} \iff G(\bar{\mathcal{F}}) \text{ is Gorenstein}
\]

\[
\iff 0 : \bar{F}_{u-i} = \bar{F}_{i+1} \quad \text{for} \quad 0 \leq i \leq u - 1 \quad \text{by Theorem 4.2}
\]

\[
\iff J : F_{u-i} = J + F_{i+1} \quad \text{for} \quad 0 \leq i \leq u - 1.
\]

This completes the proof of Theorem 4.3.

The following is an immediate consequence of Theorem 4.3 for the case of reduction number two.

**Corollary 4.4.** Let \( (R, m) \) be a \( d \)-dimensional Gorenstein local ring and let \( \mathcal{F} = \{F_i\}_{i \in \mathbb{Z}} \) be an \( F_1 \)-good filtration, where \( F_1 \) is \( m \)-primary. Assume there exists a minimal reduction \( J \) of \( \mathcal{F} \) such that \( \mu(J) = d \) and that \( r_J(\mathcal{F}) = 2 \). Then:

\[
G(\mathcal{F}) \text{ is Gorenstein} \iff J : F_2 = F_1.
\]

Corollary 4.5 deals with the problem of lifting the Gorenstein property of associated graded rings. Notice we are not assuming that \( G(\mathcal{F}) \) is Cohen-Macaulay.

**Corollary 4.5.** Let \( (R, m) \) be a \( d \)-dimensional Cohen-Macaulay local ring and let \( \mathcal{F} = \{F_i\}_{i \in \mathbb{Z}} \) be an \( F_1 \)-good filtration, where \( F_1 \) is \( m \)-primary. Assume there exists a minimal reduction \( J \) of \( \mathcal{F} \) such that \( \mu(J) = d \) and that \( F_u \not\subset J \) for \( u := r_J(\mathcal{F}) \). Set \( \bar{R} := R/J \) and \( \bar{\mathcal{F}} := \mathcal{F}/J = \{\bar{F}_i\}_{i \in \mathbb{Z}} \). If \( G(\bar{\mathcal{F}}) \) is Gorenstein, then \( G(\mathcal{F}) \) is Gorenstein.

**Proof.** If \( G(\bar{\mathcal{F}}) \) is Gorenstein, then \( \bar{R} \) is Gorenstein, and hence \( R \) is also Gorenstein, because \( (R, m) \) is Cohen-Macaulay. The condition \( F_u \not\subset J \) implies that \( \bar{F}_u \neq 0 \) and
\[ F_{u+1} = 0. \] Hence \( r_J(F) = r_{(0)}(F) \). The assertion now follows from Theorem 4.2 and Theorem 4.3. \( \square \)

The following theorem is a special case of a result of Goto and Nishida that characterizes the Gorenstein property of the Rees algebra \( R(F) \).

**Theorem 4.6.** (Goto and Nishida [GN]) Let \( (R, \mathfrak{m}) \) be a Gorenstein local ring of dimension \( d \geq 2 \) and let \( F = \{F_i\}_{i \in \mathbb{Z}} \) be an \( F_1 \)-good filtration, where \( F_1 \) is \( \mathfrak{m} \)-primary. Let \( J \) be a reduction of \( F \) with \( \mu(J) = d \). The following are equivalent:

1. The Rees algebra \( R(F) \) is Gorenstein.
2. The associated graded ring \( G(F) \) is Gorenstein and \( a(G(F)) = -2 \).
3. The associated graded ring \( G(F) \) is Gorenstein and \( r_J(F) = d - 2 \).

In Theorem 4.7 and Corollary 4.9, we generalize to the case of filtrations results of Herrmann-Huneke-Ribbe [HHR, Theorem 2.5]

**Theorem 4.7.** Let \( (R, \mathfrak{m}) \) be a regular local ring of dimension \( d \geq 2 \) and let \( F = \{F_i\}_{i \in \mathbb{Z}} \) be an \( F_1 \)-good filtration, where \( F_1 \) is \( \mathfrak{m} \)-primary. Let \( J \) be a reduction of \( F \) with \( \mu(J) = d \) and \( r_J(F) = u \). If \( G(F) \) is Cohen-Macaulay, then \( G(F/J) \) has a nonzero homogeneous socle element of degree \( \leq d - 2 \).

**Proof.** We have

\[ F_j \subseteq F_j : \mathfrak{m} \subseteq F_j : F_1 = F_j - 1 \] for all integers \( j \),

where the last equality holds by Lemma 2.4(1) because \( G(F) \) is Cohen-Macaulay. Since \( J \) is a reduction of \( F \) with \( r_J(F) = u \), we have \( F_j \subseteq J^{j-u} \) for all \( j \geq u \), hence

\[ F_j : \mathfrak{m} \subseteq J^{j-u} : \mathfrak{m} \subseteq J^{j-u} : J = J^{j-u-1} \subseteq J, \]

whenever \( j \geq u + 1 \). Thus there exists an integer \( k \geq 1 \) such that

\[ F_k : \mathfrak{m} \not\subseteq F_k + J \quad \text{and} \quad F_j : \mathfrak{m} \subseteq F_j + J, \] for all \( j \geq k + 1 \).

Let \( v \in (F_k : \mathfrak{m}) + J \setminus F_k + J \), then \( v \in F_{k-1} + J \setminus F_k + J \) by (3). Thus the image \( \overline{v} \) of \( v \) in \( R/J \) has the property that its leading form \( \overline{v}^* \in G(F/J) \) is a nonzero element in \( [G(F/J)]_{k-1} \).

**Claim 4.8.** \( : \overline{v}^* \in \text{Soc} (G(F/J)) \).

**Proof of Claim.** Let \( \alpha \) be any homogeneous element in \( \mathfrak{N} \), where \( \mathfrak{N} \) is the unique maximal (homogeneous) ideal of the zero-dimensional graded ring \( G(F/J) \). We
show that \( \alpha \cdot \mathfrak{v}^s = 0 \). We have two cases:

(Case i) : Assume that \( \deg \alpha = n \geq 1 \). Write \( \alpha = y + (F_{n+1} + J) \), where \( y \in F_n \).

Then we have

\[
\alpha \cdot \mathfrak{v}^s = yv + (F_{n+k} + J) = 0,
\]

since \( yv \in F_n((F_k : \mathfrak{m}) + J) \subseteq (F_n F_k : \mathfrak{m}) + J \subseteq (F_{n+k} : \mathfrak{m}) + J \subseteq F_{n+k} + J \), where the last inequality holds by (4).

(Case ii) : Assume that \( \deg \alpha = 0 \). Then \( \alpha = z + (F_1 + J) \), where \( z \in \mathfrak{m} \), and we have

\[
\alpha \cdot \mathfrak{v}^s = zv + (F_k + J) = 0,
\]

where the last equality holds because \( v \in (F_k : \mathfrak{m}) + J \) and \( z \in \mathfrak{m} \). This completes the proof of Claim 4.8.

Since \( \mathcal{F} \) is an \( F_1 \)-good filtration, we have \( F^n_1 \subseteq F_n \subseteq \overline{F^n_1} \) for all \( n \geq 0 \), where \( \overline{F^n_1} \) denotes the integral closure of \( F^n_1 \). Hence \( F_n \subseteq \overline{F^n_1} \) for all \( n \geq 0 \). We have

\[
F_d : \mathfrak{m} \subseteq F_d : \mathfrak{m}^{d-1} \subseteq \overline{F_d} : \mathfrak{m}^{d-1} \subseteq \overline{F_d} : \mathfrak{m}^{d-1} \subseteq J,
\]

where the last inclusion follows from a result of Lipman [L, Corollary 1.4.4]. Hence we have

\[
F_j : \mathfrak{m} \subseteq F_d : \mathfrak{m} \subseteq J \quad \text{for all} \quad j \geq d.
\]

Thus by (4), we have \( k \leq d - 1 \). Therefore \( \deg \mathfrak{v}^s = k - 1 \leq d - 2 \). Since \( \mathfrak{v}^s \in \text{Soc}(G(\mathcal{F})) \) by Claim 4.8, the proof of Theorem 4.7 is complete.

**Corollary 4.9.** Let \((R, \mathfrak{m})\) be a regular local ring of dimension \( d \geq 2 \) and let \( \mathcal{F} = \{F_i\}_{i \in \mathbb{Z}} \) be an \( F_1 \)-good filtration, where \( F_1 \) is \( \mathfrak{m} \)-primary. Let \( J \) be a reduction of \( \mathcal{F} \) with \( \mu(J) = d \). If \( F_{i+1} \subseteq \mathfrak{m} F_i \) for each \( i \geq d - 1 \) and \( G(\mathcal{F}) \) is Gorenstein, then \( r_J(\mathcal{F}) \leq d - 2 \).

**Proof.** Since \( G(\mathcal{F}) \) is Gorenstein, Proposition 3.3 shows that \( G(\mathcal{F} / J) \) is Gorenstein, as well. Hence Theorem 4.7 implies that \( [G(\mathcal{F} / J)]_i = 0 \) for all \( i \geq d - 1 \). Thus for \( i \geq d - 1 \) we have

\[
0 = [G(\mathcal{F} / J)]_i = \frac{F_i + J}{F_{i+1} + J} \cong \frac{F_i}{F_{i+1} + (J \cap F_i)} = \frac{F_i}{F_{i+1} + JF_{i-1}},
\]

where the last equality holds again by Proposition 3.3. Thus for all \( i \geq d - 1 \), we have

\[
F_i = F_{i+1} + JF_{i-1}, \tag{5}
\]
and hence by Nakayama’s Lemma, $F_i = JF_{i-1}$ since $F_{i+1} \subseteq \mathfrak{m}F_i$. Therefore $r_f(\mathcal{F}) \leq d - 2$.

5. **Integral closure filtrations of monomial parameter ideals**

In this section we examine the integral closure $\mathcal{F}$ associated to a monomial parameter ideal in a polynomial ring. We use Theorem 4.3 to give necessary and sufficient conditions in order that $G(\mathcal{F})$ be Gorenstein. We demonstrate that $G(\mathcal{F})$ and even $R(\mathcal{F})$ may be Gorenstein and yet $\mathcal{F}$ is not an ideal-adic filtration.

**Setting 5.1.** Let $R := k[x_1, \ldots, x_d]$ be a polynomial ring in $d \geq 1$ variables over the field $k$. Let $a_1, \ldots, a_d$ be positive integers and let $J := (x_1^{a_1}, \ldots, x_d^{a_d})R$ be a monomial parameter ideal. Let $L := \text{LCM}(a_1, \ldots, a_d)$ denote the least common multiple of the integers $a_1, \ldots, a_d$, and let $\mathcal{F} := \{J^n\}_{n \in \mathbb{Z}}$ be the integral closure filtration associated to $J$. The ideal $J$ has a unique Rees valuation $v$ that is defined as follows: $v(x_i) := L/a_i$ for each $i$ with $1 \leq i \leq d$. Then for every polynomial $f \in R$ one defines $v(f)$ to be the minimum of the $v$-value of a nonzero monomial occurring in $f$ (cf. [SH, (10.18), p. 209]). The Rees valuation $v$ determines the integral closure $J^n$ of every power $J^n$ of $J$. We have $J^n = \{f \in R \mid v(f) \geq nL\}$. Each of the ideals $J^n$ is again a monomial ideal. Let $\mathfrak{m} := (x_1, \ldots, x_d)R$ denote the graded maximal ideal of $R$. Notice that $s := x_1^{a_1-1} \cdots x_d^{a_d-1} \in (J : \mathfrak{m}) \setminus J$ is a socle element modulo $J$. Since $R$ is Gorenstein and $J$ is a parameter ideal, we have $(J, s)R = J : \mathfrak{m}$, and $s \in K$ for each ideal $K$ of $R$ that properly contains $J$.

**Remark 5.2.** The filtrations $\mathcal{F} = \{J^n\}_{n \geq 0}$ of Setting 5.1 may also be described as the integral closure filtrations associated to zero-dimensional monomial ideals having precisely one Rees valuation [SH, Theorem 10.3.5].

**Lemma 5.3.** Let the notation be as in Setting 5.1. For each integer $k$, let $I_k := \{f \in R \mid v(f) \geq k\}$. We have:

1. Let $\alpha \in R$ be a monomial, then $\alpha \not\in J \iff s \in \alpha R$.
2. Let $K$ be a monomial ideal, then $K \subseteq J \iff s \not\in K$.
3. Each $I_k$ is a monomial ideal, and $I_k \subseteq J \iff k \geq v(s) + 1$.
4. The reduction number $r_f(\mathcal{F})$ satisfies $r_f(\mathcal{F}) = u \iff s \in J^u \setminus J^{u+1}$.

**Proof.** For item (1), let $K = (J, \alpha)R$. If $\alpha \not\in J$ then $s \in K$. Since $K$ is a monomial ideal, $s$ is a multiple of some monomial generator of $K$. Since $s \not\in J$, we must have
s is a multiple of $\alpha$. Conversely, if $s \in \alpha R$ then $\alpha \notin J$ because $s \notin J$. Items (2) and (3) follow from item (1). For item (4), a theorem of Hochster implies that $R(\mathcal{F})$ is Cohen-Macaulay [H, Theorem 1], [BH, Theorem 6.3.5(a)]. Therefore $G(\mathcal{F})$ is Cohen-Macaulay, which gives $r_J(\mathcal{F}) = s_J(\mathcal{F}) := \min \{ n \mid J^{n+1} \subseteq J \}$. Hence by item (2), we have item (4).

\section*{Proposition 5.4.}
Let the notation be as in Setting 5.1. Write

$$v(x_1) + v(x_2) + \cdots + v(x_d) = jL + p,$$

where $j \geq 0$ and $1 \leq p \leq L$.

Then the reduction number satisfies $r_J(\mathcal{F}) = d - (j + 1)$.

\begin{proof}
Observe that

$$v(s) = dL - (v(x_1) + v(x_2) + \cdots + v(x_d))$$

$$= dL - (jL + p) \quad \text{by hypothesis}$$

$$= (d - j)L - p.$$ 

Therefore $(d - (j + 1))L \leq v(s) < (d - j)L$ and hence $s \in J^{d-(j+1)} \setminus J^{d-j}$. Thus $r_J(\mathcal{F}) = d - (j + 1)$ by Lemma 5.3(4).
\end{proof}

\section*{Lemma 5.5.}
Let the notation be as in Setting 5.1 and let $\sum_{k=1}^{d} v(x_k) = jL + p$, where $j \geq 0$ and $1 \leq p \leq L$. The following are equivalent :

1. The associated graded ring $G(\mathcal{F})$ is Gorenstein.
2. For every integer $i \geq 0$ and every monomial $\alpha \in R$ with $s \in \alpha R$ one has

$$v(\alpha) \leq (i + 1)L - 1 \iff v(\alpha) \leq (i + 1)L - p.$$ 

\begin{proof}
Let $u := r_J(\mathcal{F})$. Proposition 5.4 shows that $v(s) = (u + 1)L - p$. For any monomial $\alpha \in R$ one has

$$\alpha \notin J + J^{i+1} \iff \alpha \notin J \quad \text{and} \quad \alpha \notin J^{i+1}$$

$$\iff s \in \alpha R \quad \text{and} \quad v(\alpha) \leq (i + 1)L - 1.$$ 

Here we have used Lemma 5.3(1) and the fact that $J^{i+1}$ is a monomial ideal.

Likewise,

$$\alpha \notin J : J^{u-1} \iff \alpha J^{u-1} \notin J$$

$$\iff s \in \alpha J^{u-1}$$

$$\iff s \in \alpha R \quad \text{and} \quad \frac{s}{\alpha} \in J^{u-1}$$

$$\iff s \in \alpha R \quad \text{and} \quad v(s) - v(\alpha) \geq (u - i)L$$

$$\iff s \in \alpha R \quad \text{and} \quad v(\alpha) \leq (i + 1)L - p.$$
Thus, item (2) above holds if and only if 
\[ J + J^{i+1} = J : J^{u-i} \] for every \( i \geq 0 \) or, equivalently, for \( 0 \leq i \leq u - 1 \). But this means that \( G(\mathcal{F}) \) is Gorenstein according to Theorem 4.3. \( \square \)

We thank Paolo Mantero for showing us that \( G(\mathcal{F}) \) is Gorenstein implies \( \sum_{k=1}^{d} v(x_k) \equiv 1 \mod L \) as stated in Theorem 5.6.

**Theorem 5.6.** Let the notation be as in Setting 5.1. Then we have

\[ G(\mathcal{F}) \text{ is Gorenstein} \iff \sum_{k=1}^{d} v(x_k) \equiv 1 \mod L. \]

**Proof.** If \( p = 1 \), then \( G(\mathcal{F}) \) is Gorenstein according to Lemma 5.5. To show the converse notice that for \( i > 0 \), \( (i+1)L - 1 \) is in the numerical semigroup generated by the relatively prime integers \( v(x_1), \ldots, v(x_d) \). As \( L = a_k v(x_k) \), we may subtract a multiple of \( L \) to obtain \( (i+1)L - 1 = c_1 v(x_1) + \cdots + c_d v(x_d) \) for some integer \( i \) and \( c_k \) integers with \( 0 \leq c_k \leq a_k - 1 \). Clearly \( i \geq 0 \). Write \( \alpha := x_1^{c_1} \cdots x_d^{c_d} \). Now \( \alpha \in R \) is a monomial with \( s \in \alpha R \) and \( v(\alpha) = (i+1)L - 1 \). If \( G(\mathcal{F}) \) is Gorenstein then by Lemma 5.5, \( v(\alpha) \leq (i+1)L - p \). Therefore \( p \leq 1 \), which gives \( p = 1 \). \( \square \)

**Corollary 5.7.** Let the notation be as in Setting 5.1 and assume that \( d \geq 2 \). The following are equivalent:

1. \( \sum_{k=1}^{d} v(x_k) = L + 1 \).
2. \( G(\mathcal{F}) \) is Gorenstein and \( r_J(\mathcal{F}) = d - 2 \).
3. The Rees algebra \( R(\mathcal{F}) \) is Gorenstein.

**Proof.** The equivalence of items (1) and (2) follows from Proposition 5.4 and Theorem 5.6, whereas the equivalence of items (2) and (3) is a consequence of Theorem 4.6. \( \square \)

**Remark 5.8.** Assume notation as in Setting 5.1. Since \( G(\mathcal{F}) \) is Cohen-Macaulay, Proposition 3.8 implies that the maximal value of the reduction number \( r_J(\mathcal{F}) \) is \( d - 1 \). For every dimension \( d \), the minimal value of \( r_J(\mathcal{F}) \) is zero as can be seen by taking \( a_1 = \cdots = a_{d-1} = 1 \). If \( d \geq 2 \) and all the exponents \( a_k \) are assumed to be greater than or equal to 2, then the inequalities \( L/2 \geq L/a_k \) along with Lemma 5.3 imply that the possible values of the reduction number \( u := r_J(\mathcal{F}) \) are all integers \( u \) such that \( |\frac{d}{2}| \leq u \leq d - 1 \).
Setting 5.9. Let the notation be as in Setting 5.1. Let $e$ be a positive integer and let $y_1, \ldots, y_e$ be indeterminates over $R$. Let $S := R[y_1, \ldots, y_e]$. Let $b_1, \ldots, b_e$ be positive integers and let $K := (J, y_1^{b_1}, \ldots, y_e^{b_e})S$ be a monomial parameter ideal of $S$. Let $E := \{K^n\}_{n \geq 0}$ denote the integral closure filtration associated to the ideal $K$. Let $w$ denote the Rees valuation of $K$, and let $t := x_1^{a_1-1} \cdots x_d^{a_d-1} y_1^{b_1-1} \cdots y_e^{b_e-1}$ denote the socle element modulo the ideal $K$.

Remark 5.10 records several basic properties relating to the filtrations $\mathcal{F}$ and $\mathcal{E}$.

Remark 5.10. Assume notation as in Setting 5.1 and 5.9. Then the following hold:

1. For each positive integer $n$ we have $J^n = K^n \cap R$ and $(\mathcal{F})^n = (\mathcal{K})^n \cap R$.
2. If $\mathcal{E}$ is an ideal-adic filtration, then $\mathcal{F}$ is an ideal-adic filtration.
3. The reduction numbers satisfy the inequality $r_J(\mathcal{F}) \leq r_K(\mathcal{E})$.
4. The Rees valuation $w$ restricted to $R$ defines a valuation that is equivalent to the Rees valuation $v$, that is, these two valuations determine the same valuation ring.

Corollary 5.11. Assume notation as in Setting 5.1 and 5.9. For each monomial parameter ideal $J$ of $R$ there exists an extension $S = R[y_1, \ldots, y_e]$ and a monomial parameter ideal $K = (J, y_1^{b_1}, \ldots, y_e^{b_e})S$ such that $G(\mathcal{E})$ is Gorenstein where $\mathcal{E} = \{K^n\}_{n \geq 0}$ is the integral closure filtration associated to $K$.

Proof. Let $J = (x_1^{a_1}, \ldots, x_d^{a_d})R$, let $L$ be the least common multiple of $a_1, \ldots, a_d$ and let $v$ denote the Rees valuation of $J$. Write $\sum_{k=1}^d v(x_k) = jL + p$, where $j \geq 0$ and $1 \leq p \leq L$. If $p = 1$, then $G(\mathcal{F})$ is Gorenstein by Theorem 5.6 and we can take $S = R$. If $p > 1$, let $e = L - p + 1$ and let $S = R[y_1, \ldots, y_e]$ and $K = (J, y_1^{L_1}, \ldots, y_e^{L_e})S$. Then $w(y_k) = 1$ for each $k$ with $1 \leq k \leq e$. Also $w$ restricted to $R$ is equal to $v$ and we have $\sum_{k=1}^d w(x_k) + \sum_{k=1}^e w(y_k) = jL + p + L - p + 1 = (j + 1)L + 1$.

Therefore $G(\mathcal{E})$ is Gorenstein by Theorem 5.6. \qed

Remark 5.12. With the notation of Corollary 5.11, we have:

1. If $\sum_{k=1}^d v(x_k) = jL + p$, where $1 \leq p \leq L$, then from the construction used in the proof of Corollary 5.11 one may obtain for each positive $m$ a
polynomial extension \( S \) and a monomial parameter ideal \( K \) of \( S \) such that
\[ r_K(\mathcal{E}) = \dim S - (j + m), \]
where \( \mathcal{E} = \{K^n\}_{n \geq 0} \).

(2) If \( \sum_{k=1}^{d} v(x_k) \leq L \), then by Corollary 5.7 there exists a monomial parameter ideal \( K = (J, y_1^{b_1}, \ldots, y_e^{b_e})S \) such that the Rees algebra \( R(\mathcal{E}) \) is Gorenstein.

Example 5.13 demonstrates the existence of monomial parameter ideals \( K \) such that the integral closure filtration \( \mathcal{E} = \{K^n\}_{n \geq 0} \) has the following properties:

1. The reduction number satisfies \( r_K(\mathcal{E}) = d - 2 \).
2. The associated graded ring \( G(\mathcal{E}) \) and the Rees algebra \( R(\mathcal{E}) \) are Gorenstein.
3. The filtration \( \mathcal{E} \) is not an ideal-adic filtration.

**Example 5.13.** Let \( R = k[x_1, x_2, x_3] \) and let \( J = (x_2^2, x_2^3, x_3^3)R \). Then \( L = 42 \) and \( v(x_1) = 21, v(x_2) = 14 \) and \( v(x_3) = 6 \). Thus \( \sum_{i=1}^{3} v(x_i) = 41 = L - 1 \). Hence \( G(\mathcal{F}) \) is not Gorenstein. Notice that \( r_J(\mathcal{F}) = 2 \) and
\[ \mathcal{J} = (J, x_1x_2^4, x_1x_2x_3^2, x_1x_2^3, x_2x_3, x_2^2x_3^3)R. \]
The element \( x_1x_2^3x_3^6 \in \overline{\mathcal{J}^2} \setminus (\mathcal{J})^2 \). Hence the filtration \( \mathcal{F} = \{\overline{J^n}\}_{n \geq 0} \) is not an ideal-adic filtration. Let \( S = R[y_1, y_2] \) and let \( K = (J, y_1^{42}, y_2^{42})S \). Then we have \( w(y_1) = w(y_2) = 1 \) and \( w(x_i) = v(x_i) \) for each \( i \). Hence the sum of the \( w \)-values of the variables is equal to \( L + 1 \). Therefore \( G(\mathcal{E}) \) is Gorenstein. Notice that also the Rees algebra \( R(\mathcal{E}) \) is Gorenstein by Corollary 5.7.

Alternatively, one could let \( S = R[y_1] \) and let \( K = (J, y_1^{21})S \). Again the sum of the \( w \)-values of the variables is \( L + 1 \), so \( R(\mathcal{E}) \) and \( G(\mathcal{E}) \) are Gorenstein. In both cases \( r_K(\mathcal{E}) \) is the dimension of \( S \) minus two. In the previous case \( r_K(\mathcal{E}) = 3 \) and in this case \( r_K(\mathcal{E}) = 2 \).

### 6. The Quasi-Gorenstein Property for \( R'(\mathcal{F}) \)

Let \( (R, \mathfrak{m}) \) be a \( d \)-dimensional Gorenstein local ring and let \( \mathcal{F} = \{F_i\}_{i \in \mathbb{Z}} \) be an \( F_1 \)-good filtration in \( R \), where \( \text{ht}(F_1) = g > 0 \). Assume there exists a reduction \( J \) of \( \mathcal{F} \) with \( \mu(J) = g \) and reduction number \( u := r_J(\mathcal{F}) \). In Theorem 6.1, we prove that the extended Rees algebra \( R'(\mathcal{F}) \) is quasi-Gorenstein with \( a \)-invariant \( b \) if and only if \( J^n : F_u = F_{n+b-u+g-1} \) for every \( n \in \mathbb{Z} \). If \( G(\mathcal{F}) \) is Cohen-Macaulay, we prove in Theorem 6.2 that the maximal degree of a homogeneous minimal generator of the canonical module \( \omega_{G(\mathcal{F})} \) is at most \( g \) and that of the canonical module \( \omega_{R'(\mathcal{F})} \) is...
at most $g - 1$. With the same hypothesis, we prove in Theorem 6.3 that $R'(\mathcal{F})$ is Gorenstein if and only if $J^u : F_u = F_u$.

**Theorem 6.1.** Let $(R, \mathfrak{m})$ be a $d$-dimensional Gorenstein local ring and let $\mathcal{F} = \{F_i\}_{i \in \mathbb{Z}}$ be an $F_1$-good filtration of ideals in $R$. Let $F_1$ be an equimultiple ideal of $R$ with $\operatorname{ht} F_1 = g > 0$ and $J = (x_1, x_2, \ldots, x_g)R \subseteq F_1$ be a minimal reduction of $\mathcal{F}$. Let $R'(\mathcal{F}) = \bigoplus_{i \in \mathbb{Z}} F_it^i$. Then the following assertions are true.

1. $R'(\mathcal{F})$ has the canonical module $\omega_{R'(\mathcal{F})} = \bigoplus_{i \in \mathbb{Z}} (J^{i+u} : F_u)t^{i+(g-1)}$.

2. $R'(\mathcal{F})$ is quasi-Gorenstein with $a$-invariant $b \iff J^i : F_u = F_{i+b-u+g-1}$ for all $i \in \mathbb{Z}$.

**Proof.** (1) Let $K := \operatorname{Quot}(R)$ denote the total ring of quotients of $R$. Let $A := R[Jt, t^{-1}] \subseteq C := R'(\mathcal{F}) = \bigoplus_{i \in \mathbb{Z}} F_it^i$. Notice that $G(J) \cong A/t^{-1}A$, where $t^{-1}$ is a homogeneous $A$-regular element of degree -1. Since $J = (x_1, x_2, \ldots, x_g)R$ is generated by a regular sequence, $G(J) \cong (R/J)[X_1, X_2, \ldots, X_g]$ is a standard graded polynomial ring in $g$-variables over a Gorenstein local ring $R/J$, whence $A$ is Gorenstein and $\omega_A \cong A(-g+1) \cong A^g$. Since $C$ is a finite extension of $A$ and $\operatorname{Quot}(A) = \operatorname{Quot}(C) = K(t)$ ($\because g > 0$), we have that

$$\omega_C \cong \operatorname{Ext}^0_A(C, \omega_A) = \operatorname{Hom}_A(C, A(-g+1))$$

$$\cong \operatorname{Hom}_A(C, At^{g-1})$$

$$\cong \operatorname{Hom}_A(C, A)t^{g-1}$$

$$\cong (A :_{K(t)} C)t^{g-1}$$

$$= (A :_{R[t, t^{-1}]} C)t^{g-1},$$

where the last equality holds because

$$A :_{K(t)} C \subseteq A :_{K(t)} A \subseteq A \subseteq R[t, t^{-1}].$$

We have $\bigoplus_{i \in \mathbb{Z}} [\omega_C]_i t^i = \bigoplus_{i \in \mathbb{Z}} [A :_{R[t, t^{-1}]} C]_i t^{i+g-1}$. Since $J$ is complete intersection and $J^{i+j+1} = J^{i+j}$ for all $i$ and $j$, we have

$$[\omega_C]_i = [A :_{R[t, t^{-1}]} C]_i = \cap_j (J^{i+j} : F_j) = J^{i+u} : F_u,$$
for all $i \in \mathbb{Z}$. Therefore \( \omega_C = \bigoplus_{i \in \mathbb{Z}} [\omega_C]_i t^i = \bigoplus_{i \in \mathbb{Z}} (J^{i+u} : F_u) t^{i+g-1} \).

(2) \( C \) is quasi-Gorenstein with \( b := a(C) \) if and only if
\[
\omega_C \cong C(b) \iff \bigoplus_{i \in \mathbb{Z}} [\omega_C]_i t^i = \bigoplus_{i \in \mathbb{Z}} (J^{i+u} : F_u) t^{i+g-1} = \bigoplus_{i \in \mathbb{Z}} F_{i+b} t^i
\]
\[
\iff \bigoplus_{i \in \mathbb{Z}} (J^i : F_u) t^{i+(g-1)-u} = \bigoplus_{i \in \mathbb{Z}} F_i t^{i-b}
\]
\[
\iff J^i : F_u = F_{i+b+(g-1)-u} \quad \text{for all} \quad i \in \mathbb{Z}.
\]

This completes the proof of Theorem 6.1. \( \square \)

**Theorem 6.2.** Let \((R, \mathfrak{m})\) be a \(d\)-dimensional Gorenstein local ring and let \( \mathcal{F} = \{F_i\}_{i \in \mathbb{Z}} \) be an \( F_1 \)-good filtration of ideals in \( R \), where \( F_1 \) is an equimultiple ideal with \( \text{ht} F_1 = g > 0 \) and \( J = (x_1, x_2, \ldots, x_g)R \subseteq F_1 \) is a minimal reduction of \( \mathcal{F} \). Assume that the associated graded ring \( G(\mathcal{F}) \) is Cohen-Macaulay. Then:

1. The maximal degree of a homogeneous minimal generator of \( \omega_{G(\mathcal{F})} \) is \( \leq g \).
2. The maximal degree of a homogeneous minimal generator of \( \omega_{R^i(\mathcal{F})} \) is \( \leq g-1 \).

**Proof.** (1) Since \( J = (x_1, x_2, \ldots, x_g)R \) is an \( R \)-regular sequence, \((R/J, \mathfrak{m}/J)\) is a Gorenstein local ring of dimension \( d - g \). We may assume that \((R/J, \mathfrak{m}/J)\) is complete. By Cohen’s Structure Theorem [BH, Theorem A.21, page 373], there exists a regular local ring \( T \) that maps surjectively onto \( R/J \), say \( T \twoheadrightarrow R/J \), and hence \( R/J \cong T/K \), where \( K = \ker \phi \). Let
\[
c := \text{codim} K = \dim T - \dim T/K = \dim T - \dim R/J.
\]
Then \( \dim T = (d-g)+c \). Notice that \( G(J) = \bigoplus_{i \geq 0} J_i/J_{i+1} \cong (R/J)[X_1, X_2, \ldots, X_g] \) is a polynomial ring in \( g \)-variables over \( R/J \). Let \( S = T[X_1, X_2, \ldots, X_g] \). Then we have
\[
S \rightarrow G(J) \rightarrow G(\mathcal{F}).
\]
Since \( G(\mathcal{F}) \) is a finite \( G(J) \)-module, \( G(\mathcal{F}) \) is a finite \( S \)-module and by assumption \( G(\mathcal{F}) \) is Cohen-Macaulay. The graded version of the Auslander-Buchbaum formula implies that \( \text{pd}_S G(\mathcal{F}) = c \). Let \( \mathbb{H}_* \) be a homogeneous minimal free resolution of \( G(\mathcal{F}) \) over \( S \)
\[
\mathbb{H}_* : 0 \rightarrow H_c \rightarrow H_{c-1} \rightarrow \cdots \rightarrow H_1 \rightarrow H_0 \rightarrow G(\mathcal{F}) \rightarrow 0.
\]
Notice that $H_c \neq 0$. Let $E_1 := \text{Hom}_S(H_c, \omega_S) = \text{Hom}_S(H_c, S(-g))$. It follows [BH, Corollary 3.3.9] that

$$E_1 : 0 \rightarrow E_{c-1} \rightarrow \cdots \rightarrow E_1 \rightarrow E_0 \rightarrow \omega_G(F) \rightarrow 0.$$ 

is a homogeneous minimal free resolution of $\omega_G(F)$ over $S$, where

$$E_i = \text{Hom}_S(H_{c-i}, \omega_S) = \text{Hom}_S(H_{c-i}, S(-g))$$

for $0 \leq i \leq c$. Since $H_c = \bigoplus_j \text{finite} S(-j)^{\beta_j} (\neq 0)$, we have

$$E_0 = \text{Hom}_S(H_c, S(-g)) = \bigoplus_j \text{finite} \text{Hom}_S(S, S)(j - g)^{\beta_j} = \bigoplus_j S(j - g)^{\beta_j}.$$ 

Thus the maximal degree of a homogeneous minimal generator of $\omega_G(F)$ is $\leq g - j$ and this is $\leq g$ since $j \geq 0$.

(2) Let $C = R^1 \langle F \rangle$. Since $G(F) \cong C / t^{-1} C$ and $t^{-1}$ is a non-zero-divisor of $C$, we have

$$G(F) \text{ is Cohen-Macaulay} \iff C \text{ is Cohen-Macaulay.}$$

By [BH, Corollary 3.6.14], we have

$$\omega_G(F) = \omega_{C / t^{-1} C} \cong \left( \omega_C / t^{-1} \omega_C \right) (\text{deg } t^{-1}) = \left( \omega_C / t^{-1} \omega_C \right) (-1).$$

That is, we have

$$\bigoplus_{i \in \mathbb{Z}} \omega_{G(F)} | i = \left( \omega_C / t^{-1} \omega_C \right) (-1) = \bigoplus_{i \in \mathbb{Z}} \left[ \omega_C / t^{-1} \omega_C \right] (-1) = \bigoplus_{i \in \mathbb{Z}} \omega_{C / t^{-1} C} |_{i-1}.$$ 

Letting $g(-)$ denote maximal degree of a minimal homogeneous generator, by (1), we have

$$g(\omega_{G(F)}) \leq g \iff g\left( \omega_C / t^{-1} \omega_C \right) \leq g - 1.$$ 

Since $t^{-1}$ is a non-zero-divisor on $\omega_C$, the graded version of Nakayama’s lemma ([BH, Exercise 1.5.24]) implies that $g(\omega_C) \leq g - 1$. \qed

**Theorem 6.3.** Let $(R, \mathfrak{m})$ be a $d$-dimensional Gorenstein local ring and let $\mathcal{F} = \{F_i\}_{i \in \mathbb{Z}}$ be an $F_1$-good filtration of ideals in $R$. Let $F_1$ be an equimultiple ideal of $R$ with $\text{ht } F_1 = g > 0$, let $J = (x_1, \cdots, x_d)R \subseteq F_1$ be a minimal reduction of $\mathcal{F}$, and let $u := r_f(\mathcal{F})$ be the reduction number of the filtration $\mathcal{F}$ with respect to $J$. Let $\mathcal{C} := R^1 \langle \mathcal{F} \rangle = \bigoplus_{i \in \mathbb{Z}} F_it^i$. If $G(\mathcal{F})$ is Cohen-Macaulay, then the following conditions are equivalent.

1. $R^1 \langle \mathcal{F} \rangle$ is quasi-Gorenstein.
2. $R^1 \langle \mathcal{F} \rangle$ is Gorenstein.


Proof. Since $G(\mathcal{F})$ is Cohen-Macaulay, items (1) and (2) are equivalent.

$(1) \implies (3)$: Since $G(\mathcal{F})$ is Cohen-Macaulay and $G(\mathcal{F}) = \mathcal{C}/t^{-1}\mathcal{C}$, we have $a(G(\mathcal{F})) = a(\mathcal{C}) + \deg(t^{-1}) = b - 1$. By [HZ, Theorem 3.8], $u = r_\mathcal{F}(\mathcal{F}) = a(G(\mathcal{F})) + \ell(\mathcal{F}) = b - 1 + g$, where $\ell(\mathcal{F})$ is analytic spread of $\mathcal{F}$. By Theorem 6.1 (2), we have that $J^i : F_u = F_i$ for all $i \in \mathbb{Z}$. In particular, $J^u : F_u = F_u$.

$(3) \implies (1)$: Suppose that $J^u : F_u = F_u$. Let $b = a(\mathcal{C})$. Then we have

$C(b) = \bigoplus_{i \in \mathbb{Z}} [C]_{i+b}t^i = \bigoplus_{i \in \mathbb{Z}} [C]_{i+b+(g-1)}t^{i+(g-1)} = \bigoplus_{i \in \mathbb{Z}} [C]_{i+u}t^{i+(g-1)} = \bigoplus_{i \in \mathbb{Z}} F_{i+u}t^{i+(g-1)}.

By Theorem 6.1 (1), we have

$$\omega_\mathcal{C} = \bigoplus_{i \in \mathbb{Z}} (J^{i+u} : F_u)t^{i+(g-1)}.$$ 

To see $\omega_\mathcal{C} \cong C(b)$, we use:

**Claim 6.4.** : $J^{i+u} : F_u = F_{i+u}$ for all $i \in \mathbb{Z}$.

Proof of Claim. \(\supseteq\): For all $i \in \mathbb{Z}$, we have $F_{i+u} \cdot F_u \subseteq F_{i+u+u} = J^{i+u}F_u \subseteq J^{i+u}$, and hence $F_{i+u} \subseteq J^{i+u} : F_u$.

\(\subseteq\): We have three cases: (Case i) $i \leq -u$, (Case ii) $-u + 1 \leq i \leq -1$, and (Case iii) $i \geq 0$.

Case i: Suppose that $i \leq -u$. Then we have $J^{i+u} : F_u = R : F_u = R = F_{i+u}$.

Case ii: Suppose that $-u + 1 \leq i \leq -1$. It is enough to show that $J^{u-j} : F_u \subseteq F_{u-j}$ for $1 \leq j \leq u - 1$. In fact, let $\alpha \in J^{u-j} : F_u$ for some $j$ with $1 \leq j \leq u - 1$. Then we have $\alpha F_u \subseteq J^{u-j}$, and hence $\alpha J^j F_u \subseteq J^j J^{u-j} = J^u$. Thus we have $\alpha J^j \subseteq J^u : F_u = F_u$, by assumption (3). Therefore we have

$$\alpha \in F_u : J^j$$

$$\subseteq F_u \cdot F_n : J^j F_n \quad \text{for} \quad n \gg u \quad (\because J^j F_u = F_{u+j} \quad \text{for all} \quad j \geq 0)$$

$$\subseteq F_{u+n} : F_{j+n}$$

$$\subseteq F_{u-j} \quad \text{by Lemma 2.4.}$$

Case iii: Suppose that $i \geq 0$. It is clear for the case where $i = 0$, by assumption.

To complete the case (iii), we use:

**Claim 6.5.** : $J^{i+u} : F_u \subseteq J^i (J^u : F_u)$ for all $i \geq 1$.

Proof of Claim. Since $\omega_\mathcal{C}$ is a finite $\mathcal{C}$-module and $\mathcal{C}$ is a finite $A := R[Jt, t^{-1}]$-module, we have that $\omega_\mathcal{C}$ is a finite $A$-module. Let $\{\alpha_1, \alpha_2, \cdots, \alpha_h\}$ be a minimal set of homogeneous generator of $\omega_\mathcal{C}$ over $A$ and let $\deg \alpha_j = n_j$ for $1 \leq j \leq h$. By
Theorem 6.2 (2), deg \( \alpha_j \leq g - 1 \) for \( 1 \leq j \leq h \). That is, \( (g - 1) - n_j \geq 0 \) for \( 1 \leq j \leq h \). Hence we have

\[
[w_c]_{g-1} = \sum_{j=1}^{h} [A]_{(g-1)-n_j} \alpha_j = \sum_{j=1}^{h} J^{(g-1)-n_j} \alpha_j,
\]

\[
[w_c]_g = \sum_{j=1}^{h} [A]_{g-n_j} \alpha_j = \sum_{j=1}^{h} J^{(g-1)-n_j} J \alpha_j = J \sum_{j=1}^{h} J^{(g-1)-n_j} \alpha_j = J[w_c]_{g-1},
\]

\[
\ldots \ldots \ldots \ldots
\]

\[
[w_c]_{g+i} = \sum_{j=1}^{h} [A]_{(g+i)-n_j} \alpha_j = \sum_{j=1}^{h} J^{(g-1)-n_j} J^{i+1} \alpha_j = J^{i+1} \sum_{j=1}^{h} J^{(g-1)-n_j} \alpha_j = J^{i+1}[w_c]_{g-1}.
\]

Thus \( [w_c]_{(g-1)+i} = J^{i}[w_c]_{g-1} \) for all \( i \geq 0 \), and hence \( J^{i+u} : F_u = J^i(J^u : F_u) \), which completes the proof of Claim 6.5. The Claim 6.4 implies that

\[
\bigoplus_{i \in \mathbb{Z}} (J^{i+u} : F_u) t^{i+(g-1)} = \bigoplus_{i \in \mathbb{Z}} F_{i+b} t^i.
\]

Thus \( w_c \cong C(b) \), where \( b = a(C) \). This completes the proof of Theorem 6.3.

\[\square\]

**Corollary 6.6.** Let \( (R, \mathfrak{m}) \) be a \( d \)-dimensional Gorenstein local ring and let \( \mathcal{F} = \{F_i\}_{i \in \mathbb{Z}} \) be an \( F_1 \)-good filtration of ideals in \( R \) such that \( F_1 \) is an equimultiple ideal with \( \text{ht} F_1 = g > 0 \) and \( J = (x_1, \ldots, x_g) \mathfrak{m} R \subseteq F_1 \) is a minimal reduction of \( \mathcal{F} \) with \( u := r_J(\mathcal{F}) \). Let \( C := R^t(\mathcal{F}) = \bigoplus_{i \in \mathbb{Z}} F_i t^i \). Then the following conditions are equivalent.

1. \( G(\mathcal{F}) \) is Gorenstein.
2. \( R^t(\mathcal{F}) \) is Gorenstein.
3. \( G(\mathcal{F}) \) is Cohen-Macaulay and \( J^u : F_u = F_u \).

**Proof.** Since \( G(\mathcal{F}) \cong C/t^{-1} C \) and \( t^{-1} \) is a non-zero-divisor of \( C \), we have (1) \(\iff\) (2), and Theorem 6.3 implies (2) \(\iff\) (3).

\[\square\]

Taking the \( I \)-adic filtration \( \mathcal{F} = \{I^i\}_{i \in \mathbb{Z}} \), we get the usual definition of reduction number with respect to a minimal reduction of the ideal (i.e., \( r_J(I) = r_J(\mathcal{F}) \)). As another consequence of Theorem 6.3, we obtain a result of Goto and Iai.

**Corollary 6.7.** ([GI, Theorem 1.4]) Assume that \( (R, \mathfrak{m}) \) is a Gorenstein local ring and let \( I \) be an equimultiple ideal with \( \text{ht} I \geq 1 \). Let \( r = r_J(I) \) be a reduction number with respect to a minimal reduction \( J \) of \( I \). Then the following two conditions are equivalent.
(1) $G(I)$ is a Gorenstein ring.

(2) $G(I)$ is a Cohen-Macaulay ring and $J^r : I^r = I^r$.

**Remark 6.8.** Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with $\dim R = 1$ and let $I$ be an $\mathfrak{m}$-primary ideal. As described in Example 2.5, the Ratliff-Rush filtration $F = \{\tilde{I}^i\}_{i \in \mathbb{Z}}$ is an $I$-good filtration. Since the ideals $\tilde{I}^i$ are Ratliff-Rush ideals, $G(F) = \bigoplus_{i \geq 1} \tilde{I}^i / \tilde{I}^{i+1}$ contains a non-zero-divisor, and hence, since $\dim G(F) = 1$, $G(F)$ is Cohen-Macaulay. Let $J = xR$ be a principal reduction of $I$. The reduction number $r_F(I)$ is independent of the principal reduction $J$ by [HZ, Proposition 3.6].

Let $s_J(I) = \min\{i \mid I^{i+1} \subseteq J\}$ denote the index of nilpotency of $I$ with respect to $J$. An easy computation shows that $r_J(I) \geq r_J(F) \geq s_J(I)$.

For $R$ of dimension one, we have the following corollary to Theorem 6.3.

**Corollary 6.9.** Let $(R, \mathfrak{m})$ be a Gorenstein local ring with $\dim R = 1$, let $I$ be an $\mathfrak{m}$-primary ideal, and let $F = \{\tilde{I}^i\}_{i \in \mathbb{Z}}$ denote the Ratliff-Rush filtration associated to $I$. Let $J = xR$ be a principal reduction of $I$ and set $r = r_J(I)$ and $u = r_F(I)$. Then the following conditions are equivalent.

(1) $G(F) = \bigoplus_{i \geq 0} \tilde{I}^i / \tilde{I}^{i+1}$ is Gorenstein.

(2) $C := R'(F) = \bigoplus_{i \in \mathbb{Z}} \tilde{I}^i t^i$ is Gorenstein.

(3) $J^r : \tilde{I} = \tilde{I}^u$.

(4) $J^r : I^r = \tilde{I}^u$.

**Proof.** (1) $\iff$ (2) : Notice that $G(F) \cong C / t^{-1} C$ and $t^{-1}$ is a non-zero-divisor of $C$.

(2) $\iff$ (3) : Apply Corollary 6.6.

(2) $\implies$ (4) : Suppose that $C = \bigoplus_{i \in \mathbb{Z}} \tilde{I}^i t^i$ is Gorenstein. Then $C$ is quasi-Gorenstein with $a(C) = r_J(F) = u$. We have that

$$\omega_C \cong \bigoplus_{i \in \mathbb{Z}} (J^{i+r} : \tilde{I}) t^i = \bigoplus_{i \in \mathbb{Z}} (J^{i+r} : I^r) t^i,$$

since $I^i = \tilde{I}^i$ for all $i \geq r$. Hence $J^r : I^r = \tilde{I}^{r+b-r} = \tilde{I}^u$, where $u = a(C) = b$.

(4) $\implies$ (2) : Suppose that $J^r : I^r = \tilde{I}^u$. We have that

$$\omega_C \cong \bigoplus_{i \in \mathbb{Z}} (J^{i+r} : \tilde{I}) t^i = \bigoplus_{i \in \mathbb{Z}} (J^{i+r} : I^r) t^i.$$

To see that $C$ is Gorenstein, it suffices to show that $\omega_C \cong C(u)$. That is, we need to prove the following claim:

**Claim 6.10.** $J^{i+r} : I^r = \tilde{I}^{i+u}$ for all $i \in \mathbb{Z}$. 

Case i: Suppose that $i \leq -r$. Then $J^{i+r} : I^r = R : I^r = R = I^{i+u}$, since $r > u$.

Case ii: Suppose that $-r+1 \leq i \leq -r+p$. It is enough to show that $J^j : I^r \subseteq \widetilde{I^{u-j}}$ for all $1 \leq j \leq p$. In fact, let $\alpha \in J^j : I^r$ for all $1 \leq j \leq p$. Then $\alpha I^r \subseteq J^j$, and hence $\alpha J^{r-j} I^r \subseteq J^{r-j} J^j = J^r$. Thus we have $\alpha J^{r-j} \subseteq J^r : I^r = \widetilde{I^u}$, by assumption (4). Therefore

$$
\alpha \in \widetilde{I^u} : J^{r-j} \subseteq \widetilde{I^u} I^r : J^{r-j} I^r \\
\subseteq \widetilde{I^{u+r}} : J^{r-j} I^r \\
= \widetilde{I^{u+r}} : I^{2r-j} \\
\subseteq I^{j+u-r} \quad \text{by the fact : } \widetilde{I^k} = \bigcup_{n \geq 1} (I^{n+k} : I^n).
$$

Case iii: Suppose that $-u+1 \leq i \leq -1$. It is enough to show that $J^{r-j} : I^r \subseteq \widetilde{I^{u-j}}$ for all $1 \leq j \leq u-1$. In fact, let $\alpha \in J^{r-j} : I^r$ for all $1 \leq j \leq u-1$. Then $\alpha I^r \subseteq J^{r-j}$, and hence $\alpha J^j I^r \subseteq J^j J^{r-j} = J^r$. Thus we have $\alpha J^j \subseteq J^r : I^r = \widetilde{I^u}$, by assumption (4). Therefore

$$
\alpha \in \widetilde{I^u} : J^j \subseteq \widetilde{I^u} I^r : J^j I^r \\
\subseteq \widetilde{I^{u+r}} : J^j I^r \\
= \widetilde{I^{u+r}} : I^{r+j} \\
\subseteq \widetilde{I^{u-j}} \quad \text{by the fact : } \widetilde{I^k} = \bigcup_{n \geq 1} (I^{n+k} : I^n).
$$

Case iv: Suppose that $i \geq 0$. The claim is clear in the case where $i = 0$. For $i > 0$, we have

$$
J^{i+r} : I^r = J^i (J^r : I^r) \\
= J^i \widetilde{I^u} \quad \text{by assumption (4)} \\
= \widetilde{I^{i+u}}.
$$

This completes the proof of Claim 6.10.

By Claim 6.10, we have

$$
\omega c = \bigoplus_{I \in \mathbb{Z}} (J^{i+r} : I^r) t^i = \bigoplus_{I \in \mathbb{Z}} \widetilde{I^{i+u}} t^i \cong \bigoplus_{i \in \mathbb{Z}} [C]_{i+u} t^i = C(u).
$$
Corollary 6.9. Thus \( \mathcal{C} = \bigoplus_{i \in \mathbb{Z}} \tilde{I}^i \) is quasi-Gorenstein with \( a(\mathcal{C}) = u \). This completes the proof of Corollary 6.9.

7. Examples of filtrations

We first present three examples of one-dimensional Gorenstein local domains constructed as follows. Let \( k \) be a field and let \( 0 < n_1 < n_2 < n_3 \) be integers with \( \text{GCD}(n_1, n_2, n_3) = 1 \). Consider the subring \( R = k[[s^{n_1}, s^{n_2}, s^{n_3}]] \) of the formal power series ring \( k[[s]] \). Notice that \( R \) is a numerical semigroup ring associated to the numerical semigroup \( H = \langle n_1, n_2, n_3 \rangle \). The Frobenius number of a numerical semigroup \( H \) is the largest integer not in \( H \).

We consider the Gorenstein property of the associated graded ring \( G(\mathcal{F}_i) \) for \( i = 0, 1, 2 \), where

1. \( \mathcal{F}_0 := \{ \text{m}^i \}_{i \geq 0} \) is the integral closure filtration associated to \( \text{m} \),
2. \( \mathcal{F}_1 := \{ \text{m}^i \}_{i \geq 0} \) is the Ratliff-Rush filtration associated to \( \text{m} \),
3. \( \mathcal{F}_2 := \{ \text{m}^i \}_{i \geq 0} \) is the \( \text{m} \)-adic filtration.

The examples below will demonstrate that these filtrations are independent of each other, as far as the Gorenstein property of their associated graded rings is concerned. Notice that \( \text{m}^i \subseteq \text{m}^i \subseteq \text{m}^i \) for all \( i \geq 0 \) and \( G(\mathcal{F}_2) = G(\text{m}) = \bigoplus_{i \geq 0} \text{m}^i / \text{m}^{i+1} \). In Examples 7.1, 7.3 and 7.4, we let \( S = k[[x, y, z]] \) be the formal power series ring in three variables \( x, y, z \) over a field \( k \) and \( \text{n} := (x, y, z)S \).

Example 7.1. ([GHK, Example 5.5]) Let \( R = k[[s^{3m}, s^{3m+1}, s^{6m+3}]] \), where \( 2 \leq m \in \mathbb{Z} \) and define a homomorphism of \( k \)-algebras

\[ \varphi: S \rightarrow R \quad \text{by} \quad \varphi(x) = s^{3m}, \quad \varphi(y) = s^{3m+1}, \quad \text{and} \quad \varphi(z) = s^{6m+3}. \]

Then the ideal \( I = \ker \varphi \) is generated by \( f = zx - y^2 \) and \( g = z^m - x^{2m+1} \), whence \( R \) is a complete intersection of dimension one. We have \( G(\text{n}) = k[X, Y, Z] \) and \( I^* = (XZ, Z^{m}, Y^{3}Z^{m-1}, Y^{6}Z^{m-2}, \ldots, Y^{3(m-1)}Z, Y^{3m})G(\text{n}) \). Since \( \sqrt{I^*} : \mathbb{Z} = (X, Y, Z) \), the associated graded ring

\[ G(\text{m}) \cong k[X, Y, Z] : (XZ, Z^{m}, Y^{3}Z^{m-1}, Y^{6}Z^{m-2}, \ldots, Y^{3(m-1)}Z, Y^{3m}) \]

is not Cohen-Macaulay, see also [GHK, Theorem 5.1], and hence is not Gorenstein. Thus \( \mathcal{F}_2 \neq \mathcal{F}_1 \), by [HLS, (1.2)]. The reduction number of \( \text{m} = (s^{3m}, s^{3m+1}, s^{6m+3})R \) with respect to the principal reduction \( J = (s^{3m})R \) is \( 3m - 1 \) and the blowup of \( \text{m} \) is \( R[\frac{\text{m}}{s^{3m}}] = \frac{\text{m}^{3m-1}}{s^{3m(3m-1)}} \) ([HLS, Fact 2.1]). Since \( s = s^{3m+1}/s^{3m} \in \frac{\text{m}}{s^{3m}} \), the
blowup of \( m \) is \( \overline{R} = k[[s]] \), the integral closure of \( R \). Hence \( F_1 = F_0 \), by [HLS, Corollary 2.7]. Notice that \( \overline{m}^i = (s^{3m})^ik[[s]] \cap R \) for all \( i \geq 0 \). We observe that the reduction number \( r_J(F_1) \) of \( F_1 \) with respect to the principal reduction \( J = (s^{3m})R \) is 2m. For \( \alpha \in k[[s]] \), we denote by \( \text{ord}(\alpha) \) the order of \( \alpha \) as a power series in \( s \).

Since \( \overline{m}^i = \{ \alpha \in R \mid \text{ord}(\alpha) \geq (3m)i \} \), and the Frobenius number of the numerical semigroup of \( R \) is \( 6m^2 - 1 \), we have \( \overline{m}^{i+1} \subseteq J \) and \( J\overline{m}^i = \overline{m}^{i+1} \) for every \( i \geq 2m \). Furthermore, \( s^{6m^2+3m-1} \in \overline{m}^{2m} \), but \( s^{6m^2+3m-1} = s^{3m}s^{6m^2-1} \notin J \), which shows \( \overline{m}^{2m} \notin J \). Hence \( r_J(F_1) = 2m \).

**Claim 7.2.** \( G(F_1) \) is a Gorenstein ring.

**Proof of Claim.** By Corollary 6.9, it suffices to show that

\[ J^u : \overline{m}^u = \overline{m}^u, \quad \text{where} \quad u := r_J(F_1). \]

Since \( u := r_J(F_1) = 2m \), the inclusion \( \subseteq \) is clear. To show the reverse inclusion, it suffices to prove : \( \beta \in R \overline{m}^{2m} \implies \beta \notin (J^{2m} : \overline{m}^{2m}) \). Let \( \beta \in R \overline{m}^{2m} \), that is, \( \beta \in R \) with \( \text{ord}(\beta) < 6m^2 \). Let \( n_\beta := \text{ord}(\beta) \), where \( 0 \leq n_\beta < 6m^2 \). Then \( \sigma := s^{6m^2+6m^2-n_\beta-1} \in \overline{m}^{2m} \), since \( \text{ord}(\sigma) = 6m^2 + (6m^2 - n_\beta) - 1 \geq 6m^2 + 1 - 1 = 6m^2 \). Hence \( \beta \sigma = s^{n_\beta} \cdot s^{6m^2+6m^2-n_\beta-1} = s^{6m^2+(6m^2-1)} = (s^{3m})^{2m} \cdot s^{6m^2-1} \notin J^{2m} \), since the Frobenius number of the numerical semigroup of \( R \) is \( 6m^2 - 1 \).

**Example 7.3.** Let \( R = k[[s^4, s^6, s^7]] \) and define a homomorphism of \( k \)-algebras

\[ \varphi : S \longrightarrow R \quad \text{by} \quad \varphi(x) = s^4, \quad \varphi(y) = s^6, \quad \text{and} \quad \varphi(z) = s^7. \]

Then the ideal \( I = \ker \varphi \) is generated by \( f = x^3 - y^2 \) and \( g = z^2 - x^2y \), whence \( R \) is a complete intersection of dimension one. We have \( G(\mathfrak{m}) = k[X, Y, Z] \) and \( I^* = (Y^2, Z^2) \). Hence \( G(\mathfrak{m}) \cong k[X, Y, Z]/(Y^2, Z^2) \) is a Gorenstein ring. In particular \( F_2 = F_1 \) by [HLS, (1.2)]. The reduction number of \( \mathfrak{m} = (s^4, s^6, s^7)R \) with respect to the principal reduction \( J = (s^4)R \) is 2 and the blowup of \( \mathfrak{m} \) is \( R(\overline{\mathfrak{m}}^i) = \mathfrak{m}^{i+1} = k[[s^2, s^3]] \), which is not equal to the integral closure \( \overline{R} = k[[s]] \) of \( R \). Hence \( F_1 \neq F_0 \), by [HLS, Corollary 2.7]. Notice that \( \overline{m}^i = (s^i)^ik[[s]] \cap R \) for all \( i \geq 0 \). The reduction number \( r_J(F_0) \) of \( F_0 \) with respect to the principal reduction \( J = (s^4)R \) is 3. Indeed, since \( \overline{m}^i = \{ \alpha \in R \mid \text{ord}(\alpha) \geq 4i \} \) we conclude that \( \overline{m}^{i+1} \subset J \) for every \( i \geq 3 \) and hence \( J\overline{m}^i = \overline{m}^{i+1} \). On the other hand \( s^{13} \in \overline{m} \setminus J\overline{m}^2 \). Therefore \( r_J(F_0) = 3 \).

Since \( s^6 \in (J : \overline{m}^2) \setminus (J + \overline{m}^2) \), we have \( J : \overline{m}^2 \neq J + \overline{m}^2 \). Thus \( G(F_0) \) is not Gorenstein by Theorem 4.3.

We thank YiHuang Shen for suggesting to us Example 7.4.
**Example 7.4.** Let \( R = k[[s^6, s^{11}, s^{27}]] \) and define a homomorphism of \( k \)-algebras
\[
\varphi : S \rightarrow R \quad \text{by} \quad \varphi(x) = s^6, \quad \varphi(y) = s^{11}, \quad \text{and} \quad \varphi(z) = s^{27}.
\]
Then the ideal \( I = \ker \varphi \) is generated by \( f = z^2 - x^9 \) and \( g = xz - y^3 \), whence \( R \) is a complete intersection of dimension one. We have \( G(n) = k[X, Y, Z] \) and \( I^* = (Z^2, ZX, ZY^3, Y^6) \). Since \( \sqrt{I^*} : X = (X, Y, Z) \), the associated graded ring
\[
G(\mathfrak{m}) \cong k[X, Y, Z]/(Z^2, ZX, ZY^3, Y^6)
\]
is not a Cohen-Macaulay ring, also see [GHK, Theorem 5.1], and hence is not a Gorenstein ring. Furthermore \( \mathcal{F}_2 \neq \mathcal{F}_1 \) by [HLS, (1.2)]. The reduction number of \( \mathfrak{m} = (s^6, s^{11}, s^{27})R \) with respect to the principal reduction \( J = (s^6)R \) is 5 and the blowup of \( \mathfrak{m} \) is \( R[\overline{\mathfrak{m}}] = \frac{R[\mathfrak{m}^5]}{\mathfrak{m}^5} = k[[s^5, s^6]] \), which is not equal to the integral closure \( \overline{R} = k[[s]] \) of \( R \). Hence \( \mathcal{F}_1 \neq \mathcal{F}_0 \) by [HLS, Corollary 2.7]. We observe that
\[
\begin{align*}
\overline{\mathfrak{m}}^2 &= ks^{27} + \mathfrak{m}^2 \\
\overline{\mathfrak{m}}^3 &= ks^{38} + ks^{49} + \mathfrak{m}^3 \\
\overline{\mathfrak{m}}^4 &= ks^{49} + \mathfrak{m}^4 \\
\overline{\mathfrak{m}}^i &= \mathfrak{m}^i \quad \text{for every} \; i \geq 5.
\end{align*}
\]
The reduction number \( r_J(\mathcal{F}_1) \) of \( \mathcal{F}_1 \) with respect to the principal reduction \( J = (s^6)R \) is 4, since \( \overline{\mathfrak{m}}^i = \overline{\mathfrak{m}}^{i+1} \) for every \( i \geq 4 \), but \( s^{49} \notin \overline{\mathfrak{m}}^4 \setminus J\overline{\mathfrak{m}}^3 \). We have that \( \mathfrak{m} \mathfrak{m}^{2} \subseteq J : \overline{\mathfrak{m}}^3 \subseteq \mathfrak{m} \), where the first inclusion holds since \( r_J(\mathcal{F}_1) = 4 \). Furthermore \( \lambda(\mathfrak{m}/J + \mathfrak{m}^2) = 1 \), because \( \mathfrak{m} = ks^{11} + J + \mathfrak{m}^2 \). Since the Frobenius number of the numerical semigroup of \( R \) is 43 we have \( s^{11}s^{38} = s^{6}s^{43} \notin J \), and therefore \( s^{11} \notin J : \overline{\mathfrak{m}}^3 \). Hence \( G(\mathcal{F}_1) \) is Gorenstein by Theorem 4.3. The reduction number \( r_J(\mathcal{F}_0) \) of \( \mathcal{F}_0 \) with respect to the principal reduction \( J = (s^6)R \) is 6, since \( \overline{\mathfrak{m}}^i = \overline{\mathfrak{m}}^{i+1} \) for every \( i \geq 6 \), but \( s^{38} \notin \overline{\mathfrak{m}}^6 \setminus J\overline{\mathfrak{m}}^5 \). As \( s^{17} \in (J : \overline{\mathfrak{m}}^4) \setminus (J + \overline{\mathfrak{m}}^3) \), we obtain \( J : \mathfrak{m}^4 \supseteq J + \overline{\mathfrak{m}}^3 \). Therefore \( G(\mathcal{F}_0) \) is not Gorenstein by Theorem 4.3.

YiHuang Shen proves in [S, Theorem 4.12] that if \( (R, \mathfrak{m}) \) is a numerical semigroup ring with \( \mu(\mathfrak{m}) = 3 \) such that \( r_J(\mathfrak{m}) = s_J(\mathfrak{m}) \), then the associated graded ring \( G(\mathfrak{m}) \) is Cohen-Macaulay. The following example given by Lance Bryant shows that this does not hold for one-dimension Gorenstein local rings of embedding dimension three.

**Example 7.5.** Let \( (S, \mathfrak{n}) \) be a 3-dimensional regular local ring with \( \mathfrak{n} = (x, y, z)S \) and \( S/\mathfrak{n} = k \). Let \( I = (f, g) \), where \( f = x^3 + z^5 \) and \( g = x^2y + xz^3 \). Put \( R := S/I \) and \( \mathfrak{m} := \mathfrak{n}/I \). Then \( (R, \mathfrak{m}) \) is an 1-dimensional Gorenstein local ring. We have
A computation shows that the Gorenstein property of the associated graded rings $R$ of $m$.

Notice that (Lemma 7.6) so, in particular, $F$ is enough to show that $G(X, Y, Z)G(n)$ is not Cohen-Macaulay and hence not Gorenstein. Thus $F_2 \neq F_1$ by [HLS, (1.2)]. Let $J = (y - z)R$. Then $J$ is a minimal reduction of $m$.

A computation shows that $r_J(F_2) = r_J(F_1) = s_J(F_2) = 6$. By Corollary 6.9, to see that $G(F_1)$ is Gorenstein, it suffices to show that $(J^6 : m^6) = m^6$. To check this, it is enough to show that $\lambda(R / m^6) = 39 = \frac{(6)(13)}{2}$, where $13 = e(R)$ is the multiplicity of $R$.

Since $R$ is not reduced, the filtration $F_0$ is not a good filtration ([SH, Theorem 9.1.2]) so, in particular, $F_0 \neq F_1$.

We present examples of 2-dimensional Gorenstein local rings $(R, m)$ and consider the Gorenstein property of the associated graded rings $G(F_i)$ for $i = 0, 1, 2, 3$, where

1. $F_0 := \{\overline{m^i}\}_{i \geq 0}$ is the integral closure filtration associated to $m$,
2. $F_1 := \{(\overline{m^i})_{(1)}\}_{i \geq 0}$ is the $e_1$-closure filtration associated to $m$,
3. $F_2 := \{\overline{m^i}\}_{i \geq 0}$ is the Ratliff-Rush filtration associated to $m$,
4. $F_3 := \{m^i\}_{i \geq 0}$ is the $m$-adic filtration.

Notice that $m^i \subseteq \overline{m^i} \subseteq (m^i)_{(1)} \subseteq \overline{m}$ for all $i \geq 0$ and $G(F_3) = G(m) = \bigoplus_{i \geq 0} m^i / m^{i+1}$.

Lemma 7.6 is useful in considering the $e_1$-closure filtration in a 2-dimensional Noetherian local ring $(R, m)$. For an $m$-primary ideal $F$ of $R$, let $P_F(s)$ denote the Hilbert-Samuel polynomial having the property that $\lambda(R / F^s) = P_F(s)$ for all $s >> 0$. We write

$$P_F(s) = e_0(F) \left( \frac{s + 1}{2} \right) - e_1(F) \left( \frac{s}{1} \right) + e_2(F).$$

**Lemma 7.6.** Let $(R, m)$ be a 2-dimensional Noetherian local ring and let $F = \{F_i\}_{i \in \mathbb{Z}}$ be an $F_1$-good filtration, where $F_1$ is an $m$-primary ideal. If there exists a positive integer $c$ such that $\lambda(F_i / F^i) < c$ for all $i \geq 0$, then the Hilbert coefficients

$$G(n) = k[X, Y, Z], f^* = X^3, \text{ and } g^* = X^2Y.$$ Let $h = -yf + xg$, $\xi_4 = z^3f - xh$, and $\xi_5 = z^3g - yh$. Then $h^* = X^2Z^3$, $\xi_4^* = XYZ^5$, and $\xi_5^* = Y^2Z^5 + XZ^6$. Let

$$K = (X^3, X^2Y, X^2Z^3, XYZ^5, Y^2Z^5 + XZ^6) \subseteq I^*.$$ Then the Hilbert series of the graded ring $G(n)/K$ is

$$\frac{1 + 2t + 3t^2 + 2t^3 + 2t^4 + t^5 + 2t^6}{1 - t} = 1 + 3t + 6t^2 + 8t^3 + 10t^4 + 11t^5 + 13t^6 + 13t^7 + \cdots$$

and these values are the same as those in the Hilbert series of $G(n) = G(n)/I^*$, so that $K = I^*$. Since $(I^* : X)$ is primary to the unique homogeneous maximal ideal $(X, Y, Z)G(n)$, $G(m)$ is not Cohen-Macaulay and hence not Gorenstein. Thus $F_2 \neq F_1$ by [HLS, (1.2)]. Let $J = (y - z)R$. Then $J$ is a minimal reduction of $m$.
of the polynomials \( P_{F_i}(s) \) and \( P_{F_i}(s) \) satisfy

\[
e_0(F_i) = e_0(F_i) \quad \text{and} \quad e_1(F_i) = e_1(F_i) \quad \text{for all} \quad i \geq 0.
\]

Therefore \( (F_i)_{11} = (F_i)_{11} \) for all \( i \geq 0 \).

**Proof.** Fix \( i \geq 1 \), we have \( (F_i)_{11} \subseteq (F_i)_{11} \subseteq F_{is} \) for all \( s \geq 1 \). Our hypothesis implies

\[
c > \lambda(F_{is}/(F_i)^s) \geq \lambda((F_i)^s/(F_i)^s) \geq 0 \quad \text{for all} \quad s \geq 1.
\]

For all sufficiently large \( s \), we have

\[
c > \lambda((F_i)^s/(F_i)^s) = \lambda(R/(F_i)^s) - \lambda(R/(F_i)^s)
\]

\[= P_{F_i}(s) - P_{F_i}(s).
\]

Thus \( P_{F_i}(s) - P_{F_i}(s) \) is a constant polynomial, which implies \( e_0(F_i) = e_0(F_i) \) and \( e_1(F_i) = e_1(F_i) \).

**Example 7.7.** Let \( k \) be a field of characteristic other than 2 and set \( S = k[[x, y, z, w]] \) and \( n = (x, y, z, w)S \), where \( x, y, z, w \) are indeterminates over \( k \). Let

\[
f = x^2 - w^4,
\]

\[
g = xy - z^3.
\]

Let \( I = (f, g)S, R = S/I, \) and \( m = n/I \). Since \( f, g \) is a regular sequence, \( R \) is a 2-dimensional Gorenstein local ring. We have:

1. \( \mathcal{F}_3 = \mathcal{F}_2 \neq \mathcal{F}_1 = \mathcal{F}_0 \).

2. \( G(\mathcal{F}_3) \) is not Gorenstein and \( r_J(\mathcal{F}_3) = 5 \), where \( J = (y, w)R \).

3. \( G(\mathcal{F}_0) \) is Gorenstein and \( r_J(\mathcal{F}_0) = 4 \), where \( J = (y, w)R \).

**Proof.** The associated graded ring \( G := \text{gr}_n(S) = k[X, Y, Z, W] \) is a polynomial ring in 4 variables over the field \( k \), and \( G(\mathcal{F}_3) = G(m) = G/I^* \), where \( I^* \) is the leading form ideal of \( I \) in \( G = \text{gr}_n(S) \). One computes that

\[I^* = (X^2, XY, XZ^3, Z^6 + Y^2W^4)G.
\]

Thus \( G/I^* = G(m) \) is a 2-dimensional standard graded ring of depth one. Notice that \( W \) is \( G(m) \)-regular. The ring \( G(m) \) is not Cohen-Macaulay, and hence \( G(m) \) is not Gorenstein. We also have \( \mathcal{F}_3 = \mathcal{F}_2 \) by [HLS, (1.2)], and \( r_J(m) = 5 \), where \( J = (y, w)R \).
Set

\[ T = \frac{k[x, y, z, w]}{(x^2 - w^4, xy - z^3)}, \]

\[ L_1 = ((y, z, w) + (x))T, \]

\[ L_2 = ((y, z, w)^2 + (x))T, \]

\[ L_3 = ((y, z, w)^3 + x(z, w))T, \]

\[ L_n = ((y, z, w)^n + xw^{n-4}(z, w)^2)T, \quad \text{for all } n \geq 4. \]

Then \( T \) is 2-dimensional, Gorenstein, excellent and reduced, since the characteristic of the field \( k \) is other than 2. The ring \( T \) becomes a positively graded \( k \)-algebra if we set

\[ \deg(x) = 2, \quad \deg(y) = \deg(z) = \deg(w) = 1. \]

With this grading it turns out that \( L_n = \bigoplus_{i \geq n} [T]_i \), for all \( n \geq 1 \). In particular \( L^n_1 \subseteq L_n \), and since the image in \( T \) of \( x \) is integral over \( L_2 \) it follows that \( L_n \) is integral over \( L^n_1 \). As \( T \) is reduced, the ideal \( L_n = \bigoplus_{i \geq n} [T]_i \) is integrally closed, and since \( T \) is excellent, \( L_nR \) remains integrally closed in \( R \), the completion of \( T \) with respect to the homogeneous maximal ideal. We conclude that \( \overline{m}^n = L^n_1R = L_nR \) for every \( n \geq 1 \).

The reduction number \( r_J(F_0) \) of \( F_0 \) with respect to \( J = (y, w)R \) is 4, since \( J\overline{m}^i = \overline{m}^{i+1} \) for all \( i \geq 4 \), whereas \( zx^2 \in \overline{m}^3 \setminus J\overline{m}^3 \). We have that \( J + \overline{m}^2 \subseteq J : \overline{m}^3 \subseteq J + \overline{m} \), where the first inclusion holds because \( r_J(F_0) = 4 \). Notice that \( J + \overline{m}^2 = (x, y, w, z^2)R \) and \( J + \overline{m} = (x, y, w, z)R \). This implies that \( \lambda(J + \overline{m}/J + \overline{m}^2) = 1 \). Since \( z \cdot xz \notin J \) and \( xz \in \overline{m}^2, z \notin J : \overline{m}^2 \) and hence \( J : \overline{m}^2 = J + \overline{m}^2 \). Thus \( G(F_0) \) is a Gorenstein ring, by Theorem 4.3. One computes that \( \lambda(\overline{m}^i/\overline{m}^j) \leq 3 \) for all \( i \geq 0 \).

By Lemma 7.6, we have \( (\overline{m}^i)_1 = (\overline{m}^j)_1 \) for all \( i \geq 1 \). Since \( \overline{m}^2 \subseteq (\overline{m}^2)_1 \subseteq \overline{m}^1 \), it follows that \( (\overline{m}^i)_1 = \overline{m}^1 \) for all \( i \geq 1 \). That is, \( F_1 = F_0 \). Since \( G(F_0) \) is Gorenstein, but \( G(F_3) \) is not, we also deduce that \( F_0 \neq F_3 \). \( \square \)

Example 7.8. Let \( S = k[[x, y, z, w]] \) be a formal power series ring over a field \( k \) and \( n = (x, y, z, w)S \), where \( x, y, z, w \) are indeterminates over \( k \). Let

\[ f = x^2 - w^5, \]

\[ g = xy - z^3. \]
Let \( I = (f, g)S \), \( R = S/I \), and \( \mathfrak{m} = \mathfrak{n}/I \). Since \( f, g \) is a regular sequence, \( R \) is a 2-dimensional Gorenstein local ring. Set \( \mathcal{F} = \{ F_i \}_{i \geq 0} \), where

\[
\begin{align*}
F_0 &= R, \\
F_1 &= \mathfrak{m}, \\
F_2 &= ((y, z, w)^2 + (x))R, \\
F_3 &= ((y, z, w)^3 + x(z, w))R, \\
F_i &= ((y, z, w)^i + xw^{i-4}(z, w)^2)R, \quad \text{for all } i \geq 4.
\end{align*}
\]

Then:

1. \( \mathcal{F} \) is a \( F_1 \)-good filtration.
2. \( G(\mathfrak{m}) \) is not Gorenstein and \( r_J(\mathfrak{m}) = 5 \), where \( J = (y, w)R \).
3. \( G(\mathcal{F}) \) is Gorenstein and \( r_J(\mathcal{F}) = 4 \), where \( J = (y, w)R \) and \( G(\mathcal{F}) \) is not reduced.
4. \( \mathcal{F} = \{(\mathfrak{m}^i)_{(1)}\}_{i \geq 0} \) is the \( e_1 \)-closure filtration associated to \( \mathfrak{m} \).

**Proof.** The associated graded ring \( G := \text{gr}_n(S) = k[X, Y, Z, W] \) is a polynomial ring in 4 variables over the field \( k \), and \( G(\mathfrak{m}) = G/I^* \), where \( I^* \) is the leading form ideal of \( I \) in \( G = \text{gr}_n(S) \). One computes that

\[
I^* = (X^2, XY, XZ^3, Z^6)G.
\]

Thus \( G/I^* = G(\mathfrak{m}) \) is a 2-dimensional standard graded ring of depth one. Notice that \( W \) is \( G(\mathfrak{m}) \)-regular. The ring \( G(\mathfrak{m}) \) is not Cohen-Macaulay, and hence \( G(\mathfrak{m}) \) is not Gorenstein. Also we have \( \mathfrak{m}^i = \widetilde{\mathfrak{m}}^i \) for all \( i \geq 1 \), by [HLS, (1.2)] and \( r_J(\mathfrak{m}) = 5 \), where \( J = (y, w)R \). One computes that \( F_1 F_1 \subseteq F_2 \) and \( F_i F_j = F_{i+j} \) for all \( i, j \geq 1 \) with \( i + j \geq 3 \), by using the relations \( x^2 = w^5 \) and \( xy = z^3 \) in \( R \). Hence \( \mathcal{F} \) is a \( F_1 \)-good filtration. The reduction number \( r_J(\mathcal{F}) \) of \( \mathcal{F} \) with respect to \( J = (y, w)R \) is 4 and \( G(\mathcal{F}) \) is a Gorenstein ring, by the same argument in the proof of Example 7.7. \( G(\mathcal{F}) \) is not reduced, since \( x^* \in F_2/F_3 \) is a non-zero nilpotent element in \( G(\mathcal{F}) \).

For \( x \in F_2 \setminus F_3 \), \( (x^*)^2 = x^2 + F_5 = w^5 + F_5 = 0 \), since \( w^5 \in F_5 \). One computes that \( \lambda(F_i/F_i^j) \leq 3 \) for all \( i \geq 0 \). By Lemma 7.6, we have \( (F_i^j)_{(1)} = (F_i)_{(1)} \) for all \( i \geq 1 \). Since \( G(\mathcal{F}) \) is Cohen-Macaulay, the extended Rees ring \( R'(\mathcal{F}) \) is Cohen-Macaulay and hence satisfies \( (S_2) \). Therefore by [CPV, Theorem 4.2], we have \( F_i = (F_i)_{(1)} = (F_i^1)_{(1)} = (\mathfrak{m}^i)_{(1)} \) for all \( i \geq 1 \). \( \square \)
Example 7.9. ([CHRR, Example 5.1]) Let $k$ be a field of characteristic other than 2 or 3 and set $S = k[[x, y, z, w]]$ and $\mathfrak{n} = (x, y, z, w)S$, where $x, y, z, w$ are indeterminates over $k$. Let
\[ f = z^2 - (x^3 + y^3), \]
\[ g = w^2 - (x^3 - y^3). \]
Let $I = (f, g)S$, $R = S/I$, and $\mathfrak{m} = \mathfrak{n}/I$. Since $f, g$ is a regular sequence, $R$ is a 2-dimensional Gorenstein local ring. Notice that $R$ is also a normal domain. We have:
1. $\mathcal{F}_3 = \mathcal{F}_2 = \mathcal{F}_1 \neq \mathcal{F}_0$.
2. $G(\mathcal{F}_3)$ is Gorenstein and $r_J(\mathcal{F}_3) = 2$, where $J = (x, y)R$.
3. $G(\mathcal{F}_0)$ is not Gorenstein and $r_J(\mathcal{F}_0) = 3$, where $J = (x, y)R$.

Proof. The associated graded ring $G(\mathfrak{n}) = k[X, Y, Z, W]$ is a polynomial ring in 4 variables over the field $k$, and the associated graded ring $G(\mathcal{F}_3) = G(\mathfrak{n}) = G/I^*$, where $I^*$ is the leading form ideal of $I$ in $G$. One computes that $I^* = (Z^2, W^2)G$. Thus $G/I^* = G(\mathfrak{n})$ is Gorenstein. In particular the extended Rees ring $R'(\mathcal{F})$ is Cohen-Macaulay, and hence by [CPV, Theorem 4.2], $\mathcal{F}_3 = \mathcal{F}_2 = \mathcal{F}_1$. Also we have $r_J(\mathfrak{m}^2) = 2$, where $J = (x, y)R$, since $zw \in \mathfrak{m}^2 \setminus J \mathfrak{m}$ and $J \mathfrak{m}^2 = \mathfrak{m}^3$.

Set
\[ T = \frac{k[x, y, z, w]}{(z^2 - (x^3 + y^3), w^2 - (x^3 - y^3))}, \]
\[ L_1 = ((x, y) + (z, w))T, \]
\[ L_2 = ((x, y)((x, y) + (z, w)) + (zw))T, \]
\[ L_n = ((x, y)^{n-1}((x, y) + (z, w)) + (x, y)^{n-3}(zw))T \quad \text{for all} \quad n \geq 3. \]
The ring $T$ becomes a positively graded $k$-algebra if we set
\[ \deg(x) = \deg(y) = 2 \quad \text{and} \quad \deg(z) = \deg(w) = 3. \]
Since the characteristic of the field $k$ is not equal to 2 or 3, the ring $T$ is a 2-dimensional Gorenstein excellent normal domain. Notice that
\[ [T]_0 = k, \quad [T]_1 = (0), \quad [T]_2 = (x, y), \quad [T]_3 = (z, w), \quad [T]_4 = (x, y)^2, \]
\[ [T]_{2n-1} = (x, y)^{n-2}(z, w), \quad [T]_{2n} = (x, y)^n + (x, y)^{\frac{n}{2}}(zw) \quad \text{for all} \quad n \geq 3, \]
where $[\cdot]$ denotes the floor function, $\langle \cdot \rangle$ stands for $k$ vector space spanned by $\cdot$, and power denotes symmetric power. From this one sees that $L_n = \bigoplus_{i \geq 2n}[T]_i$. In particular $L_1^n \subseteq L_n$, and since the image in $T$ of $zw$ is integral over $L_1^2$ it follows that $L_n$ is integral over $L_1^n$. We deduce, as in the proof of Example 7.7, that $\mathcal{T}_{L_1^2} = L_n$. 

and then $\overline{m^r} = L_nR$ for every $n \geq 1$. The reduction number $r_J(F_0)$ of $F_0$ with respect to $J = (x, y)R$ is 3, since $J\overline{m^i} = \overline{m^{i+1}}$ for all $i \geq 3$, but $zw \in \overline{m^2 \setminus Jm^2}$.

Since $z$ and $w$ are in $J : m^2$, we obtain $J : m^2 = m$. We have $J + \overline{m^2} = (x, y, zw)R$, whereas $J : \overline{m^2} = m$ because $z$ and $w$ are in $J : \overline{m^2}$. Therefore $J + \overline{m^2} \subseteq J : \overline{m^2}$, and then Theorem 4.3 shows that $G(F_0)$ is not Gorenstein. In particular $F_3 \neq F_0$ since $G(F_3)$ is Gorenstein.

**Remark 7.10.** Let $(R, m)$ be a 2-dimensional regular local ring.

1. Let $F = \{ F_i \}_{i \in \mathbb{Z}}$ be an $F_1$-good filtration, where $F_1$ is $m$-primary. If $G(F)$ is Gorenstein, then $F$ is the $F_1$-adic filtration and $F_1$ is a complete intersection.

2. Let $I$ be an $m$-primary ideal. If $G(I)$ is Gorenstein, then the coefficient ideal filtrations $F_3 \subseteq F_2 \subseteq F_1 \subseteq F_0$ associated to $I$ are all the same.

**Proof.** (1): We may assume that the residue field of $R$ is infinite., in which case $F$ has a reduction $J$ which is a complete intersection. If $G(F)$ is Cohen-Macaulay then $r_J(F) \leq 1$ according to Proposition 3.8, hence $F$ is the $F_1$-adic filtration by Remark 3.4. If in addition $G(F)$ is Gorenstein, we claim that $r_J(I) \neq 1$ for $I = F_1$. Indeed, suppose $r_J(I) = 1$. In this case Theorem 4.3 implies that $J : I = I$, hence $\overline{I^J} = I$. However, $\overline{I^J} \cong \text{Hom}_R(R/I, R/J) \cong \text{Ext}^2_R(R/I, R)$, and using a minimal free $R$-resolution of $R/I$ one sees that the minimal number of generators of the latter module is $\mu(I) - 1$. On the other hand, $\mu(I/J) = \mu(I) - 2$ since $J$ is a minimal reduction of $I$. This contradiction proves that $r_J(I) = 0$, hence $I = J$ is a complete intersection.

(2): We apply part (1) to the filtration $F = \{ \overline{I_i} \}_{i \in \mathbb{Z}}$ and use the fact that a complete intersection has no proper reduction.

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