Integral Domains Inside Noetherian Power Series Rings: Constructions and Examples May 2015

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1991 *Mathematics Subject Classification.* Primary 13A15, 13B35, 13F25; Secondary 13B02, 13B22, 13B40, 13C15, 13H05

Key words and phrases. power series, Noetherian integral domain, completion, generic fiber, flatness, prime spectra.

ABSTRACT. In this monograph the authors gather together results and examples from their work of the past two decades related to power series rings and to completions of Noetherian integral domains.

A major theme is the creation of examples that are appropriate intersections of a field with a homomorphic image of a power series ring over a Noetherian domain. The creation of examples goes back to work of Akizuki and Schmidt in the 1930s and Nagata in the 1950s.

In certain circumstances, the intersection examples may be realized as a directed union, and the Noetherian property for the associated directed union is equivalent to a flatness condition. This flatness criterion simplifies the analysis of several classical examples and yields other examples such as

- A catenary Noetherian local integral domain of any specified dimension of at least two that has geometrically regular formal fibers and is not universally catenary.
- A three-dimensional non-Noetherian unique factorization domain B such that the unique maximal ideal of B has two generators; B has precisely n prime ideals of height two, where n is an arbitrary positive integer; and each prime ideal of B of height two is not finitely generated but all the other prime ideals of B are finitely generated.
- A two-dimensional Noetherian local domain that is a birational extension of a polynomial ring in three variables over a field yet fails to have Cohen-Macaulay formal fibers. This example also demonstrates that Serre's condition S_1 need not lift to the completion; the example is related to an example of Ogoma.

Another theme is an analysis of extensions of integral domains $R \hookrightarrow S$ having trivial generic fiber, that is, every nonzero prime ideal of S has a nonzero intersection with R. Motivated by a question of Hochster and Yao, we present results about

- The height of prime ideals maximal in the generic fiber of certain extensions involving mixed power series/polynomial rings.
- The prime ideal spectrum of a power series ring in one variable over a one-dimensional Noetherian domain.
- The dimension of S if $R \hookrightarrow S$ is a local map of complete local domains having trivial generic fiber.

A third theme relates to the questions:

- What properties of a Noetherian domain extend to a completion?
- What properties of an ideal pass to its extension in a completion?
- What properties extend for a more general multi-adic completion?

We give an example of a three-dimensional regular local domain R having a prime ideal P of height two with the property that the extension of P to the completion of R is not integrally closed.

All of these themes are relevant to the study of prime spectra of Noetherian rings and of the induced spectral maps associated with various extensions of Noetherian rings. We describe the prime spectra of various extensions involving power series.

Contents

Preface

The authors have had as a long-term project the creation of examples using power series to analyze and distinguish several properties of commutative rings and their spectra. This monograph is our attempt to expose the results that have been obtained in this endeavor, to put these results in better perspective and to clarify their proofs. We hope in this way to assist current and future researchers in commutative algebra in utilizing the techniques described here.

Dedication

This monograph is dedicated to Mary Ann Heinzer, to Maria Rotthaus, to Roger Wiegand, and to the past, present and future students of the authors.

William Heinzer, Christel Rotthaus, Sylvia Wiegand

CHAPTER 1

Introduction

When we started to collaborate on this work about twenty years ago, we were inspired by expository talks Judy Sally gave on the following question:

QUESTION 1.1. What rings lie between a Noetherian integral domain and its field of fractions?

We were also inspired by Shreeram Abhyankar's research such as that in his paper [**1**] to ask the following related question:[1]

QUESTION 1.2. Let I be an ideal of a Noetherian integral domain R and let R^* denote the I-adic completion of R. What rings lie between R and R^*? For example, if x and y are indeterminates over a field k, what rings lie between the polynomial ring $k[x, y]$ and the mixed polynomial-power series ring $k[y][[x]]$?

In this book we encounter a wide variety of integral domains fitting the descriptions of Question 1.1 and Question 1.2.

Over the past eighty years, important examples of Noetherian integral domains have been constructed that are an intersection of a field with a homomorphic image of a power series ring. The basic idea is that, starting with a typical Noetherian integral domain R such as a polynomial ring over a field, we look for more unusual Noetherian and non-Noetherian extension rings inside a homomorphic image S of an ideal-adic completion of R. An ideal-adic completion of R is a homomorphic image of a power series ring over R; see Section 3.1 of Chapter 3.[2]

BASIC CONSTRUCTION EQUATION 1.3. This construction features an "intersection" domain A of the form:

$$A := L \cap S,$$

where R and S are as in the preceding paragraph, and L is a field between the field of fractions of R and the total quotient ring of S.

We have the following major goals:

(1) To construct new examples of Noetherian rings, continuing a tradition that goes back to Akizuki and Schmidt in the 1930s and Nagata in the 1950s.
(2) To construct new non-Noetherian integral domains that illustrate recent advances in ideal theory.

[1] Ram's work demonstrates the vastness of power series rings; a power series ring in two variables over a field k contains for each positive integer n an isomorphic copy of the power series ring in n variables over k. The authors have fond memories of many pleasant conversations with Ram concerning power series.

[2] Most terminology used in this introduction, such as "ideal-adic completion", "coefficient field", "essentially finitely generated" and "integral closure", are defined in Chapters 2 and 3.

(3) To study birational extensions of Noetherian integral domains as in Question 1.1.
(4) To consider the fibers of an extension $R \hookrightarrow R^*$, where R is a Noetherian domain and R^* is the completion of R with respect to an ideal-adic topology, and to relate these fibers to birational extensions of R.

These objectives form a complete circle, since (4) is used to accomplish (1).

We have been captivated by these topics and have been examining ways to create new rings from well-known ones for a number of years. Several chapters of this monograph, such as Chapters 4, 5 6, 15, 17, and 22, contain a reorganized development of previous work on this technique.

Basic Construction Equation 1.3 as presented here is *universal* in the following sense: Every Noetherian local domain A having a coefficient field k and with field of fractions L finitely generated over k is an intersection $L \cap S$, as in Basic Construction Equation 1.3, where $S = \widehat{R}/I$ and I is a suitable ideal of the $\mathbf{m}$-adic completion $\widehat{R}$ of a Noetherian local domain $(R, \mathbf{m})$. Furthermore we can choose R so that k is also a coefficient field for the ring R, L is the field of fractions of R and R is essentially finitely generated over k; see Section 4.1 of Chapter 4.

Classical examples of Noetherian integral domains with interesting properties are constructed by Akizuki, Schmidt, and Nagata in [**9**], [**140**], and [**115**]. This work is continued by Brodmann-Rotthaus, Ferrand-Raynaud, Heitmann, Lequain, Nishimura, Ogoma, Rotthaus, Weston and others in [**20**], [**21**], [**42**], [**81**], [**82**], [**83**], [**90**], [**118**], [**123**], [**124**], [**131**], [**132**], and [**156**].

What are the classical examples?

Classical Examples 1.4. Many of the classical examples concern integral closures. Akizuki's 1935 example is a one-dimensional Noetherian local domain R of characteristic zero such that the integral closure of R is not a finitely generated R-module [**9**]. Schmidt's 1936 example is a one-dimensional normal Noetherian local domain R of positive characteristic such that the integral closure of R in a finite purely inseparable extension field is not a finitely generated R-module [**140**, pp. 445-447]. In relation to integral closure, Nagata's classic examples include (1) a two-dimensional Noetherian local domain with a non-Noetherian birational integral extension and (2) a three-dimensional Noetherian local domain such that the integral closure is not Noetherian [**117**, Examples 4 and 5, pp. 205-207].

In Example 4.14 of Chapter 4, we consider another example constructed by Nagata. This is the first occurence of a two-dimensional regular local domain containing a field of characteristic zero that fails to be a Nagata domain, and hence is not excellent. For the definition and information on Nagata rings and excellent rings; see Definitions 2.11 and 3.37 in Chapters 2 and 3, and see Chapter 13. We describe in Example 4.16 a construction due to Rotthaus of a Nagata domain that is not excellent.

In the foundational work of Akizuki, Nagata and Rotthaus (and indeed in most of the papers cited above) the description of the constructed ring A as the basic intersection domain of Equation 1.3 is not explicitly stated. Instead A is defined as a direct limit or directed union of subrings. In Chapters 4 to 6, we expand the basic construction to include an additional integral domain, also associated to the ideal-adic completion of R with respect to a principal ideal. Our expanded "Basic Construction" consists of two integral domains that fit with these examples:

Basic Construction 1.5. This construction consists of two integral domains described as follows:

(BC1) The "intersection" integral domain A of Basic Construction Equation 1.3: $A = L \cap S$, the intersection of a field L with a homomorphic image S of the completion of R with respect to a principal ideal, and

(BC2) An "approximation" domain B, that is a directed union inside A that approximates A and is more easily understood; sometimes B is a nested union of localized polynomial rings over R.

The details of the construction of B as in (BC2) are given in Chapters 5 and 17. Construction Properties Theorems 5.14 and 17.11 describe essential properties of the construction and are used throughout this book.

In certain circumstances the approximation domain B of (BC2) is equal to the intersection domain A of (BC1). In this case, the intersection domain A is a directed union. This yields more information about A. The description of A as an intersection is often unfathomable! In case $A = B$, the critical elements of B that determine L are called *limit-intersecting* over R; see Chapter 5 (Definition 5.10) and Chapters 8, 22 and 23 where we discuss the limit-intersecting condition further.

To see a specific example of the construction, consider the ring $R := \mathbb{Q}[x, y]$, the polynomial ring in the variables x and y over the field $\mathbb{Q}$ of rational numbers. Let S be the formal power series ring $\mathbb{Q}[[x, y]]$ and let L be the field $\mathbb{Q}(x, y, e^x, e^y)$.[3] Then Equation 1.3 yields that

$$\alpha = \frac{e^x - e^y}{x - y} \in A = \mathbb{Q}(x, y, e^x, e^y) \cap \mathbb{Q}[[x, y]], \tag{1.3.a}$$

but $\alpha \notin B$, the approximation domain. In this example, the intersection domain A is Noetherian, whereas the approximation domain B is not Noetherian. More details about this example are given in Example 4.10 of Chapter 4 and in Theorem 12.3 and Example 12.6 of Chapter 12.

A primary task of our study is to determine, for a given Noetherian domain R, whether the ring $A = L \cap S$ of Basic Construction Equation 1.3 is Noetherian. An important observation related to this task is that the Noetherian property for the associated direct limit ring B is equivalent to a flatness condition; see Noetherian Flatness Theorems 6.3 and 17.13. Whereas it took only about a page for Nagata [**117**, page 210] to establish the Noetherian property of his example, the proof of the Noetherian property for the example of Rotthaus took 7 pages [**131**, pages 112-118]. The results presented in Chapter 6 establish the Noetherian property rather quickly for this and other examples.

The construction of B as in (BC2) is related to an interesting construction introduced by Ray Heitmann [**81**, page 126]. Let x be a nonzero nonunit in a Noetherian integral domain R, and let R^* denote the (x)-adic completion of R. Heitmann describes a procedure for associating, to each element τ in R^* that is transcendental over R, an extension ring T of $R[\tau]$ having the property that the

[3]This example with power series in two variables does not come from one principal ideal-adic completion of R as in (BC1) above, but it may be realized, for example, by taking first the (x)-adic completion R^* and then taking the (y)-adic completion of R^*, an "iterative" process.

(x)-adic completion of T is R^*.[4] Heitmann uses this technique to construct interesting examples of non-catenary Noetherian rings. In a 1997 article, the present authors adapt the construction of Heitmann to prove a version of Noetherian Flatness Theorem 6.3 of Chapter 6 that applies for one transcendental element τ over a semilocal Noetherian domain R: If the element τ satisfies a flatness condition we call *primarily limit-intersecting*, then the constructed intersection domain A is equal to the approximation domain B and is Noetherian [**62**, Theorem 2.8]; see Remark 6.7.2.

This "primarily limit-intersecting" concept from [**62**] extends to more than one transcendental element τ; see Definition 22.9. This permits the extension of Heitmann's construction to finitely many algebraically independent elements of R^*; see [**62**, Theorem 2.12]. Thus, with Basic Construction Equation 1.3 as presented in Chapter 5, we are able to prove Noetherian Flatness Theorem 6.3 in the case where the base ring R is an arbitrary Noetherian integral domain with field of fractions K, the extension ring S is the (x)-adic completion R^* of R, and the field L is generated over K by a finite set of algebraically independent power series in S.

In the case where S is the ideal-adic completion R^* of R and L is a field between R and the total quotient ring of R^*, the integral domain $A = L \cap R^*$ sometimes inherits nice properties from R^*, for example, the Noetherian property. If the approximation domain B is Noetherian, then B is equal to the intersection domain A. The converse fails however; it is possible for B to be equal to A and not be Noetherian; see Example 10.9. If B is not Noetherian, we can sometimes determine the prime ideals of B that are not finitely generated; see Example 15.1. If a ring has exactly one prime ideal that is not finitely generated, that prime ideal contains all nonfinitely generated ideals of the ring.

In Section 6.2 of Chapter 6 and Chapter 12, we adjust the construction from Chapters 4 and 5. An "insider" technique is introduced in Section 6.2 of Chapter 6, and generalized in Chapter 10 for building new examples inside more straightforward examples constructed as above. Using Insider Construction 10.1, the verification of the Noetherian property for the constructed rings is streamlined. Even if one of the constructed rings is not Noetherian, the proof is simplified. We analyze classical examples of Nagata and others from this viewpoint in Section 6.3 and 6.4 of Chapter 6. Chapter 12 contains an investigation of more general rings that involve power series in two variables x and y over a field k, as is the case with the specific example given above in Equation 1.3.a.

In Chapters 15 and 16, we use Insider Construction 10.1 to construct low-dimensional non-Noetherian integral domains that are strangely close to being Noetherian: One example is a three-dimensional local unique factorization domain B inside $k[[x, y]]$; the ring B has maximal ideal $(x, y)B$ and exactly one prime ideal that is not finitely generated; see Example 15.1.

There has been considerable interest in non-Noetherian analogues of Noetherian notions such as the concept of a "regular" ring; see the book by Glaz [**48**]. Rotthaus and Sega in [**137**] show that the approximation domains B constructed in Chapters 15 and 16, even though non-Noetherian, are *coherent regular* local

[4]Heitmann remarks in [**81**] that this type of extension also occurs in [**117**, page 203]. The ring T is not finitely generated over $R[\tau]$ and no proper $R[\tau]$-subring of T has R^* as its (x)-adic completion. Necessary and sufficient conditions are given in order that T be Noetherian in Theorem 4.1 of [**81**].

rings by showing that every finitely generated submodule of a free module over B has a finite free resolution; see [**137**] and Remark 15.12.[5]

One of our additional goals is to consider the question: "What properties of a ring extend to a completion?" Chapter 11 contains an example of a three-dimensional regular local domain $(A, \mathbf{n})$ with a height-two prime ideal P such that the extension $P\widehat{A}$ to the $\mathbf{n}$-adic completion of A is not integrally closed. In Chapter 18 we prove that the Henselization of a Noetherian local ring having geometrically normal formal fibers is universally catenary; we also present for each integer $n \geq 2$ a catenary Noetherian local integral domain having geometrically normal formal fibers that is not universally catenary.

We consider excellence in regard to the question: "What properties of the base ring R are preserved by the construction?" Since excellence is an important property satisfied by most of our rings, we present in Chapter 13 a brief exposition of excellent rings. In some cases we determine when the constructed ring is excellent; for example, see Chapter 9 (Prototype Theorems 9.2, 17.25 and 17.28), Chapter 10 and Chapter 19. Assume the ring R is a unique factorization domain (UFD) and R^* is the (a)-adic completion of R with respect to a prime element a of R. We observe that the approximation domain B is then a UFD; see Theorem 5.17 of Chapter 5.

Since the Noetherian property for the approximation domain is equivalent to the flatness of a certain homomorphism, we devote considerable time and space to exploring flat extensions. We present results involving flatness in Chapters 6, 7, 8, 9, 20, 21, 22 and 23.

The application of Basic Construction Equation 1.3 in Chapters 20 and 21 yields "idealwise" examples that are of a different nature from the examples in earlier chapters. Whereas the base ring $(R, \mathbf{m})$ is an excellent normal local domain with $\mathbf{m}$-adic completion $(\widehat{R}, \widehat{\mathbf{m}})$, the field L is more general than in Chapter 5. We take L to be a purely transcendental extension of the field of fractions K of R such that L is contained in the field of fractions of $\widehat{R}$; say $L = K(G)$, where G is a set of elements of $\widehat{\mathbf{m}}$ that are algebraically independent over K. Define $D := L \cap \widehat{R}$. The set G is said to be *idealwise independent* if $K(G) \cap \widehat{R}$ equals the localized polynomial ring $R[G]_{(\mathbf{m},G)}$. The results of Chapters 20 and 21 show that the intersection domain can sometimes be small or large, depending on whether expressions in the power series allow additional prime divisors as denominators. The consideration of idealwise independence leads us to examine other related flatness conditions. The analysis and properties related to idealwise independence are summarized in Summaries 20.6 and 21.1.

In Chapters 22 and 23, we consider properties of the constructed rings A and B in the case where R is an excellent normal local domain. We draw connections with Cohen-Macaulay fibers and discuss properties of an example, due to Ogoma, of a three-dimensional normal Nagata local domain whose generic formal fiber is not equidimensional.

Let R be a Noetherian ring with Jacobson radical $\mathcal{J}$. In Chapter 19 we consider the multi-ideal-adic completion R^* of R with respect to a filtration $\mathcal{F} = \{Q_n\}_{n \geq 0}$,

[5]Rotthaus and Sega show more generally that the approximation domains constructed with Insider Construction 10.1 are coherent regular if $R = k[x, y_1 \ldots, y_r]_{(x, y_1 \ldots, y_r)}$ is a localized polynomial ring over a field k, $m = 1$, $r, n \in \mathbb{N}$ and $\tau_1, \ldots, \tau_n$ are algebraically independent elements of $xk[[x]]$. The approximation domains used by Rotthaus and Sega can have arbitrarily large Krull dimension, whereas the rings constructed in Chapters 15 and 16 have dimension 3 or 4.

where $Q_n \subseteq \mathcal{J}^n$ and $Q_{nk} \subseteq Q_n^k$ for each $n, k \in \mathbb{N}$. We prove that R^* is Noetherian. If R is an excellent local ring, we prove that R^* is excellent. If R is a Henselian local ring, we prove that R^* is Henselian.

In Chapter 26, we study prime ideals and their relations in mixed polynomial-power series extensions of low-dimensional rings. For example, we determine the prime ideal structure of the power series ring $R[[x]]$ over a one-dimensional Noetherian domain R and the prime ideal structure of $k[[x]][y]$, where x and y are indeterminates over a field k. We analyze the generic fibers of mixed polynomial-power series ring extensions in Chapter 24. Motivated by a question of Hochster and Yao, we consider in Chapter 27 extensions of integral domains $S \hookrightarrow T$ having *trivial generic fiber*; that is, every nonzero prime ideal of T intersects S in a nonzero prime ideal.

The topics of this book include the following:

(1) An introduction and glossary for the terms and tools used in the book, Chapters 2 and 3.
(2) The development of the construction of the intersection domain A and the approximation domain B, Chapters 4, 5, 6, 10, 17.
(3) Flatness properties of maps of rings, Chapters 6, 7, 8, 9, 20-23.
(4) Preservation of properties of rings and ideals under passage to completion, Chapters 11, 19.
(5) The catenary and universally catenary property of Noetherian rings, Chapter 9, 17, 18.
(6) Excellent rings and geometrically regular and geometrically normal formal fibers, Chapters 3, 13, 7, 10, 9, 17, 18.
(7) Examples of non-Noetherian local rings having Noetherian completions, Chapters 4, 5, 10, 12, 15-20, 28.
(8) Examples of Noetherian rings, Chapters 4, 5, 9, 11, 12, 14, 16, 17, 28.
(9) Prime ideal structure, Chapters 15, 24-27.
(10) Approximating a discrete rank-one valuation domain using higher-dimensional regular local rings, Chapter 14.
(11) Trivial generic fiber extensions, Chapters 24-27.
(12) Transfer of excellence, Chapters 10, 9, 19.
(13) Birational extensions of Noetherian domains, Chapters 6, 15, 16, 22, 23.
(14) Completions and multi-ideal-adic completions, Chapters 3, 19.
(15) Exercises to engage the reader in these topics and to lead to further extensions of the material presented here.

We thank Bruce Olberding for carefully reading this manuscript and for his many helpful suggestions.

The authors are grateful for the hospitality, cooperation and support of Michigan State, Nebraska, Purdue, CIRM in Luminy and MSRI in Berkeley, where we worked on this research.

CHAPTER 2

Tools

In this chapter we review conventions and terminology, state several basic theorems and review the concept of flatness of modules and homomorphisms.

2.1. Conventions and terminology

We generally follow the notation of Matsumura [**103**]. Thus by a ring we mean a commutative ring with identity, and a ring homomorphism $R \to S$ maps the identity element of R to the identity element of S. For commutative rings, when we write $R \subseteq S$, we mean that R is a subring of S, and that R contains the identity element of S. We use the words "map", "morphism", and "homomorphism" interchangeably.

The set of prime ideals of a ring R is called the *prime spectrum* of R and is denoted $\operatorname{Spec} R$. The set $\operatorname{Spec} R$ is naturally a partially ordered set with respect to inclusion. For an ideal I of a ring R, let

$$\mathcal{V}(I) \;=\; \{P \in \operatorname{Spec} R \mid I \subseteq P \;\}.$$

The *Zariski topology* on $\operatorname{Spec} R$ is obtained by defining the *closed* subsets to be the sets of the form $\mathcal{V}(I)$ as I varies over all the ideals of R. The *open* subsets are the complements $\operatorname{Spec} R \setminus \mathcal{V}(I)$.

We use $\mathbb{Z}$ to denote the ring of integers, $\mathbb{N}$ for the positive integers, $\mathbb{N}_0$ the non-negative integers, $\mathbb{Q}$ the rational numbers, $\mathbb{R}$ the real numbers and $\mathbb{C}$ the complex numbers.

Regular elements, regular sequence. An element r of a ring R is said to be a *zerodivisor* if there exists a nonzero element $a \in R$ such that $ar = 0$, and r is a *regular element* if r is not a zerodivisor.

A sequence of elements $x_1, \ldots, x_d$ in R is called a *regular sequence* if (i) $(x_1, \ldots, x_d)R \neq R$, and (ii) x_1 is a regular element of R, and, for i with $2 \leq i \leq d$, the image of x_i in $R/(x_1, \ldots, x_{i-1})R$ is a regular element; see [**103**, pages 123].

The *total ring of fractions* of the ring R, denoted $\mathcal{Q}(R)$, is the localization of R at the multiplicatively closed set of regular elements, thus $\mathcal{Q}(R) := \{\, a/b \mid a, b \in R$ and b is a regular element$\}$. There is a natural embedding $R \hookrightarrow \mathcal{Q}(R)$ of a ring R into its total ring of fractions $\mathcal{Q}(R)$, where $r \mapsto \frac{r}{1}$ for every $r \in R$.

An *integral domain*, sometimes called a *domain* or an *entire* ring, is a nonzero ring in which every nonzero element is a regular element. If R is a subring of an integral domain S and S is a subring of $\mathcal{Q}(R)$, we say S is *birational over* R, or a *birational extension* of R.

Krull dimension, height. The *Krull dimension*, or briefly *dimension*, of a ring R, denoted $\dim R$, is n if there exists a chain $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$ of prime ideals of R and there is no such chain of length greater than n. We say that $\dim R = \infty$ if there exists a chain of prime ideals of R of length greater than n

for each $n \in \mathbb{N}$. For a prime ideal P of a ring R, the *height* of P, denoted $\operatorname{ht} P$, is $\dim R_P$, where R_P is the localization of R at the multiplicatively closed set $R \setminus P$. The *height* of a proper ideal I, denoted $\operatorname{ht} I$, is defined to be

$$\operatorname{ht} I = \min\{\operatorname{ht} P \mid P \in \operatorname{Spec} R \text{ and } I \subseteq P \}.$$

We sometimes refer to $\dim(R/P)$ as the *dimension* of P.

Unique factorization domains. An integral domain R is a *unique factorization domain* (UFD), sometimes called a *factorial ring*, if every nonzero nonunit of R is a finite product of prime elements; an element $p \in R$ is *prime* if pR is a prime ideal.

In a UFD every height-one prime ideal is principal; this is Exercise 2.1.

Local rings. If a ring R (not necessarily Noetherian) has a unique maximal ideal $\mathbf{m}$, we say R is *local* and write $(R, \mathbf{m})$ to denote that R is local with maximal ideal $\mathbf{m}$. If $(R, \mathbf{m})$ and $(S, \mathbf{n})$ are local rings, a ring homomorphism $f : R \to S$ is a *local homomorphism* if $f(\mathbf{m}) \subseteq \mathbf{n}$.

Let $(R, \mathbf{m})$ be a local ring. A subfield k of R is said to be a *coefficient field for* R if the composite map $k \hookrightarrow R \to R/\mathbf{m}$ defines an isomorphism of k onto $R/\mathbf{m}$.

If $(R, \mathbf{m})$ is a subring of a local ring $(S, \mathbf{n})$, then S is said to *dominate* R if $\mathbf{m} = \mathbf{n} \cap R$, or equivalently, if the inclusion map $R \hookrightarrow S$ is a local homomorphism.

The local ring $(S, \mathbf{n})$ is said to *birationally dominate* $(R, \mathbf{m})$ if S is an integral domain that dominates R and S is contained in the field of fractions of R.

Nilradical, reduced. For an ideal I of a ring R, the *radical* of I, denoted $\sqrt{I}$, is the ideal $\sqrt{I} = \{a \in R \mid a^n \in I \text{ for some } n \in \mathbb{N}\}$. The ideal I is said to be a *radical ideal* if $\sqrt{I} = I$. The *nilradical* of a ring R is $\sqrt{(0)}$. The nilradical of R is the intersection of all the prime ideals of R. The ring R is said to be *reduced* if (0) is a radical ideal. Sometimes an ideal I of a ring R is said to have a property if R/I has that property. For example, sometimes a radical ideal I of R is called a *reduced ideal* since R/I is a reduced ring.

Jacobson radical. The *Jacobson radical* $\mathcal{J}(R)$ of a ring R is the intersection of all maximal ideals of R. An element z of R is in $\mathcal{J}(R)$ if and only if $1 + zr$ is a unit of R for all $r \in R$.

If I is a proper ideal of R, then $1 + I := \{\ 1 + a \mid a \in I\ \}$ is a multiplicatively closed subset of R that does not contain 0. Let $(1 + I)^{-1}R$ denote the localization $R_{(1+I)}$ of R at the multiplicatively closed set $1 + I$, [**103**, Section 4]. If P is a prime ideal of R and $P \cap (1 + I) = \emptyset$, then $(P + I) \cap (1 + I) = \emptyset$. Therefore I is contained in every maximal ideal of $(1 + I)^{-1}R$, so $I \subseteq \mathcal{J}((1 + I)^{-1}R)$. In particular for the principal ideal $I = zR$, where z is a nonunit of R, we have $z \in \mathcal{J}((1 + zR)^{-1}R)$.

Finite, finite type, finite presentation. Let R be a ring, let M be an R-module and let S be an R-algebra.

(1) M is said to be a *finite* R-module if M is finitely generated as an R-module.
(2) S is said to be *finite* over R if S is a finitely generated R-module.
(3) S is of *finite type* over R if S is finitely generated as an R-algebra. Equivalently, S is an R-algebra homomorphic image of a polynomial ring in finitely many variables over R.
(4) S is *finitely presented* as an R-algebra if, for some polynomial ring $R[x_1, \ldots, x_n]$ in variables $x_1, \ldots, x_n$ and R-algebra homomorphism φ :

$R[x_1, \ldots, x_n] \to S$ that is surjective, $\ker \varphi$ is a finitely generated ideal of $R[x_1, \ldots, x_n]$.

(5) S is *essentially finite* over R if S is a localization of a finite R-module.
(6) S is *essentially of finite type* over R if S is a localization of a finitely generated R-algebra. We also say that S is *essentially finitely generated* in this case.
(7) S is *essentially finitely presented* over R if S is a localization of a finitely presented R-algebra.

Symbolic powers. If P is a prime ideal of a ring R and e is a positive integer, the e^{th} *symbolic power of* P, denoted $P^{(e)}$, is defined as

$$P^{(e)} := \{a \in R \mid ab \in P^e \text{ for some } b \in R \setminus P\}.$$

Valuation domains. An integral domain R is a *valuation domain* if for each element $a \in \mathcal{Q}(R) \setminus R$, we have $a^{-1} \in R$. A valuation domain R is called a *discrete rank-one valuation ring* or a *discrete valuation ring* (DVR) if R is Noetherian and not a field; equivalently, R is a local principal ideal domain (PID) and not a field.

REMARKS 2.1. (1) If R is a valuation domain with field of fractions K and F is a subfield of K, then $R \cap F$ is again a valuation domain and has field of fractions F [**117**, (11.5)]. If R is a DVR and the field F is not contained in R, then $R \cap F$ is again a DVR [**117**, (33.7)].

(2) Every valuation domain R has an associated *valuation* v and *value group* G; the valuation v is a function $v : R \to G$ satisfying properties 1 and 2 of Remark 2.5, where the order function $ord_{R,I}$ is replaced by v in the equations of properties 1 and 2. See [**103**, p. 75] for more information about the value group and valuation associated to a valuation domain.

Algebraic independence. For a a subring R of a commutative ring S, we say that elements $a_1, \ldots, a_m \in S$ are *algebraically independent* over R if, for indeterminates $x_1, \ldots, x_m$ over R, the only polynomial $f(x_1, \ldots, x_m) \in R[x_1, \ldots, x_m]$ with $f(a_1, \ldots, a_m) = 0$ is the zero polynomial.

Integral ring extensions, integral closure, normal domains. Let R be a subring of commutative ring S.

(1) An element $a \in S$ is said to be *integral* over R if a is a root of some monic polynomial in the polynomial ring $R[x]$.
(2) The ring S is said to be *integral* over R, or an *integral extension* of R, if every element $a \in S$ is integral over R.
(3) The *integral closure* of R in S is the set of all elements of S that are integral over R.
(4) The ring R is said to be *integrally closed* in S if every element of S that is integral over R is in R.
(5) An integral domain R is said to be *integrally closed* if R is integrally closed in its field of fractions $\mathcal{Q}(R)$.
(6) The *integral closure* or *derived normal ring* of an integral domain R is the integral closure of R in its field of fractions.
(7) As in [**103**, page 64], we define the ring R to be a *normal ring* if for each $P \in \operatorname{Spec} R$ the localization R_P is an integrally closed domain. Since every localization of an integrally closed domain is again an integrally

closed domain [**103**, Example 3, page 65], an integrally closed domain is a normal ring.

REMARK 2.2. If R is a Noetherian normal ring and $\mathbf{p}_1, \ldots, \mathbf{p}_r$ are the minimal primes of R, then R is isomorphic to the direct product $R/\mathbf{p}_1 \times \cdots \times R/\mathbf{p}_r$ and each $R/\mathbf{p}_i$ is an integrally closed domain; see [**103**, page 64]. Since a nontrivial direct product is not local, a normal Noetherian local ring is a normal domain.

We record in Theorem 2.3 an important result about the integral closure of a normal Noetherian domain in a finite separable algebraic field extension; see [**103**, Lemma 1, page 262], [**117**, (10.16)], [**164**, Corollary 1, page 265], or [**4**, page 522].

THEOREM 2.3. *Let R be a normal Noetherian integral domain with field of fractions K. If L/K is a finite separable algebraic field extension, then the integral closure of R in L is a finite R-module. Thus, if R has characteristic zero, then the integral closure of R in a finite algebraic field extension is a finite R-module.*

REMARK 2.4. Let R be a normal integral domain with field of fractions K and let L/K be a finite separable algebraic field extension. The integral closure of R in L is always contained in a finitely generated R-module. Two different proofs of this are given in [**164**, Theorem 7, page 264]; both proofs involve a vector space basis for L/K of elements integral over R. The first proof uses the discriminant of this basis, while the second proof uses the dual basis determined by the trace map of L/K.

The order function associated to an ideal. Let I be a nonzero ideal of an integral domain R such that $\bigcap_{n=0}^{\infty} I^n = (0)$. Adopt the convention that $I^0 = R$, and for each nonzero element $r \in R$ define

$$\operatorname{ord}_{R,I}(r) \ := \ n \quad \text{if} \quad r \in I^n \ \setminus \ I^{n+1}.$$

In the case where $(R, \mathbf{m})$ is a local ring, we abbreviate $\operatorname{ord}_{R,\mathbf{m}}$ by ord_R.

REMARK 2.5. With R, I and $\operatorname{ord}_{R,I}$ as above, consider the following two properties for nonzero elements a, b in R:

(1) If $a + b \neq 0$, then $\operatorname{ord}_{R,I}(a+b) \geq \min\{\operatorname{ord}_{R,I}(a), \operatorname{ord}_{R,I}(b)\}$.
(2) $\operatorname{ord}_{R,I}(ab) = \operatorname{ord}_{R,I}(a) + \operatorname{ord}_{R,I}(b)$.

Clearly the function $\operatorname{ord}_{R,I}$ always satisfies property 1.

Assume $\operatorname{ord}_{R,I}$ satisfies property 2 for all nonzero a, b in R. Then the function $\operatorname{ord}_{R,I}$ extends uniquely to a function on $\mathcal{Q}(R) \setminus (0)$ by defining

$$\operatorname{ord}_{R,I}(\frac{a}{b}) \ := \ \operatorname{ord}_{R,I}(a) \ - \ \operatorname{ord}_{R,I}(b)$$

for nonzero elements $a, b \in R$, and the set

$$V \ := \ \{q \ \in \mathcal{Q}(R) \ \setminus \ (0) \mid \operatorname{ord}_{R,I}(q) \ \geq \ 0 \ \} \ \cup \ \{0\}$$

is a DVR. Moreover, if $\mathbf{m}_V$ denotes the maximal ideal of V, then $R \cap \mathbf{m}_V = I$.

Thus, if $\operatorname{ord}_{R,I}$ satisfies property 2 for all nonzero a, b in R, then I is a prime ideal of R, the function $\operatorname{ord}_{R,I}$ is the valuation on V described in Remark 2.1.2, and the value group is the integers viewed as an additive group.

Let A be a commutative ring and let $R := A[[x]] = \{f = \sum_{i=0}^{\infty} f_i x^i \mid f_i \in A\}$, the *formal power series ring* over A in the variable x. With $I := xR$ and f a

nonzero element in R, we write $\operatorname{ord} f$ for $\operatorname{ord}_{R,I}(f)$. Thus $\operatorname{ord} f$ is the least integer $i \geq 0$ such that $f_i \neq 0$. The element f_i is called *leading form* of f.

Regular local rings. A local ring $(R, \mathbf{m})$ is a *regular local ring*, often abbreviated *RLR*, if R is Noetherian and $\mathbf{m}$ can be generated by $\dim R$ elements. If $(R, \mathbf{m})$ is a regular local ring, then R is an integral domain; thus we may say R is a *regular local domain*. The order function ord_R of a RLR satisfies the properties of Remark 2.5, and the associated valuation domain

$$V := \{q \in \mathcal{Q}(R) \setminus \{0\} \mid \operatorname{ord}_R(q) \geq 0\} \cup \{0\}$$

is a DVR that birationally dominates R. If $x \in \mathbf{m} \setminus \mathbf{m}^2$, then $V = R[\mathbf{m}/x]_{xR[\mathbf{m}/x]}$, where $\mathbf{m}/x = \{y/x \mid y \in \mathbf{m}\}$.

REMARKS 2.6. (1) A regular local ring is a normal Noetherian local domain; normality is proved in [**103**, Theorem 19.4] using a result of Serre.

(2) A regular local ring is a UFD; see [**103**, Theorem 20.3]. This result, first proved in 1959 by Auslander and Buchsbaum [**13**], was a significant triumph for homological methods in commutative algebra.

Krull domains. We record the definition of Krull domain:

DEFINITION 2.7. An integral domain R is said to be a *Krull domain* if there exists a family $\mathcal{F} = \{V_\lambda\}_{\lambda \in \Lambda}$ of DVRs of its field of fractions $\mathcal{Q}(R)$ such that

- $R = \bigcap_{\lambda \in \Lambda} V_\lambda$, and
- Every nonzero element of $\mathcal{Q}(R)$ is a unit in all but finitely many of the V_λ.

We give some properties of Krull domains in Remarks 2.8.

REMARKS 2.8. (1) A unique factorization domain (UFD) is a Krull domain, and a Noetherian integral domain is a Krull domain if and only if it is integrally closed. An integral domain R is a Krull domain if and only if it satisfies the following three properties:
   - R_P is a DVR for each prime ideal P of R of height one.
   - $R = \bigcap\{\ R_P \mid P \text{ is a height-one prime } \}$.
   - Every nonzero element of R is contained in only finitely many height-one primes of R.

(2) If R is a Krull domain, then $\mathcal{F} = \{R_P \mid P \text{ is a height-one prime ideal } \}$ is the unique minimal set of DVRs satisfying the properties in the definition of a Krull domain [**103**, Theorem 12.3]. The family $\mathcal{F}$ is called the family of *essential valuation rings* of R. For each nonzero nonunit a of R the principal ideal aR has no embedded associated prime ideals and a unique irredundant primary decomposition $aR = q_1 \cap \cdots \cap q_t$. If $p_i = \sqrt{(q_i)}$, then $R_{p_i} \in \mathcal{F}$ and q_i is a symbolic power of p_i; that is, $q_i = p_i^{(e_i)}$, where $e_i \in \mathbb{N}$; see [**103**, Corollary, page 88].

Krull domains have an approximation property with respect to the family of DVRs and valuations (as in Remarks 2.1.2) obtained by localizing at height-one primes.

THEOREM 2.9. *(Approximation Theorem* [**103**, Theorem 12.6]*) For A a Krull domain with field of fractions K, let $P_1, \ldots, P_r$ be height-one primes of A, and let*

v_i denote the valuation with value group $\mathbb{Z}$ associated to the DVR A_{P_i}, for each i with $1 \leq i \leq r$. For arbitrary integers $e_1, \ldots, e_r$, there exists $x \in K$ such that

$$v_i(x) \;=\; e_i \quad \textit{for} \quad 1 \;\leq\; i \;\leq\; r \quad \textit{and} \quad v(x) \;\geq\; 0,$$

for every valuation v associated to a height-one prime ideal of A that is not in the set $\{P_1, \ldots, P_r\}$.

DEFINITION 2.10. Let R be a Krull domain and let $R \hookrightarrow S$ be an inclusion map of R into a Krull domain S. The extension $R \hookrightarrow S$ satisfies the **PDE** condition ("pas d'éclatement", or in English "no blowing up") provided that for every height-one prime ideal Q in S, the height of $Q \cap R$ is at most one [**41**, page 30].

Nagata rings. In the 1950s Nagata introduced and investigated a class of Noetherian rings that behave similarly to rings that arise in algebraic geometry [**110**], [**112**]. In Nagata's book, Local Rings [**117**], the rings in this class are called *pseudo-geometric*. Following Matsumura, we call these rings *Nagata rings*:

DEFINITION 2.11. A commutative ring R is called a *Nagata* ring if R is Noetherian and, for every $P \in \operatorname{Spec} R$ and every finite extension field L of $\mathcal{Q}(R/P)$, the integral closure of R/P in L is finitely generated as a module over R/P.

It is clear from the definition that a homomorphic image of a Nagata ring is again a Nagata ring. We refer to the following non-trivial theorem due to Nagata as *Nagata's Polynomial Theorem*.

THEOREM 2.12. *(Nagata's Polynomial Theorem)* [**117**, Theorem 36.5, page 132] *If A is a Nagata ring and $x_1, \ldots, x_n$ are indeterminates over A, then the polynomial ring $A[x_1, \ldots, x_n]$ is a Nagata ring. It follows that every algebra essentially of finite type over a Nagata ring is again a Nagata ring.*

By Theorem 2.12, every algebra of finite type over a field, over the ring of integers, or over a discrete valuation ring of characteristic 0 is a Nagata ring.

Henselian rings. It is mentioned in [**117**, p. 221], that the notion of Henselian rings was introduced by Azumaya [**15**].

DEFINITION 2.13. [**117**, p. 103] A local ring $(R, \mathbf{m})$ is *Henselian* provided the following holds: for every monic polynomial $f(x) \in R[x]$ satisfying $f(x) \equiv g_0(x)h_0(x)$ modulo $\mathbf{m}[x]$, where g_0 and h_0 are monic polynomials in $R[x]$ such that

$$g_0R[x] + h_0R[x] + \mathbf{m}[x] = R[x],$$

there exist monic polynomials $g(x)$ and $h(x)$ in $R[x]$ such that $f(x) = g(x)h(x)$ and such that both

$$g(x) \;-\; g_0(x) \quad \text{and} \quad h(x) \;-\; h_0(x) \;\in\; \mathbf{m}[x].$$

Thus Henselian rings are precisely those local rings that satisfy the property asserted for complete local rings in Hensel's Lemma 2.14.

LEMMA 2.14. Hensel's Lemma [**103**, Theorem 8.3] *Let $(R, \mathbf{m})$ be a complete local ring, let x be an indeterminate over R, let $f(x) \in R[x]$ be a monic polynomial and let $\overline{f}$ be the polynomial obtained by reducing the coefficients mod $\mathbf{m}$. If $\overline{f}(x)$ factors modulo $\mathbf{m}[x]$ into two comaximal factors, then this factorization can be lifted back to $R[x]$.*

The concept of the Henzelization of a local ring was introduced by Natata [**109**], [**111**], [**116**]. We list results concerning Henselian rings and Henselization from [**117**], where proofs are given for these results.[1]

REMARKS 2.15. (1) Associated with every local ring $(R, \mathbf{m})$, there exists an extension ring that is Henselian and local, called the *Henselization* of R and denoted $(R^h, \mathbf{m}^h)$; see [**117**, Theorem 43.5 and the four paragraphs preceding it, p. 180]. By [**117**, (43.3), p. 180, and Theorem 43.5, p. 181], R^h dominates R, R^h has the same residue field as R and $\mathbf{m}R^h = \mathbf{m}^h$. Moreover, by [**117**, page 182], the Henselization R^h of R is unique up to an R-isomomorphism.

(2) The Henselization R^h of a local ring R is faithfully flat over R [**117**, Theorem 43.8]; the concept of faithful flatness is defined in Definitions 2.30. It follows that $R/\mathbf{m}^n$ is canonically isomorphic to $R^h/(\mathbf{m}^h)^n$, for each $n \in \mathbb{N}$. Thus the $\mathbf{m}$-adic completion $\widehat{R}$ of R is also the $\mathbf{m}^h$-adic completion of R^h; the $\mathbf{m}$-adic topology and completion are defined in Definitions 3.1.

(3) If $(R, \mathbf{m})$ is a Noetherian local ring, then $(R^h, \mathbf{m}^h)$ is a Noetherian local ring such that with respect to the topologies on R and R^h defined by $\mathbf{m}$ and $\mathbf{m}^h$, respectively, R is a dense subspace of R^h [**117**, Theorem 43.10]. Thus we have $R \hookrightarrow R^h \hookrightarrow \widehat{R}$. Every complete Noetherian local ring is Henselian [**117**, Theorem 30.3].

(4) If R is Henselian, then $R^h = R$ [**117**, (43.11)].

(5) If $(R, \mathbf{m})$ is a local integral domain, then R is Henselian $\iff$ for every integral domain S that is an integral extension of R, S is a local domain [**117**, Theorem 43.12].

(6) If R is a Henselian ring and R' is a local ring that is integral over R, then R' is Henselian [**117**, Corollary 43.16].

(7) If $(R', \mathbf{m}')$ is a local ring that is integral over a local ring $(R, \mathbf{m})$, then $R' \otimes_R R^h = (R')^h$ [**117**, Theorem 43.17].

(8) If $(R', \mathbf{m}')$ is a local ring that dominates the local ring $(R, \mathbf{m})$ and if R' is a localization of a finitely generated integral extension, then $(R')^h$ is a finitely generated module over R^h [**117**, Theorem 43.18].

We give more information about Nagata rings, Henselian rings and the Henselization of a local ring in Chapter 13.

2.2. Basic theorems

Theorem 2.16 is a famous result proved by Krull that is now called the Krull Intersection Theorem.

THEOREM 2.16 (Krull [**103**, Theorem 8.10]). *Let I be an ideal of a Noetherian ring R.*

(1) *If I is contained in the Jacobson radical $\mathcal{J}(R)$ of R, then $\bigcap_{n=1}^{\infty} I^n = 0$, and, for each finite R-module M, we have $\bigcap_{n=1}^{\infty} I^n M = 0$.*

(2) *If I is a proper ideal of a Noetherian integral domain, then $\bigcap_{n=1}^{\infty} I^n = 0$.*

Theorem 2.17 is another famous result of Krull that is now called the Krull Altitude Theorem. It involves the concept of a minimal prime divisor of an ideal

[1]The notation in [**117**], in particular the meaning of "local ring" and "finite type", differ from our usage in this book. We have adjusted these results to our terminology.

I of a ring R, where $P \in \operatorname{Spec} R$ is a *minimal prime divisor* of I if $I \subseteq P$ and if $P' \in \operatorname{Spec} R$ and $I \subseteq P' \subseteq P$, then $P' = P$.

THEOREM 2.17 (Krull [**103**, Theorem 13.5]). *Let R be a Noetherian ring and let $I = (a_1, \ldots, a_r)R$ be an ideal generated by r elements. If P is a minimal prime divisor of I, then $\operatorname{ht} P \leq r$. Hence the height of a proper ideal of R is finite.*

Theorem 2.18 is yet another famous result that is now called the Krull-Akizuki Theorem.

THEOREM 2.18 (Krull-Akizuki [**103**, Theorem 11.7]). *Let A be a one-dimensional Noetherian integral domain with field of fractions K, let L be a finite algebraic field extension of K, and let B be a subring of L with $A \subseteq B$. Then*

(1) *The ring B is Noetherian of dimension at most one.*
(2) *If J is a nonzero ideal of B, then B/J is an A-module of finite length.*

To prove that a ring is Noetherian, it suffices by the following well-known result of Cohen to prove that every prime ideal of the ring is finitely generated.

THEOREM 2.19 (Cohen [**29**]). *If each prime ideal of the ring R is finitely generated, then R is Noetherian.*

Theorem 2.20 is another important result proved by Cohen.

THEOREM 2.20 (Cohen [**30**]). *Let R be a Noetherian integral domain and let S be an extension domain of R. For $P \in \operatorname{Spec} S$ and $\mathbf{p} = P \cap R$, we have*

$$\operatorname{ht} P \ + \ tr.deg._{k(\mathbf{p})}k(P) \ \leq \ \operatorname{ht} \mathbf{p} \ + \ tr.deg._R S,$$

where $k(\mathbf{p})$ is the field of fractions of $R/\mathbf{p}$ and $k(P)$ is the field of fractions of S/P.

Theorem 2.21 is a useful result due to Nagata about Krull domains and UFDs.

THEOREM 2.21. [**138**, Theorem 6.3, p. 21] *Let R be a Krull domain. If S is a multiplicatively closed subset of R generated by prime elements and $S^{-1}R$ is a UFD, then R is a UFD.*

We use the following:

FACT 2.22. If D is an integral domain and c is a nonzero element of D such that cD is a prime ideal, then $D = D[1/c] \cap D_{cD}$.

PROOF. Let $\beta \in D[1/c] \cap D_{cD}$. Then $\beta = \frac{b}{c^n} = \frac{b_1}{s}$ for some $b, b_1 \in D$, $s \in D \setminus cD$ and integer $n \geq 0$. If $n > 0$, we have $sb = c^n b_1 \implies b \in cD$. Thus we may reduce to the case where $n = 0$; it follows that $D = D[1/c] \cap D_{cD}$. □

REMARKS 2.23. (1) If R is a Noetherian integral domain and S is a multiplicatively closed subset of R generated by prime elements, then $S^{-1}R$ a UFD implies that R is a UFD [**138**, Theorem 6.3] or [**103**, Theorem 20.2].

(2) If x is a nonzero prime element in an integral domain R such that R_{xR} is a DVR and $R[1/x]$ is a Krull domain, then R is a Krull domain by Fact 2.22; and, by Theorem 2.21, R is a UFD if $R[1/x]$ is a UFD.

(3) Let R be a valuation domain with value group $\mathbb{Z} \oplus \mathbb{Z}$ ordered lexicographically; that is, for every pair $(a, b), (c, d)$ of elements of $\mathbb{Z} \oplus \mathbb{Z}$, $(a, b) > (c, d) \iff a > c$, or $a = c$ and $b > d$. Then the maximal ideal $\mathbf{m}$ of R is principal, say $\mathbf{m} = xR$. It follows that $R[1/x]$ is a DVR; however R is not a Krull domain.

The Eakin-Nagata Theorem is useful for proving descent of the Noetherian property.

THEOREM 2.24 (Eakin-Nagata [**103**, Theorem 3.7(i)]). *If B is a Noetherian ring and A is a subring of B such that B is a finitely generated A-module, then A is Noetherian.*

An interesting result proved by Nishimura is

THEOREM 2.25 (Nishimura [**118**, Theorem, page 397], or [**103**, Theorem 12.7]). *Let R be a Krull domain. If R/P is Noetherian for every height-one prime ideal P of R, then R is Noetherian.*

REMARK 2.26. It is observed in [**60**, Lemma 1.5] that the conclusion of Theorem 2.25 still holds if it is assumed that R/P is Noetherian for all but at most finitely many of the height-one primes P of R.

Theorem 2.27 is useful for describing the maximal ideals of a power series ring $R[[x]]$. It is related to the fact that an element $f = a_0 + a_1x + a_2x^2 + \cdots \in R[[x]]$ with the $a_i \in R$ is a unit of $R[[x]]$ if and only if a_0 is a unit of R.

THEOREM 2.27 ([**117**, Theorem 15.1]). *Let $R[[x]]$ be the formal power series ring in a variable x over a commutative ring R. There is a one-to-one correspondence between the maximal ideals $\mathbf{m}$ of R and the maximal ideals $\mathbf{m}^*$ of $R[[x]]$ where $\mathbf{m}^*$ corresponds to $\mathbf{m}$ if and only if $\mathbf{m}^*$ is generated by $\mathbf{m}$ and x.*

As an immediate corollary of Theorem 2.27, we have

COROLLARY 2.28. *The element x is in the Jacobson radical $\mathcal{J}(R[[x]])$ of the power series ring $R[[x]]$. In the formal power series ring $S := R[[x_1, \ldots, x_n]]$, the ideal $(x_1, \ldots, x_n)S$ is contained in the Jacobson radical $\mathcal{J}(S)$ of S.*

Theorem 2.29 is an important result first proved by Chevalley.

THEOREM 2.29 (Chevalley [**27**]). *If $(R, \mathbf{m})$ is a Noetherian local domain, then there exists a DVR that birationally dominates R.*

More generally, let P be a prime ideal of a Noetherian integral domain R. There exists a DVR V that birationally contains R and has center P on R, that is, the maximal ideal of V intersects R in P.

2.3. Flatness

The concept of flatness was introduced by Serre in the 1950's in an appendix to his paper [**141**]. Mumford writes in [**105**, page 424]: "The concept of flatness is a riddle that comes out of algebra, but which technically is the answer to many prayers."

DEFINITIONS 2.30. A module M over a ring R is *flat* over R if tensoring with M preserves exactness of every exact sequence of R-modules. The R-module M is said to be *faithfully flat* over R if, for every sequence $\mathcal{S}$ of R-modules,

$$\mathcal{S}: \quad 0 \longrightarrow M_1 \longrightarrow M_2,$$

the sequence $\mathcal{S}$ is exact if and only if its tensor product with M, $\mathcal{S} \otimes_R M$, is exact.

A ring homomorphism $\phi : R \to S$ is said to be a *flat homomorphism* if S is flat as an R-module.

Flatness is preserved by several standard ring constructions as we record in Remarks 2.31. There is an interesting elementwise criterion for flatness that is stated as item 2 of Remarks 2.31.

REMARKS 2.31. The following facts are useful for understanding flatness. We use these facts to obtain the results in Chapters 6 and 15.

(1) Since localization at prime ideals commutes with tensor products, the module M is flat as an R-module $\iff$ M_Q is flat as an R_Q-module, for every prime ideal Q of R.
(2) An R-module M is flat over R if and only if for every $m_1, \ldots, m_n \in M$ and $a_1, \ldots, a_n \in R$ such that $\sum a_i m_i = 0$, there exist a positive integer k, a subset $\{b_{ij}\}_{i=1,j=1}^{n,k} \subseteq R$, and elements $m'_1, \ldots, m'_k \in M$ such that $m_i = \sum_{j=1}^{k} b_{ij} m'_j$ for each i and $\sum_{i=1}^{n} a_i b_{ij} = 0$ for each j; see [**103**, Theorem 7.6] or [**101**, Theorem 1]. Thus every free module is flat, and a nested union of flat modules is flat.
(3) A finitely generated module over a local ring is flat if and only if it is free [**101**, Proposition 3.G].
(4) If the ring S is a localization of R, then S is flat as an R-module [**101**, (3.D), page 19].
(5) Let S be a flat R-algebra. Then S is faithfully flat over R $\iff$ one has $JS \neq S$ for every proper ideal J of R; see [**101**, Theorem 3, page 28] or [**103**, Theorem 7.2].
(6) If the ring S is a flat R-algebra, then every regular element of R is regular on S [**101**, (3.F)].
(7) Let S be a faithfully flat R-algebra and let I be an ideal of R. Then $IS \cap R = I$ [**103**, Theorem 7.5].
(8) Let R be a subring of a ring S. If S is Noetherian and faithfully flat over R, then R is Noetherian; see Exercise 8 at the end of this chapter.
(9) Let R be an integral domain with field of fractions K and let S be a faithfully flat R-algebra. By item 6, every nonzero element of R is regular on S and so K naturally embeds in the total quotient ring $\mathcal{Q}(S)$ of S. By item 7, all ideals in R extend and contract to themselves with respect to S, and thus $R = K \cap S$. In particular, if $S \subseteq K$, then $R = S$ [**101**, page 31].
(10) If $\phi : R \to S$ is a flat homomorphism of rings, then ϕ satisfies the Going-down Theorem [**101**, (5.D), page 33]. This implies for each $P \in \operatorname{Spec} S$ that the height of P in S is greater than or equal to the height of $\phi^{-1}(P)$ in R.
(11) Let $R \to S$ be a flat homomorphism of rings and let I and J be ideals of R. Then $(I \cap J)S = IS \cap JS$. If J is finitely generated, then $(I :_R J)S = IS :_S JS$; see [**103**, Theorem 7.4] or [**101**, (3.H) page 23].
(12) Consider the following short exact sequence of R-modules:
$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0\,.$$
The modules A and C are flat over R if and only if B is [**103**, Theorem 7.9].
(13) If S is a flat R-algebra and M is a flat S-module, then M is a flat R-module [**103**, page 46].
(14) If S is an R-algebra and M is faithfully flat over both R and S, then S is faithfully flat over R [**103**, page 46].

The following standard result about flatness follows from what Matsumura calls "change of coefficient ring". It is convenient to refer to both the module and homomorphism versions.

FACT 2.32. Let C be a commutative ring, let D, E and F be C-algebras.

(1) If $\psi : D \to E$ is a flat, respectively faithfully flat, C-algebra homomorphism, then $\psi \otimes_C 1_F : D \otimes_C F \to E \otimes_C F$ is a flat, respectively faithfully flat, C-algebra homomorphism.
(2) If E is a flat, respectively faithfully flat, D-module via the C-algebra homomorphism ψ, then $E \otimes_C F$ is a flat, respectively faithfully flat, $D \otimes_C F$-module via the C-algebra homomorphism $\psi \otimes_C 1_F$.

PROOF. By the definition of flat, respectively faithfully flat, homomorphism in Definitions 2.30, the two statements are equivalent. Since E is a flat, respectively faithfully flat, D-module, $E \otimes_D (D \otimes_C F)$ is a flat, respectively faithfully flat, $(D \otimes_C F)$-module by [**103**, p. 46, Change of coefficient ring]. Since $E \otimes_D (D \otimes_C F) = E \otimes_C F$, Fact 2.32 follows. □

We use Remark 2.33.3 in Chapter 12.

REMARKS 2.33. Let R be an integral domain.

(1) Every flat R-module M is torsionfree, i.e., if $r \in R, x \in M$ and $rx = 0$, then $r = 0$ or $x = 0$; see [**101**, (3.F), page 21]
(2) Every finitely generated torsionfree module over a PID is free; see for example [**35**, Theorem 5, page 462].
(3) Every torsionfree module over a PID is flat. This follows from item 2 and Remark 2.31.2.
(4) Every injective homomorphism of R into a field is flat. This follows from Remarks 2.31.13 and 2.31.4.

In Chapter 3 we discuss other tools we will be using involving ideal-adic completions and properties of excellent rings .

Exercises

(1) Prove that every height-one prime ideal of a UFD is principal.
(2) Let V be a local domain with nonzero principal maximal ideal yV. Prove that V is a DVR if $\bigcap_{n=1}^{\infty} y^n V = (0)$.
Comment: It is not being assumed that V is Noetherian, so it needs to be established that V has dimension one.
(3) Prove as stated in Remark 2.1 that if R is a valuation domain with field of fractions K and F is a subfield of K, then $R \cap F$ is again a valuation domain and has field of fractions F; also prove that if R is a DVR and the field F is not contained in R, then $R \cap F$ is again a DVR.
(4) Prove that a unique factorization domain is a Krull domain.
(5) Let R be a Noetherian ring. Let $P_1 \subset P_2$ be prime ideals of R. If there exists a prime ideal Q of R with Q distinct from P_1 and P_2 such that $P_1 \subset Q \subset P_2$, prove that there exist infinitely many such prime ideals Q.

Suggestion: Apply Krull's Altitude Theorem 2.17, and use the fact that an ideal contained in a finite union of primes is contained in one of them; see for example [**11**, Proposition 1.11, page 8].

(6) Prove as asserted in Remark 2.5 that, if $\mathrm{ord}_{R,I}(ab) = \mathrm{ord}_{R,I}(a) + \mathrm{ord}_{R,I}(b)$, for all nonzero a, b in R, and if we define $\mathrm{ord}_{R,I}(\frac{a}{b}) := \mathrm{ord}_{R,I}(a) - \mathrm{ord}_{R,I}(b)$ for nonzero elements $a, b \in R$, then:
   (a) The function $\mathrm{ord}_{R,I}$ extends uniquely to a function on $\mathcal{Q}(R) \setminus (0)$ with this definition.
   (b) $V := \{q \in \mathcal{Q}(R) \setminus (0) \mid \mathrm{ord}_{R,I}(q) \geq 0\} \cup \{0\}$ is a DVR, and
   (c) R is an integral domain and I is a prime ideal.

(7) Let $R[[x]]$ be the formal power series ring in a variable x over a commutative ring R.
   (i) Prove that $a_0 + a_1x + a_2x^2 + \cdots \in R[[x]]$, where the $a_i \in R$, is a unit of $R[[x]]$ if and only if a_0 is a unit of R.
   (ii) Prove that x is contained in every maximal ideal of $R[[x]]$.
   (iii) Prove Theorem 2.27 that the maximal ideals $\mathbf{m}$ of R are in one-to-one correspondence with the maximal ideals $\mathbf{m}^*$ of $R[[x]]$, where $\mathbf{m}^*$ corresponds to $\mathbf{m}$ if and only if $\mathbf{m}^*$ is generated by $\mathbf{m}$ and x.

(8) Prove items 4-8 of Remarks 2.31.

Suggestion: For the proof of item 8, use item 7.

(9) Let $f : A \to B$ be a ring homomorphism and let P be a prime ideal of A. Prove that there exists a prime ideal Q in B that contracts in A to P if and only if the extended ideal $f(P)B$ contracts to P in A, i.e., $P = f(P)B \cap A$. (Here we are using the symbol $\cap$ as in Matsumura [**103**, item (3), page xiii].)

(10) Let $f : A \hookrightarrow B$ be an injective ring homomorphism and let P be a minimal prime of A.
   (i) Prove that there exists a prime ideal Q of B that contracts in A to P.
   (ii) Deduce that there exists a minimal prime Q of B that contracts in A to P.

Suggestion: Consider the multiplicatively closed set $A \setminus P$ in B.

(11) Let P be a height-one prime of a Krull domain A and let v denote the valuation with value group $\mathbb{Z}$ associated to the DVR A_P. If A/P is Noetherian, prove that $A/P^{(e)}$ is Noetherian for every positive integer e.

Suggestion: Using Theorem 2.9, show there exists $x \in \mathcal{Q}(A)$ such that $v(x) = 1$ and $1/x \in A_Q$ for every height-one prime Q of A different from P. Let $B = A[x]$.
   (i) Show that $P = xB \cap A$ and $B = A + xB$.
   (ii) Show that $A/P \cong B/xB \cong x^iB/x^{i+1}B$ for every positive integer i.
   (iii) Deduce that B/x^eB is a Noetherian B-module and thus a Noetherian ring.
   (iv) Prove that $x^eB \cap A \subseteq x^eA_P \cap A = P^{(e)}$ and B/x^eB is a finite $A/(x^eB \cap A)$-module generated by the images of $1, x, \ldots, x^{e-1}$.
   (v) Apply Theorem 2.24 to conclude that $A/(x^eB \cap A)$ and hence $A/P^{(e)}$ is Noetherian.

(12) Let A be a Krull domain having the property that A/P is Noetherian for all but at most finitely many of the $P \in \mathrm{Spec}\, A$ with $\mathrm{ht}\, P = 1$. Prove that A is Noetherian.

Suggestion: By Nishimura's result Theorem 2.25, and Cohen's result Theorem 2.19, it suffices to prove each prime ideal of A of height greater than one is finitely generated. Let $P_1, \ldots, P_n$ be the height-one prime ideals of A for which A/P_i may fail to be Noetherian. For each nonunit $a \in A \setminus (P_1 \cup \cdots \cup P_n)$, observe that $aA = Q_1^{(e_1)} \cap \cdots \cap Q_s^{(e_s)}$, where $Q_1, \ldots, Q_s$ are height-one prime ideals of A not in the set $\{P_1, \ldots, P_n\}$. Consider the embedding $A/aA \hookrightarrow \prod (A/Q_i^{(e_i)})$. By Exercise 11, each $A/Q_i^{(e_i)}$ is Noetherian. Apply Theorem 2.24 to conclude that A/aA is Noetherian. Deduce that every prime ideal of A of height greater than one is finitely generated.

(13) Let R be a two-dimensional Noetherian integral domain. Prove that every Krull domain that birationally dominates R is Noetherian.

Comment: It is known that the integral closure of a two-dimensional Noetherian integral domain is Noetherian [**117**, (33.12)]. A proof of Exercise 13 is given in [**54**, Theorem 9]. An easier proof may be obtained using Nishimura's result Theorem 2.25.

CHAPTER 3

More tools

In this chapter we discuss ideal-adic completions. We describe several results concerning complete local rings. We review the definitions of catenary and excellent rings and record several results about these rings.

3.1. Introduction to ideal-adic completions

DEFINITIONS 3.1. Let R be a commutative ring with identity. A *filtration* on R is a descending sequence $\{I_n\}_{n=0}^{\infty}$ of ideals of R. Since $I_{n+1} \subset I_n$, the natural maps $R/I_{n+1} \to R/I_n$ form an inverse system. Associated to the filtration $\{I_n\}$, there is a well-defined completion R^* that may be defined to be the inverse limit [1]

$$R^* \;=\; \varprojlim_n \; R/I_n. \tag{3.1.1}$$

There is a canonical homomorphism $\psi : R \to R^*$ [**121**, Chapter 9], and the map ψ induces a map $R \to R^*/I_nR^*$ such that

$$R^*/I_nR^* \;\cong\; R/I_n; \tag{3.1.2}$$

see [**121**, page 412] or [**103**, page 55] for more details.

Regarding the filtration $\{I_n\}_{n=0}^{\infty}$ as a system of neighborhoods of 0, and defining for each $x \in R$ the family $\{x+I_n\}$ to be a system of neighborhoods of x, makes R a topological group under addition. This type of topology is called a *linear topology* on R. For more details and an extension to R-modules, see [**103**, Section 8].

If $\bigcap_{n=0}^{\infty} I_n = (0)$, then this linear topology is Hausdorff [**103**, page 55] and gives rise to a metric on R: For $x \neq y \in R$, the distance from x to y is $d(x,y) = 2^{-n}$, where n is the largest n such that $x - y \in I_n$. In particular, the map ψ is injective, and R may be regarded as a subring of R^*.

In the terminology of Northcott, a filtration $\{I_n\}_{n=0}^{\infty}$ is said to be *multiplicative* if $I_0 = R$ and $I_nI_m \subseteq I_{n+m}$, for all $m \geq 0$, $n \geq 0$ [**121**, page 408]. A well-known example of a multiplicative filtration on R is the I-adic filtration $\{I^n\}_{n=0}^{\infty}$, where I is a fixed ideal of R. In this case we say $R^* := \varprojlim_n \; R/I^n$ is the *I-adic completion* of R. If the canonical map $R \to R^*$ is an isomorphism, we say that R is *I-adically complete*. An ideal L of R is *closed in the* I-adic topology on R if $\bigcap_{n=1}^{\infty}(L+I^n) = L$.

We reserve the notation $\widehat{R}$ for the situation where R is a local ring with maximal ideal $\mathbf{m}$ such that $\bigcap_{n=0}^{\infty} \mathbf{m}^n = (0)$ and $\widehat{R}$ is the $\mathbf{m}$-adic completion of R. For a local ring $(R, \mathbf{m})$, we say that $\widehat{R}$ is "the" *completion* of R. If $\mathbf{m}$ is generated by elements

[1] We refer to Appendix A of [**103**] for the definition of direct and inverse limits. Also see the discussion of inverse limits in [**11**, page 103].

$a_1, \ldots, a_n$, then $\widehat{R}$ is realizable by taking the a_1-adic completion R_1^* of R, then the a_2-adic completion R_2^* of R_1^*, ..., and then the a_n-adic completion of R_{n-1}^*.

We record the following results about ideal-adic completions.

REMARKS 3.2. Let I be an ideal of a ring R.

(1) If R is I-adically complete, then I is contained in the Jacobson radical $\mathfrak{J}(R)$; see [**103**, Theorem 8.2] or [**101**, 24.B, pages 73-74].
(2) If R is a Noetherian ring, then the I-adic completion R^* of R is flat over R [**103**, Theorem 8.8], and R^* is Noetherian by [**103**, Theorem 8.12].
(3) If R is Noetherian, then the I-adic completion R^* of R is faithfully flat over $R \iff$ for each proper ideal J of R we have $JR^* \neq R^*$.
(4) If R is a Noetherian ring and $I \subseteq \mathfrak{J}(R)$, then the I-adic completion R^* is faithfully flat over R, and $\dim R = \dim R^*$ [**101**, Theorem 56, page 172] and [**101**, pages 173-175]. Moreover, if R is an integral domain with field of fractions K, then $R = K \cap R^*$ by Remark 2.31.9.
(5) If $I = (a_1, \ldots, a_n)R$ is an ideal of a Noetherian ring R, then the I-adic completion R^* of R is isomorphic to a quotient of the formal power series ring $R[[x_1, \ldots, x_n]]$; namely,

$$R^* \;=\; \frac{R[[x_1, \ldots, x_n]]}{(x_1 - a_1, \ldots, x_n - a_n)R[[x_1, \ldots, x_n]]}$$

[**103**, Theorem 8.12].

REMARKS 3.3. Assume $z \in R$ and $\bigcap_{n=1}^{\infty} z^n R = (0)$. Then the canonical map $\psi : R \to R^*$ is injective, where $R^* := \varprojlim_n R/z^n R$ is the *z-adic completion* of R.

(1) Let y be an indeterminate over R. If the ideal $(y - z)R[[y]]$ is closed in the J-adic topology on $R[[y]]$, where $J := (y, z)R[[y]]$, then the z-adic completion R^* also has the form

$$R^* \;=\; \frac{R[[y]]}{(y - z)R[[y]]}. \tag{3.3.0}$$

This follows from [**117**, (17.5)].
(2) If R is Noetherian, then the ideal $(y - z)R[[y]]$ is closed in the J-adic topology on $R[[y]]$ and the representation of R^* as in Equation 3.3.0 holds by Remark 3.2.5. A direct proof of this statement may also be given as follows: let $\bar{}$ denote image in $R[[y]]/(y - z)R[[y]]$. It suffices to show that $\bigcap_{n=1}^{\infty} \overline{(y, z)^n R[[y]]} = (\overline{0})$.

We have $\overline{(y, z)^n R[[y]]} = \overline{y^n R[[y]]}$, for every $n \in \mathbb{N}$. By Corollary 2.28, the element y is in the Jacobson radical of $R[[y]]$. Hence $\overline{y}$ is in the Jacobson radical of $\overline{R[[y]]}$, a Noetherian ring. We have

$$\bigcap_{n=1}^{\infty} \overline{(y, z)^n R[[y]]} = \bigcap_{n=1}^{\infty} \overline{y^n R[[y]]} = (\overline{0}).$$

The second equality follows from Theorem 2.16.1. Therefore $(y - z)R[[y]]$ is closed in the J-adic topology. Thus if R is Noetherian then R^* has the form of Equation 3.3.0.
(3) If R^* has the form of Equation 3.3.0, then the elements of R^* are power series in z with coefficients in R, but without the uniqueness of expression

as power series that occurs in the formal power series ring $R[[y]]$. If R is already complete in its (z)-adic topology, then $R = R^*$, but often it is the case for a Noetherian integral domain R that there exist elements of R^* that are transcendental over the field of fractions of R; see Section 3.2.

3.2. Uncountable transcendence degree for a completion

In this section, we make a small excursion to consider some cases where the transcendence degree of completions and power series rings are uncountable over a base integral domain. We call these results "facts", because they appear to be well known. We include brief proofs here to make the results more accessible.

We begin with a useful fact about uncountable Noetherian commutative rings.

FACT 3.4. If R is an uncountable Noetherian commutative ring, then there exists a prime ideal P of R such that R/P is uncountable. Hence there exists a minimal prime P_0 of R such that R/P_0 is uncountable.

PROOF. The ring R contains a finite chain of ideals

$$0 \;=\; I_0 \;\subset\; I_1 \;\subset\; \cdots \;\subset\; I_\ell \;=\; R$$

such that each quotient $I_{i+1}/I_i \cong R/P_i$, for some prime ideal P_i of R, [**103**, Theorem 6.4]. If each of the quotients were countable then R would be countable. Thus R/P is uncountable for some prime ideal P of R, and hence R/P_0 is uncountable, for each minimal prime P_0 contained in P. □

FACT 3.5. If R is a countable Noetherian integral domain and z is a nonzero nonunit of R, then the (z)-adic completion R^* of R contains an uncountable subset that is algebraically independent over R. That is, R^* has uncountable transcendence degree over R.

PROOF. We first observe that the (z)-adic completion

$$R^* \;:=\; \varprojlim_n \; R/z^nR$$

of R is uncountable. For each $n \in \mathbb{N}$, let $\theta_{n+1} : \frac{R}{z^{n+1}R} \to \frac{R}{z^nR}$ be the canonical homomorphism. Elements of R^* may be identified with coherent sequences $\{\zeta_n\}_{n\in\mathbb{N}}$ in the sense that $\theta_{n+1}(\zeta_{n+1}) = \zeta_n$ for each $n \in \mathbb{N}$; see [**11**, page 103]. Since for each n and each ζ_n, there are at least two choices for the element ζ_{n+1} such that $\theta_{n+1}(\zeta_{n+1}) = \zeta_n$, the cardinality of R^* is at least $2^{\aleph_0}$ and hence is uncountable.

By Fact 3.4 there exists a minimal prime P_0 of R^* such that R^*/P_0 is uncountable. Since R is a Noetherian integral domain, R^* is flat over R by Remark 3.2.2. Thus, by Remark 2.31.9, $P_0 \cap R = 0$. Since a countably generated extension domain of R is countable and the algebraic closure of the field of fractions of a countable integral domain is countable, there exists an uncountable subset Λ of R^*/P_0 such that Λ is algebraically independent over R. Let $\Lambda^* \subset R^*$ be such that the elements of Λ^* map in a one-to-one way onto the elements in Λ under the residue class map $R^* \to R^*/P_0$. Then $R[\Lambda^*] \subset R^*$, and $R[\Lambda^*]$ is a polynomial ring over R in an uncountable set of indeterminates.[2] □

[2]It may happen, however, that there exist nonzero elements in the subring $R[\Lambda^*]$ of R^* that are zero-divisors in R^*.

In relation to transcendence degree and filtrations, Joe Lipman brought Remark 3.6 and Fact 3.7 to our attention; he also indicated the proofs sketched below.

REMARK 3.6. Let k be a field and let R be a ring containing k. Let $(I_a)_{a\in A}$ be a family of ideals of R with index set A such that the family is closed under finite intersection and the intersection of all of the I_a is (0). If $m \in \mathbb{N}$ and $v_1, \dots, v_m \in R$ are linearly independent vectors over k, then for some a their images in R/I_a are linearly independent. Otherwise, if V is the vector space generated by the v_i, then $(V \cap I_a)_{a\in A}$ would be an infinite family of nonzero vector subspaces of the finite-dimensional vector space V that is closed under finite intersection and such that the intersection of all of them is (0), a contradiction.

FACT 3.7. Let y be an indeterminate over a field k. Then the power series ring $k[[y]]$ has uncountable transcendence degree over k.

PROOF. We show the k-vector space dimension of $k[[y]]$ is uncountable. For this, let k_0 be the prime subfield of k. We consider the family $\{I_n := y^n k_0[[y]]\}_{n\in\mathbb{N}}$ of ideals of $k_0[[y]]$ and the corresponding family $\{I'_n := y^n k[[y]]\}_{n\in\mathbb{N}}$ of ideals of $k[[y]]$. For every $n \in \mathbb{N}$, the k-homomorphism $\varphi : k \otimes_{k_0} k_0[[y]] \to k[[y]]$ induces a map $\overline{\varphi} : k \otimes_{k_0} (k_0[[y]]/y^n k_0[[y]]) \to k[[y]]/y^n k[[y]]$ that is an isomorphism of two n-dimensional vector spaces over k.

Since $k_0[[y]]$ is uncountable and k_0 is countable, the k_0-vector space dimension of $k_0[[y]]$ is uncountable, and so there is an uncountable subset $\mathcal{B}$ of $k_0[[y]]$ that is linearly independent over k_0. Let $v_1, \dots v_m$ be a finite subset of $\mathcal{B}$. Then by Remark 3.6 the images of $v_1, \dots, v_m$ in $k_0[[y]]/(y^n k_0[[y]])$ are linearly independent over k_0, for some n. Since $\overline{\varphi}$ is a k-isomorphism, the images of $v_1, \dots, v_m$ in $k[[y]]/(y^n k[[y]])$ are linearly independent over k. Thus $v_1, \dots, v_m$ must be linearly independent over k. Therefore $\mathcal{B}$ is linearly independent over k. □

3.3. Basic results about completions

In Proposition 3.8 we give conditions for an ideal to be closed with respect to an I-adic topology.

PROPOSITION 3.8. *Let I be an ideal in a ring R and let R^* denote the I-adic completion of R.*

1. *Let L be an ideal of R such that LR^* is closed in the I-adic topology on R^*. Then L is closed in the I-adic topology on R if and only if $LR^* \cap R = L$.* [3]
2. *If R is Noetherian and I is contained in the Jacobson radical of R, then every ideal L of R is closed in the I-adic topology on R.*
3. *If R^* is Noetherian, then every ideal $\mathfrak{A}$ of R^* is closed in the I-adic topology on R^*.*

PROOF. For item 1, we have $LR^* = \bigcap_{n=1}^{\infty}(L + I^n)R^*$, since the ideal LR^* is closed in R^*. By Equation 3.1.2, $R/I^n \cong R^*/I^n R^*$, for each $n \in \mathbb{N}$. It follows that

$$R/(L + I^n) \cong R^*/(L + I^n)R^*, \quad \text{and} \quad L + I^n = (L + I^n)R^* \cap R, \tag{3.8.0}$$

[3]Here, as in [**103**, page xiii], we interpret $LR^* \cap R$ to be the preimage $\psi^{-1}(LR^*)$, where $\psi : R \to R^*$ is the canonical map of R to its I-adic completion R^*.

for each $n \in \mathbb{N}$. By Equation 3.8.0, L is closed in R if and only if $LR^* \cap R = L$. This proves item 1. Item 2 now follows from statements 3 and 4 of Remark 3.2.

Item 3 follows from item 2, since IR^* is contained in the Jacobson radical of R^* by Remark 3.2.1. □

In Theorem 8 of Cohen's famous paper [**28**] on the structure and ideal theory of complete local rings a result similar to Nakayama's lemma is obtained without the usual finiteness condition of Nakayama's lemma [**103**, Theorem 2.2]. As formulated in [**103**, Theorem 8.4], the result is:

THEOREM 3.9. (A version of Cohen's Theorem 8) *Let I be an ideal of a ring R and let M be an R-module. Assume that R is complete in the I-adic topology and $\bigcap_{n=1}^{\infty} I^n M = (0)$. If M/IM is generated over R/I by elements $\overline{w}_1, \ldots, \overline{w}_s$ and w_i is a preimage in M of $\overline{w}_i$ for $1 \leq i \leq s$, then M is generated over R by $w_1, \ldots, w_s$.*

Let K be a field and let $R = K[[x_1, \ldots, x_n]]$ be a formal power series ring in n variables over K. It is well-known that there exists a K-algebra embedding of R into the formal power series ring $K[[y, z]]$ in two variables over K [**165**, page 219]. We observe in Corollary 3.10 restrictions on such an embedding.

COROLLARY 3.10. *Let $(R, \mathbf{m})$ be a complete Noetherian local ring and assume that the map $\varphi : (R, \mathbf{m}) \to (S, \mathbf{n})$ is a local homomorphism.*

(1) *If $\mathbf{m}S$ is $\mathbf{n}$-primary and $S/\mathbf{n}$ is finite over $R/\mathbf{m}$, then S is a finitely generated R-module.*
(2) *If $\mathbf{m}S = \mathbf{n}$ and $R/\mathbf{m} = S/\mathbf{n}$, then φ is surjective.*
(3) *Assume that $R = K[[x_1, \ldots, x_n]]$ is a formal power series ring in $n > 2$ variables over the field K and $S = K[[y, z]]$ is a formal power series ring in two variables over K. If φ is injective, then $\varphi(\mathbf{m})S$ is not $\mathbf{n}$-primary.*

We record in Remarks 3.12 several consequences of Cohen's structure theorems for complete local rings. We use the following definitions.

DEFINITIONS 3.11. Let $(R, \mathbf{m})$ be a local ring.

(1) $(R, \mathbf{m})$ is said to be *equicharacteristic* if R has the same characteristic as its residue field $R/\mathbf{m}$.
(2) A subfield k of R is a *coefficient field* of R if the canonical map of $R \to R/\mathbf{m}$ restricts to an isomorphism of k onto $R/\mathbf{m}$.

REMARKS 3.12.

(1) Every equicharacteristic complete Noetherian local ring has a coefficient field; see [**28**], [**103**, Theorem 28.3], [**117**, (31.1)].
(2) If k is a coefficient field of a complete Noetherian local ring $(R, \mathbf{m})$ and $x_1, \ldots, x_n$ are generators of $\mathbf{m}$, then every element of R can be expanded as a power series in $x_1, \ldots, x_n$ with coefficients in k; see [**117**, (31.1)]. Thus R is a homomorphic image of a formal power series ring in n variables over k.
(3) (i) Every complete Noetherian local ring is a homomorphic image of a complete regular local ring.
(ii) Every complete regular local ring is a homomorphic image of a power series ring over either a field or a complete discrete valuation ring [**28**], [**117**, (31.12)].
(4) Let $(R, \mathbf{m})$ be a complete Noetherian local domain. Then:

(a) R is a finite integral extension of a complete regular local domain [**117**, (31.6)].
(b) The integral closure of R in a finite algebraic field extension is a finite R-module [**117**, (32.1)].

Historically the following terminology has been used for local rings to indicate properties of the completion.

DEFINITIONS 3.13. A Noetherian local ring R is said to be

(1) *analytically unramified* if the completion $\widehat{R}$ is reduced, i.e., has no nonzero nilpotent elements;
(2) *analytically irreducible* if the completion $\widehat{R}$ is an integral domain;
(3) *analytically normal* if the completion $\widehat{R}$ is an integrally closed (i.e., normal) domain.

If a Noetherian local ring R is analytically irreducible or analytically normal, then R is analytically unramified. If R is analytically normal, then R is analytically irreducible. If $\widehat{R}$ is not reduced, we say that R is *analytically ramified*; if $\widehat{R}$ is not an integral domain, we say that R is *analytically reducible*.

A classical theorem of Rees describes necessary and sufficient conditions in order that a Noetherian local ring be analytically unramified. We refer to this result as the *Rees Finite Integral Closure Theorem*.

THEOREM 3.14. (Rees Finite Integral Closure Theorem) [**130**] *Let $(R, \mathbf{m})$ be a reduced Noetherian local ring with total ring of fractions $\mathcal{Q}(R)$. Then the following are equivalent.*

(1) *The ring R is analytically unramified.*
(2) *For every choice of finitely many elements $\lambda_1, \ldots, \lambda_n$ in $\mathcal{Q}(R)$, the integral closure of $R[\lambda_1, \ldots, \lambda_n]$ is a finite $R[\lambda_1, \ldots, \lambda_n]$-module.*

The following is an immediate corollary of Theorem 3.14.

COROLLARY 3.15. (Rees) [**130**] *Let $(R, \mathbf{m})$ be an analytically unramified Noetherian local ring and let $\lambda_1, \ldots, \lambda_n$ be elements of $\mathcal{Q}(R)$. For every prime ideal P of $A = R[\lambda_1, \ldots, \lambda_n]$, the local ring A_P is also analytically unramified.*

REMARKS 3.16. Let R be a Noetherian local ring.

(1) If R is analytically unramified, then the integral closure of R is a finite R-module by Rees Finite Integral Closure Theorem 3.14 or [**117**, (32.2)].
(2) If $(R, \mathbf{m})$ is one-dimensional and an integral domain, then the following two statements hold [**117**, Ex. 1 on page 122] and [**86**].
   (i) The integral closure $\overline{R}$ of R is a finite R-module if and only if R is analytically unramified.
   (ii) The minimal primes of $\widehat{R}$ are in one-to-one correspondence with the maximal ideals of $\overline{R}$.

3.4. Chains of prime ideals, fibers of maps

We begin by discussing chains of prime ideals.

DEFINITIONS 3.17. Let P and Q be prime ideals of a ring A.

(1) If $P \subsetneq Q$, we say that the inclusion $P \subsetneq Q$ is *saturated* if there is no prime ideal of A strictly between P and Q.

(2) A possibly infinite chain of prime ideals $\cdots \subsetneq P_i \subsetneq P_{i+1} \subsetneq \cdots$ is called *saturated* if every inclusion $P_i \subsetneq P_{i+1}$ is saturated.
(3) A ring A is *catenary* provided for every pair of prime ideals $P \subsetneq Q$ of A, every chain of prime ideals from P to Q can be extended to a saturated chain and every two saturated chains from P to Q have the same number of inclusions.
(4) A ring A is *universally catenary* provided every finitely generated A-algebra is catenary.
(5) A ring A is said to be *equidimensional* if $\dim A = \dim A/P$ for every minimal prime P of A.

Theorem 3.18 is a well-known result of Ratliff that we call Ratliff's Equidimension Theorem.

THEOREM 3.18. (Ratliff's Equidimension Theorem) [**103**, Theorem 31.7] *A Noetherian local domain A is universally catenary if and only if its completion $\widehat{A}$ is equidimensional.*

Ratliff's sharper result, also called Ratliff's Equidimension Theorem, relates the universally catenary property to properties of the completion, even if the Noetherian local ring is not a domain.

THEOREM 3.19. (Ratliff's Equidimension Theorem) [**127**, Theorem 2.6] *A Noetherian local ring $(R, \mathbf{m})$ is universally catenary if and only if the completion of $R/\mathbf{p}$ is equidimensional for every minimal prime ideal $\mathbf{p}$ of R.*

REMARK 3.20. Every Noetherian local ring that is a homomorphic image of a regular local ring, or even a homomorphic image of a Cohen-Macaulay local ring, is universally catenary [**103**, Theorem 17.9, page 137].

We record in Proposition 3.21 an implication of the Krull Altitude Theorem 2.17.

PROPOSITION 3.21. *Let $(R, \mathbf{m})$ be a catenary Noetherian local domain and let $P \in \operatorname{Spec} R$ with $\dim R/P = n \geq 1$. Let d be an integer with $1 \leq d \leq n$, and let*

$$\mathcal{A} \;:=\; \{Q \in \operatorname{Spec} R \mid P \subseteq Q \text{ and } \dim R/Q \;=\; d\;\}.$$

Then $P = \bigcap_{Q \in \mathcal{A}} Q$.

PROOF. If $d = n$, then $P \in \mathcal{A}$ and the statement is true. To prove the assertion for d with $1 \leq d < n$, it suffices to prove it in the case where $\dim R/P = d + 1$; for if the statement holds in the case where $n = d + 1$, then by an iterative procedure on intersections of prime ideals, the statement also holds for $n = d + 2, \cdots$.

Thus we assume $n = d + 1$. Since $\operatorname{ht}(\mathbf{m}/P) \geq 2$, Krull's Altitude Theorem 2.17 implies that there exist infinitely many prime ideals properly between P and $\mathbf{m}$; see Exercise 5 in Chapter 2. Theorem 2.17 also implies that for each element $a \in \mathbf{m} \setminus P$ and each minimal prime Q of $P + aR$, we have $\operatorname{ht}(Q/P) = 1$. Since R is catenary, it follows that $\dim(R/Q) = \dim(R/P) - 1 = d$. Therefore the set $\mathcal{A}$ is infinite. Since an ideal in a Noetherian ring has only finitely many minimal primes, we have $P = \bigcap_{Q \in \mathcal{A}} Q$. □

DISCUSSION 3.22. Let $f : A \to B$ be a ring homomorphism. The map f can always be factored as the composite of the surjective map $A \to f(A)$ followed by the

inclusion map $f(A) \hookrightarrow B$. This is often helpful for understanding the relationship of A and B. If J is an ideal of B, then $f^{-1}(J)$ is an ideal of A called the *contraction* of J to A with respect to f. As in [**103**, page xiii], we often write $J \cap A$ for $f^{-1}(J)$. If Q is a prime ideal of B, then $P := f^{-1}(Q) = Q \cap A$ is a prime ideal of A. Thus associated with the ring homomorphism $f : A \to B$, there is a well-defined *spectral map* $f^* : \operatorname{Spec} B \to \operatorname{Spec} A$ of topological spaces, where for $Q \in \operatorname{Spec} B$ we define $f^*(Q) = f^{-1}(Q) = Q \cap A = P \in \operatorname{Spec} A$.

Let A be a ring and let $P \in \operatorname{Spec}(A)$. The *residue field of* A at P, denoted $k(P)$, is the field of fractions $\mathcal{Q}(A/P)$ of A/P. By permutability of localization and residue class formation we have $k(P) = A_P/PA_P$.

Given a ring homomorphism $f : A \to B$ and an ideal I of A, the ideal $f(I)B$ is called the *extension* of I to B with respect to f. For $P \in \operatorname{Spec} A$, the extension ideal $f(P)B$ is, in general, not a prime ideal of B. The *fiber* over P in $\operatorname{Spec} B$ is the set of all $Q \in \operatorname{Spec} B$ such that $f^*(Q) = P$. Exercise 7 of Chapter 2 asserts that the fiber over P is nonempty if and only if P is the contraction of the extended ideal $f(P)B$. The *fiber ring* of the map f over P is the ring C defined as:

$$C := B \otimes_A k(P) = S^{-1}(B/f(P)B) = (S^{-1}B)/(S^{-1}f(P)B), \tag{3.22.0}$$

where S is the multiplicatively closed set $A \setminus P$; see [**103**, last paragraph, p. 47]. In general, the fiber over P in $\operatorname{Spec} B$ is the spectrum of the ring C. That is, the fiber of f over P in $\operatorname{Spec} B$ is the set of prime ideals of C with the Zariski topology. Notice that a prime ideal Q of B contracts to P in A if and only if $f(P) \subseteq Q$ and $Q \cap S = \emptyset$. This describes exactly the prime ideals of C as in Equation 3.22.0.

For $Q^* \in \operatorname{Spec} C$, and $Q = Q^* \cap B$, we have $P = Q \cap A$ and

$$Q^* = QC, \quad \text{and} \quad C_{Q^*} = B_Q/PB_Q = B_Q \otimes_A k(P); \tag{3.22.1}$$

see [**103**, top, p. 48].

THEOREM 3.23. [**103**, Theorem 23.7 and Corollary, p. 184] *Let $(A, \mathbf{m})$ and $(B, \mathbf{n})$ be Noetherian local rings and $\varphi : A \to B$ a flat local homomorphism. Then*

1. *If B is regular, normal, or reduced, then so is A.*
2. *If A and $B/\mathbf{m}B$ are regular, then B is regular.*
3. *If both A and the fiber rings of φ are normal, respectively, reduced, then B is normal, respectively, reduced.*

COROLLARY 3.24. *Let $(R, \mathbf{m})$ be a Noetherian local ring and let $\widehat{R}$ denote its $\mathbf{m}$-adic completion. Then R is an RLR if and only if $\widehat{R}$ is an RLR.*

PROOF. By Remark 3.2.4, the extension $R \hookrightarrow \widehat{R}$ is faithfully flat. Thus Theorem 3.23 applies. □

We consider more properties of completions in Discussion 3.26. The notion of "depth" is relevant for that discussion and is defined in Definition 3.25.

DEFINITION 3.25. Let I be an ideal in a Noetherian ring R and let M be a finitely generated R-module such that $IM \neq M$. Elements $x_1, \ldots, x_d$ in I are said to form a *regular sequence on* M, or an *M-sequence*, if x_1 is not a zerodivisor on M and for i with $2 \leq i \leq d$, the element x_i is not a zerodivisor on $M/(x_1, \ldots, x_{i-1})M$. It is known that maximal M-sequences of elements of I exist and all maximal M-sequences of elements of I have the same length n; see [**103**, Theorem 16.7] or [**85**, Theorem 121]. This integer n is called the *grade of* I on M and denoted $G(I, M)$.

If R is a Noetherian local ring with maximal ideal $\mathbf{m}$, and M is a nonzero finitely generated R-module, then the grade of $\mathbf{m}$ on M is also called the *depth* of M. In particular the depth of R is $G(\mathbf{m}, R)$.

Discussion 3.26. Related to Corollary 3.24, one of our goals is the study of the relationship between a Noetherian local ring $(R, \mathbf{m})$ and its $\mathbf{m}$-adic completion $\widehat{R}$. Certain properties of the ring R may fail to hold in $\widehat{R}$. For example,

(1) The rings A/fA and D of Remarks 4.15.2 and 4.15.1 are Noetherian local domains, whereas the completion of the one-dimensional domain A/fA is not reduced and the completion of the two-dimensional normal ring D is not an integral domain.
(2) Let T be a complete Noetherian local ring of depth at least two such that no nonzero element of the prime subring of T is a zero divisor on T. Ray Heitmann has shown the remarkable result that every such ring T is the completion of a Noetherian local UFD [**82**, Theorem 8]. Let $T = k[[x, y, z]]/(z^2)$, where x, y, z are indeterminates over a field k. By Heitmann's result there exists a two-dimensional Noetherian local UFD $(R, \mathbf{m})$ such that the completion of R is T. Thus there exists a two-dimensional normal Noetherian local domain for which the completion is not reduced.

Remark 3.27. Shreeram Abhyankar and Ben Kravitz in [**7**, Example 3.5] use Heitmann's construction mentioned in Discussion 3.26.2 along with Rees Finite Integral Closure Theorem 3.14 to give a counterexample to an erroneous theorem on page 125 of the book *Commutative Algebra II* by Oscar Zariski and Pierre Samuel [**165**]. Abhyankar and Kravitz also note that a related lemma on the previous page of [**165**] is incorrect.

With R and $\widehat{R}$ as in Discussion 3.26, if $Q \in \operatorname{Spec} \widehat{R}$ and $P = Q \cap R$, then the natural map $\varphi : R \to \widehat{R}$ induces a flat local homomorphism $\varphi_Q : R_P \to \widehat{R}_Q$. Theorem 3.23 applies in this situation with $A = R_P$ and $B = \widehat{R}_Q$. This motivates interest in the ring $\widehat{R}_Q/P\widehat{R}_Q$.

Definitions 3.28. Let $f : A \to B$ be a ring homomorphism of Noetherian rings, let $P \in \operatorname{Spec} A$, and let $k(P)$ be as in Discussion 3.22.

(1) The fiber over P with respect to the map f is said to be *regular* if the ring $B \otimes_A k(P)$ is a Noetherian regular ring, i.e., $B \otimes_A k(P)$ is a Noetherian ring with the property that its localization at every prime ideal is a regular local ring.
(2) The fiber over P with respect to the map f is said to be *normal* if the ring $B \otimes_A k(P)$ is a normal Noetherian ring, i.e., $B \otimes_A k(P)$ is a Noetherian ring with the property that its localization at every prime ideal is a normal Noetherian local domain.
(3) The fiber over P with respect to the map f is said to be *reduced* if the ring $B \otimes_A k(P)$ is a Noetherian reduced ring.
(4) The map f has *regular*, respectively, *normal, reduced, fibers* if the fiber over P is regular, respectively, normal, reduced, for every $P \in \operatorname{Spec} A$.

Definitions 3.29. Let $f : A \to B$ be a ring homomorphism of Noetherian rings, and let $P \in \operatorname{Spec} A$.

(1) The fiber over P with respect to the map f is said to be *geometrically regular* if for every finite extension field F of $k(P)$ the ring $B \otimes_A F$ is a Noetherian regular ring. The map $f : A \to B$ is said to have *geometrically regular* fibers if for each $P \in \operatorname{Spec} A$ the fiber over P is geometrically regular.
(2) The fiber over P with respect to the map f is said to be *geometrically normal* if for every finite extension field F of $k(P)$ the ring $B \otimes_A F$ is a Noetherian normal ring. The map $f : A \to B$ is said to have *geometrically normal* fibers if for each $P \in \operatorname{Spec} A$ the fiber over P is geometrically normal.
(3) The fiber over P with respect to the map f is said to be *geometrically reduced* if for every finite extension field F of $k(P)$ the ring $B \otimes_A F$ is a Noetherian reduced ring. The map $f : A \to B$ is said to have *geometrically reduced* fibers if for each $P \in \operatorname{Spec} A$ the fiber over P is geometrically reduced.

REMARK 3.30. Let $f : A \to B$ be a ring homomorphism with A and B Noetherian rings and let $P \in \operatorname{Spec} A$. To check that the fiber of f over P is geometrically regular as in Definition 3.29, it suffices to show that $B \otimes_A F$ is a Noetherian regular ring for every finite purely inseparable field extension F of $k(P)$, [**51**, No 20, Chap. 0, Théorème 22.5.8, p. 204]. Thus, if the characteristic of the field $k(P) = A_P/PA_P$ is zero, then, for every ring homomorphism $f : A \to B$ with B Noetherian, the fiber over P is geometrically regular if and only if it is regular. A similar statement is true with "regular" replaced by "normal" or "reduced". That is, in characteristic zero, if the homomorphism f is normal, resp. reduced, then f is geometrically normal, resp. geometrically reduced [**51**, No 24, Ch. IV, Prop. 6.7.4 and Prop. 6.7.7].

DEFINITIONS 3.31. Let $f : A \to B$ be a ring homomorphism, where A and B are Noetherian rings.

(1) The homomorphism f is said to be *regular* if it is flat with geometrically regular fibers. See Definition 2.30 for the definition of flat.
(2) The homomorphism f is said to be *normal* if it is flat with geometrically normal fibers.

REMARK 3.32. Let $f : A \to B$ be a ring homomorphism of Noetherian rings and $P \in \operatorname{Spec} A$. By Remark 2.6, every regular local ring is a normal Noetherian local domain. Thus, if the fiber over P with respect to f is geometrically regular, then the fiber over P is geometrically normal; if f has geometrically regular fibers, then f has geometrically normal fibers; and if f is a regular homomorphism, then f is a normal homomorphism.

EXAMPLE 3.33. Let x be an indeterminate over a field k of characteristic zero, and let

$$A := k[x(x-1), x^2(x-1)]_{(x(x-1),x^2(x-1))} \subset k[x]_{(x)} =: B.$$

Then $(A, \mathbf{m}_A)$ and $(B, \mathbf{m}_B)$ are one-dimensional local domains with the same field of fractions $k(x)$ and with $\mathbf{m}_A B = \mathbf{m}_B$. Hence the inclusion map $f : A \hookrightarrow B$ has geometrically regular fibers. Since $A \neq B$, the map f is not flat by Remark 2.31.8. Hence f is not a regular morphism.

We present in Chapter 7 examples of maps of Noetherian rings that are regular, and other examples of maps that are flat but fail to be regular.

The formal fibers of a Noetherian local ring as in Definition 3.34 play an important role in the concepts of excellent Noetherian rings, defined in Definition 3.37 and Nagata rings, defined in Definition 2.11.

DEFINITION 3.34. Let $(R, \mathbf{m})$ be a Noetherian local ring and let $\widehat{R}$ be the $\mathbf{m}$-adic completion of R. The *formal fibers* of R are the fibers of the canonical inclusion map $R \hookrightarrow \widehat{R}$.

DEFINITION 3.35. A Noetherian ring A is called a *G-ring* if, for each prime ideal P of A, the map of A_P to its PA_P-adic completion is regular, or, equivalently, the formal fibers of A_P are geometrically regular for each prime ideal P of A.

REMARK 3.36. In Definition 3.35 it suffices that, for every maximal ideal $\mathbf{m}$ of A, the map from $A_{\mathbf{m}}$ to its $\mathbf{m}A_{\mathbf{m}}$-adic completion is regular, by [**103**, Theorem 32.4]

DEFINITION 3.37. A Noetherian ring A is *excellent* if

(i) A is universally catenary,
(ii) A is a G-ring, and
(iii) for every finitely generated A-algebra B, the set $\mathrm{Reg}(B)$ of prime ideals P of B for which B_P is a regular local ring is an open subset of $\operatorname{Spec} B$.

REMARKS 3.38. The class of excellent rings includes the ring of integers as well as all fields and all complete Noetherian local rings [**103**, page 260]. All Dedekind domains of characteristic zero are excellent [**101**, (34.B)]. Every excellent ring is a Nagata ring by [**101**, Theorem 78, page 257].

The usefulness of the concept of excellent rings is enhanced by the fact that the class of excellent rings is stable under the ring-theoretic operations of localization and passage to a finitely generated algebra [**51**, Chap. IV], [**101**, (33.G) and (34.A)]. Therefore excellence is preserved under homomorphic images.

REMARKS 3.39. As shown in Proposition 9.4, there exist DVRs in positive characteristic that are not excellent. In Corollary 18.14, we prove that the two-dimensional Noetherian local ring B of characteristic zero constructed in Example 18.13 has the property that the map $f : B \to \widehat{B}$ has geometrically regular fibers. This ring B of Example 18.13 is also an example of a catenary ring that is not universally catenary. Thus the property of having geometrically regular formal fibers does not imply that a Noetherian local ring is excellent.

REMARK 3.40. In order to discuss early examples using the techniques of this book, we have included in Chapters 2 and 3 brief definitions of deep, technically demanding concepts, such as geometric regularity and excellence. These concepts are discussed in more detail in Chapters 7 and 13.

Exercises

(1) ([**36**]) Let R be a commutative ring and let P be a prime ideal of the power series ring $R[[x]]$. Let $P(0)$ denote the ideal in R of constant terms of elements of P.

(i) If $x \notin P$ and $P(0)$ is generated by n elements of R, prove that P is generated by n elements of $R[[x]]$.
(ii) If $x \in P$ and $P(0)$ is generated by n elements of R, prove that P is generated by $n+1$ elements of $R[[x]]$.
(iii) If R is a PID, prove that every prime ideal of $R[[x]]$ of height one is principal.

(2) Let R be a DVR with maximal ideal yR and let $S = R[[x]]$ be the formal power series ring over R in the variable x. Let $f \in S$. Recall that f is a unit in S if and only if the constant term of f is a unit in R by Exercise 4 of Chapter 2.
(a) Show that S is a 2-dimensional RLR with maximal ideal $(x, y)S$.
(b) If g is a factor of f and S/fS is a finite R-module, then S/gS is a finite R-module.
(c) If n is a positive integer and $f := x^n + y$, then S/fS is a DVR. Moreover, S/fS is a finite R-module if and only if $R = \widehat{R}$, i.e., R is complete.
(d) If f is irreducible and $fS \neq xS$, then S/fS is a finite R-module implies that R is complete.
(e) If R is complete, then S/fS is a finite R-module for each nonzero f in S.

Suggestion: For item (d) use that if R is not complete, then by Nakayama's lemma, the completion of R is not a finite R-module. For item (e) use Theorem 3.9.

Let f be a monic polynomial in x with coefficients in R.

What are necessary and sufficient conditions in order that the residue class ring S/fS is a finite R-module?

(3) (Related to Dumitrescu's article [**34**]) Let R be an integral domain and let $f \in R[[x]]$ be a nonzero nonunit of the formal power series ring $R[[x]]$. Prove that the principal ideal $fR[[x]]$ is closed in the (x)-adic topology, that is, $fR[[x]] = \bigcap_{m\geq 0}(f, x^m)R[[x]]$.

Suggestion: Reduce to the case where $c = f(0)$ is nonzero. Then f is a unit in the formal power series ring $R[\frac{1}{c}][[x]]$. If $g \in \bigcap_{m\geq 0}(f, x^m)R[[x]]$, then $g = fh$ for some $h \in R[\frac{1}{c}][[x]]$, say $h = \sum_{n\geq 0} h_n x^n$, with $h_n \in R[\frac{1}{c}]$. Let $m \geq 1$. As $g \in (f, x^m)R[[x]]$, $g = fq + x^m r$, for some $q, r \in R[[x]]$. Thus $g = fh = fq + x^m r$, hence $f(h-q) = x^m r$. As $f(0) \neq 0$, $h - q = x^m s$, for some $s \in R[\frac{1}{c}][[x]]$. Hence $h_0, h_1, \ldots, h_{m-1} \in R$.

(4) Let R be a commutative ring and let $f = \sum_{n\geq 0} f_n x^n \in R[[x]]$ be a power series having the property that its leading form f_r is a regular element of R, that is, $\operatorname{ord} f = r$, so $f_0 = f_1 = \cdots = f_{r-1} = 0$, and f_r is a regular element of R. As in the previous exercise, prove that the principal ideal $fR[[x]]$ is closed in the (x)-adic topology.

(5) Let $f : A \hookrightarrow B$ be as in Example 3.33.
(i) Prove as asserted in the text that f has geometrically regular fibers but is not flat.
(ii) Prove that the inclusion map of $C := k[x(x-1)]_{(x(x-1))} \hookrightarrow k[x]_{(x)} = B$ is flat and has geometrically regular fibers. Deduce that the map $C \hookrightarrow B$ is a regular map.

(6) Let $\phi : (R, \mathbf{m}) \hookrightarrow (S, \mathbf{n})$ be an injective local map of the Noetherian local ring $(R, \mathbf{m})$ into the Noetherian local ring $(S, \mathbf{n})$. Let $\widehat{R} = \varprojlim_n R/\mathbf{m}^n$ denote the

$\mathbf{m}$-adic completion of R and let $\widehat{S} = \varprojlim_n S/\mathbf{n}^n$ denote the $\mathbf{n}$-adic completion of S.

(i) Prove that there exists a map $\widehat{\phi} : \widehat{R} \to \widehat{S}$ that extends the map $\phi : R \hookrightarrow S$.

(ii) Prove that $\widehat{\phi}$ is injective if and only if for each positive integer n there exists a positive integer s_n such that $\mathbf{n}^{s_n} \cap R \subseteq \mathbf{m}^n$.

(iii) Prove that $\widehat{\phi}$ is injective if and only if for each positive integer n the ideal $\mathbf{m}^n$ is closed in the topology on R defined by the ideals $\{\mathbf{n}^n \cap R\}_{n\in\mathbb{N}}$, i.e., the topology on R that defines R as a *subspace* of S.

Suggestion: For each $n \in \mathbb{N}$, we have $\mathbf{m}^n \subseteq \mathbf{n}^n \cap R$. Hence there exists a map $\phi_n : R/\mathbf{m}^n \to R/(\mathbf{n}^n \cap R) \hookrightarrow S/\mathbf{n}^n$, for each $n \in N$. The family of maps $\{\phi_n\}_{n\in\mathbb{N}}$ determines a map $\widehat{\phi} : \widehat{R} \to \widehat{S}$. Since $R/\mathbf{m}^n$ is Artinian, the descending chain of ideals $\{\mathbf{m}^n + (\mathbf{n}^s \cap R)\}_{s\in\mathbb{N}}$ stabilizes, and $\mathbf{m}^n$ is closed in the subspace topology if and only if there exists a positive integer s_n such that $\mathbf{n}^{s_n} \cap R \subseteq \mathbf{m}^n$. This holds for each $n \in \mathbb{N}$ if and only if the $\mathbf{m}$-adic topology on R is the subspace topology from S.

(7) Let $(R, \mathbf{m})$, $(S, \mathbf{n})$ and $(T, \mathbf{q})$ be Noetherian local rings. Assume there exist injective local maps $f : R \hookrightarrow S$ and $g : S \hookrightarrow T$, and let $h := gf : R \hookrightarrow T$ be the composite map. For $\widehat{f} : \widehat{R} \to \widehat{S}$ and $\widehat{g} : \widehat{S} \to \widehat{T}$ and $\widehat{h} : \widehat{R} \to \widehat{T}$ as in the previous exercise, prove that $\widehat{h} = \widehat{g}\widehat{f}$.

(8) Let $(R, \mathbf{m})$ and $(S, \mathbf{n})$ be Noetherian local rings such that S dominates R and the $\mathbf{m}$-adic completion $\widehat{R}$ of R dominates S.

(i) Prove that R is a subspace of S.

(ii) Prove that $\widehat{R}$ is an algebraic retract of $\widehat{S}$, i.e., $\widehat{R} \hookrightarrow \widehat{S}$ and there exists a surjective map $\pi : \widehat{S} \to \widehat{R}$ such that π restricts to the identity map on the subring $\widehat{R}$ of $\widehat{S}$.

(9) Let k be a field and let R be the localized polynomial ring $k[x]_{xk[x]}$, and thus $\widehat{R} = k[[x]]$. Let $n \geq 2$ be a positive integer. If $\operatorname{char} k = p > 0$, assume that n is not a multiple of p.

(i) Prove that there exists $y \in k[[x]]$ such that $y^n = 1 + x$.

(ii) For y as in (i), let $S := R[yx] \hookrightarrow k[[x]]$. Prove that S is a local ring integral over R with maximal ideal $(x, yx)S$. By the previous exercise, $\widehat{R} = k[[x]]$ is an algebraic retract of $\widehat{S}$.

(iii) Prove that the integral closure $\overline{S}$ of S is not local. Indeed, if the field k contains a primitive n-th root of unity, then $\overline{S}$ has n distinct maximal ideals. Deduce that $\widehat{R} \neq \widehat{S}$, so $\widehat{R}$ is a nontrivial algebraic retract of $\widehat{S}$.

Suggestion: Use Remark 3.12.3 and Remark 3.16.2ii.

(10) (Cohen) Let $(B, \mathbf{n})$ be a local ring that is not necessarily Noetherian. If the maximal ideal $\mathbf{n}$ is finitely generated and $\bigcap_{n=1}^{\infty} \mathbf{n}^n = (0)$, prove that the completion $\widehat{B}$ of B is Noetherian [**28**] or [**117**, (31.7)]

Suggestion: Use Theorem 3.9.

Comment: In [**28**, page 56] Cohen defines $(B, \mathbf{n})$ to be a *generalized local ring* if $\mathbf{n}$ is finitely generated and $\bigcap_{n=1}^{\infty} \mathbf{n}^n = (0)$. He proves that the completion of a generalized local ring is Noetherian, and that a complete generalized local ring is Noetherian [**28**, Theorems 2 and 3]. Cohen mentions that he does not know

whether there exists a generalized local ring that is not Noetherian. Nagata in [**108**] gives such an example of a non-Noetherian generalized local ring $(B, \mathbf{n})$. In Nagata's example $\widehat{B} = k[[x, y]]$ is a formal power series ring in two variables over a field. Heinzer and Roitman in [**57**] survey properties of generalized local rings including this example of Nagata.

CHAPTER 4

First examples of the construction,

In this chapter, we describe elementary and historical examples of Noetherian rings. In Section 4.1, we justify that Basic Construction Equation 1.3 is universal in the sense described in Chapter 1. In Sections 4.2, 4.3 and 4.4, several examples are described in terms of Equation 1.3, using a form of Basic Construction 1.5.

The basic idea of Inclusion Construction 5.3 defined in the next chapter is: Start with a well understood Noetherian domain R, then take an ideal-adic completion R^* of R and intersect R^* with an appropriate field L between R and the total quotient ring of R^*.[1]

4.1. Universality

In this section we describe how Basic Construction Equation 1.3 can be regarded as universal for the construction of many Noetherian local domains.

Consider the following general question.

QUESTION 4.1. Let k be a field and let L/k be a finitely generated field extension. What are the Noetherian local domains $(A, \mathbf{n})$ such that

(1) L is the field of fractions A, and
(2) k is a coefficient field for A?

Recall from Section 2.1, that k is a coefficient field of $(A, \mathbf{n})$ if the composite map $k \hookrightarrow A \to A/\mathbf{n}$ defines an isomorphism of k onto $A/\mathbf{n}$.

In relation to Question 4.1, we obtain in Theorem 4.2 the following general facts.

THEOREM 4.2. *Let $(A, \mathbf{n})$ be a Noetherian local domain having a coefficient field k. Then there exists a Noetherian local subring $(R, \mathbf{m})$ of A such that:*

(1) *The local ring R is essentially finitely generated over k.*
(2) *If $\mathcal{Q}(A) = L$ is finitely generated over k, then R has field of fractions L.*
(3) *The field k is a coefficient field for R.*
(4) *The local ring A dominates R and $\mathbf{m}A = \mathbf{n}$.*
(5) *The inclusion map $\varphi : R \hookrightarrow A$ extends to a surjective homomorphism $\widehat{\varphi} : \widehat{R} \to \widehat{A}$ of the $\mathbf{m}$-adic completion $\widehat{R}$ of R onto the $\mathbf{n}$-adic completion $\widehat{A}$ of A.*
(6) *For the ideal $I := \ker(\widehat{\varphi})$ of the completion $\widehat{R}$ of R from item 5, we have:*
   (a) *$\widehat{R}/I \cong \widehat{A}$, so $\widehat{R}/I$ dominates A, and*
   (b) *$P \cap A = (0)$ for every $P \in \operatorname{Ass}(\widehat{R}/I)$, and so the field of fractions $\mathcal{Q}(A)$ of A embeds in the total ring of quotients $\mathcal{Q}(\widehat{R}/I)$ of $\widehat{R}/I$, and*
   (c) *$A = \mathcal{Q}(A) \cap (\widehat{R}/I)$.*

[1] This is made more explicit in Section 5.1 of Chapter 5.

PROOF. Since A is Noetherian, there exist elements $t_1, \ldots, t_n \in \mathbf{n}$ such that $(t_1, \ldots, t_n)A = \mathbf{n}$. For item 2, we may assume that $L = k(t_1, \ldots, t_n)$, since every element of $\mathcal{Q}(A)$ has the form a/b, where $a, b \in \mathbf{n}$. To see the existence of the integral domain $(R, \mathbf{m})$ and to establish item 1, we set $T := k[t_1, \ldots, t_n]$ and $\mathbf{p} := \mathbf{n} \cap T$. Define $R := T_{\mathbf{p}}$ and $\mathbf{m} := \mathbf{n} \cap R$. Then $k \subseteq R \subseteq A$, $\mathbf{m}A = \mathbf{n}$, R is essentially finitely generated over k and k is a coefficient field for R. Thus we have established items 1- 4. Even without the assumption that $\mathcal{Q}(A)$ is finitely generated over k, there is a relationship between R and A that is realized by passing to completions. Let φ be the inclusion map $R \hookrightarrow A$. The map φ extends to a map $\widehat{\varphi} : \widehat{R} \to \widehat{A}$, and by Corollary 3.10.2, the map $\widehat{\varphi}$ is surjective; thus item 5 holds. Let $I := \ker \widehat{\varphi}$. Then $\widehat{R}/I \cong \widehat{A}$, for the first part of item 6. The remaining assertions in item 6 follow from the fact that A is a Noetherian local domain and $\widehat{A} \cong \widehat{R}/I$. Applying Remarks 3.2, we have $\widehat{R}/I$ is faithfully flat over A, and by Remark 2.31.6 the nonzero elements of A are regular on $\widehat{R}/I$.

The following commutative diagram, where the vertical maps are injections, displays the relationships among these rings:

$$
\begin{array}{ccccccc}
 & & \widehat{R} & \xrightarrow{\hat{\varphi}} & \widehat{A} \cong \widehat{R}/I & \longrightarrow & \mathcal{Q}(\widehat{R}/I) \\
 & & \uparrow & & \uparrow & & \uparrow \\
k & \xrightarrow{\subseteq} & R & \xrightarrow{\varphi} & A := \mathcal{Q}(A) \cap (\widehat{R}/I) & \longrightarrow & \mathcal{Q}(A) \quad .
\end{array}
\tag{4.1}
$$

This completes the proof of Theorem 4.2. □

Theorem 4.2 implies Corollary 4.3, yielding further information regarding Question 4.1.

COROLLARY 4.3. *Every Noetherian local domain $(A, \mathbf{n})$ having a coefficient field k, and having the property that the field of fractions L of A is finitely generated over k is realizable as an intersection $L \cap (\widehat{R}/I)$, where R is a Noetherian local domain essentially finitely generated over k with $\mathcal{Q}(R) = L$, and I is an ideal in the completion $\widehat{R}$ of R such that $P \cap R = (0)$ for each associated prime P of $\widehat{R}/I$.*

REMARK 4.4. In connection with Corollary 4.3, a result proved in [**55**, Corollary 2] implies that a d-dimensional Noetherian local domain $(R, \mathbf{m})$ that is essentially finitely generated over a field k has the following property: every d-dimensional Noetherian local domain S that is either normal or quasi-unmixed, and that birationally dominates R is essentially finitely generated over R. Thus S is essentially finitely generated over k. A Noetherian local domain $(S, \mathbf{n})$ is said to be *quasi-unmixed* if its $\mathbf{n}$-adic completion $\widehat{S}$ is equidimensional in the sense of Definition 3.17.5.

A modification of the question raised by Judy Sally from Chapter 1 is:

QUESTION 4.5. Let R be a Noetherian integral domain. What Noetherian overrings of R exist inside the field of fractions of R?

In connection with Question 4.5, the Krull-Akizuki theorem (see Theorem 2.18) implies that every birational overring of a one-dimensional Noetherian integral domain is Noetherian and of dimension at most one. On the other hand, every Noetherian domain of dimension greater than one admits birational overrings that are

not Noetherian. Indeed, if R is an integral domain with $\dim R > 1$, then by [**117**, (11.9)] there exists a valuation ring V that is birational over R with $\dim V > 1$. Since a Noetherian valuation ring has dimension at most one, if $\dim R > 1$, then there exist birational overrings of R that are not Noetherian.

REMARK 4.6. Corollary 4.3 is a first start towards a classification of the Noetherian local domains A having a given coefficient field k, and having the property that the field of fractions of A is finitely generated over k. A drawback with Corollary 4.3 is that it is not true for every triple R, L, I as in Corollary 4.3 that $L \cap (\widehat{R}/I)$ is Noetherian (see Examples 10.9 below). In order to have a more satisfying classification an important goal is to identify necessary and sufficient conditions that $L \cap (\widehat{R}/I)$ is Noetherian for R, L, I as in Corollary 4.3.

4.2. Elementary examples

We first consider examples where R is a polynomial ring over a field k. In the case of one variable the situation is well understood:

EXAMPLE 4.7. Let x be a variable over a field k, let $R := k[x]$, and let L be a subfield of the field of fractions of $k[[x]]$ such that $k(x) \subseteq L$. Then the intersection domain $A := L \cap k[[x]]$ is a rank-one discrete valuation domain (DVR) with field of fractions L (see Remark 2.1), maximal ideal xA and (x)-adic completion $A^* = k[[x]]$. For example, if we work with the field $\mathbb{Q}$ of rational numbers and our favorite transcendental function e^x, and we put $L = \mathbb{Q}(x, e^x)$, then A is a DVR having residue field $\mathbb{Q}$ and field of fractions L of transcendence degree 2 over $\mathbb{Q}$.

The integral domain A of Example 4.7 with $k = \mathbb{Q}$ is perhaps the simplest example of a Noetherian local domain on an algebraic function field $L/\mathbb{Q}$ of two variables that is not essentially finitely generated over its ground field $\mathbb{Q}$, i.e., A is not the localization of a finitely generated $\mathbb{Q}$-algebra. For an appropriate choice of the field L, however, the ring A does have a nice description as an infinite nested union of localized polynomial rings in two variables over $\mathbb{Q}$; see Section 4.4. Thus in a certain sense there is a good description of the elements of the intersection domain A in this case.

The case where the base ring R involves two variables is more interesting. The following theorem of Valabrega [**154**] is useful in considering this case.

THEOREM 4.8. (Valabrega) *Let C be a DVR, let x be an indeterminate over C, and let L be a subfield of $\mathcal{Q}(C[[x]])$ such that $C[x] \subset L$. Then the integral domain $D = L \cap C[[x]]$ is a two-dimensional regular local domain having completion $\widehat{D} = \widehat{C}[[x]]$, where $\widehat{C}$ is the completion of C.*

Exercise 4 of this chapter outlines a proof for Theorem 4.8. Applying Valabrega's Theorem 4.8, we see that the intersection domain is a two-dimensional regular local domain with the "right" completion in the following two examples:

EXAMPLE 4.9. Let x and y be indeterminates over $\mathbb{Q}$ and let C be the DVR $\mathbb{Q}(x, e^x) \cap \mathbb{Q}[[x]]$. Then $A_1 := \mathbb{Q}(x, e^x, y) \cap C[[y]] = C[y]_{(x,y)}$ is a two-dimensional regular local domain with maximal ideal $(x, y)A_1$ and completion $\mathbb{Q}[[x, y]]$.

EXAMPLE 4.10. This example is related to the iterative examples of Chapter 12. Let x and y be indeterminates over $\mathbb{Q}$ and let E be the DVR $\mathbb{Q}(x, e^x) \cap \mathbb{Q}[[x]]$ as in Example 4.7. Then $A_2 := \mathbb{Q}(x, y, e^x, e^y) \cap E[[y]]$ is a two-dimensional regular local domain with maximal ideal $(x, y)A_2$ and completion $\mathbb{Q}[[x, y]]$. See Theorem 12.3.

REMARKS 4.11. (1) There is a significant difference between the integral domains A_1 of Example 4.9 and A_2 of Example 4.10. As is shown in Theorem 17.25, the two-dimensional regular local domain A_1 of Example 4.9 is, in a natural way, a nested union of three-dimensional regular local domains. It is possible therefore to describe A_1 rather explicitly. On the other hand, the two-dimensional regular local domain A_2 of Example 4.10 contains, for example, the element $\frac{e^x-e^y}{x-y}$. There is an integral domain B naturally associated with A_2 that is a nested union of four-dimensional RLRs, is three-dimensional and is not Noetherian; see Example 12.6. Notice that the two-dimensional regular local ring A_1 is a subring of an algebraic function field in three variables over $\mathbb{Q}$, while A_2 is a subring of an algebraic function field in four variables over $\mathbb{Q}$. Since the field $\mathbb{Q}(x, e^x, y)$ is contained in the field $\mathbb{Q}(x, e^x, y, e^y)$, the local ring A_1 is dominated by the local ring A_2.

(2) It is shown in Theorem 20.20 and Corollary 20.23 of Chapter 20 that if we go outside the range of Valabrega's theorem, that is, if we take more general subfields L of the field of fractions of $\mathbb{Q}[[x, y]]$ such that $\mathbb{Q}(x, y) \subseteq L$, then the intersection domain $A = L \cap \mathbb{Q}[[x, y]]$ can be, depending on L, a localized polynomial ring in $n \geq 3$ variables over $\mathbb{Q}$ or even a localized polynomial ring in infinitely many variables over $\mathbb{Q}$. In particular, $A = L \cap \mathbb{Q}[[x, y]]$ need not be Noetherian. Theorem 12.3 describes possibilities for the intersection domain A in this setting.

4.3. Historical examples

There are classical examples, related to singularities of algebraic curves, of one-dimensional Noetherian local domains $(R, \mathbf{m})$ such that the $\mathbf{m}$-adic completion $\widehat{R}$ is not an integral domain, that is, R is analytically reducible. We demonstrate this in Example 4.12.

EXAMPLE 4.12. Let X and Y be variables over $\mathbb{Q}$ and consider the localized polynomial ring

$$S := \mathbb{Q}[X, Y]_{(X,Y)} \text{ and the quotient ring } R := \frac{S}{(X^2 - Y^2 - Y^3)S}.$$

Since the polynomial $X^2 - Y^2 - Y^3$ is irreducible in the polynomial ring $\mathbb{Q}[X, Y]$, the ring R is a one-dimensional Noetherian local domain. Let x and y denote the images in R of X and Y, respectively. The principal ideal yR is primary for the maximal ideal $\mathbf{m} = (x, y)R$, and so the $\mathbf{m}$-adic completion $\widehat{R}$ is also the y-adic completion of R. Thus

$$\widehat{R} = \frac{\mathbb{Q}[X][[Y]]}{(X^2 - Y^2(1 + Y))}.$$

Since $1 + Y$ has a square root $(1 + Y)^{1/2} \in \mathbb{Q}[[Y]]$, we see that $X^2 - Y^2(1 + Y)$ factors in $\mathbb{Q}[X][[Y]]$ as

$$X^2 - Y^2(1 + Y) = (X - Y(1 + Y)^{1/2}) \cdot (X + Y(1 + Y)^{1/2}).$$

Thus $\widehat{R}$ is not an integral domain. Since the polynomial $Z^2 - (1 + y) \in R[Z]$ has x/y as a root and $x/y \notin R$, the integral domain R is not normal; see Section 2.1. The birational integral extension $\overline{R} := R[\frac{x}{y}]$ has two maximal ideals,

$$\mathbf{m}_1 := (\mathbf{m}, \frac{x}{y} - 1)\overline{R} = (\frac{x - y}{y})\overline{R} \quad \text{and} \quad \mathbf{m}_2 := (\mathbf{m}, \frac{x}{y} + 1)\overline{R} = (\frac{x + y}{y})\overline{R}.$$

To see, for example, that $\mathbf{m}_1 = (\frac{x-y}{y})\overline{R}$, it suffices to show that $\mathbf{m} \subset (\frac{x-y}{y})\overline{R}$. It is obvious that $x - y \in (\frac{x-y}{y})\overline{R}$. We also clearly have $\frac{x^2-y^2}{y^2} \in (\frac{x-y}{y})\overline{R}$, and $x^2 - y^2 = y^3$. Hence $\frac{y^3}{y^2} = y \in (\frac{x-y}{y})\overline{R}$, and so $\mathbf{m}_1$ is principal and generated by $\frac{x-y}{y}$. Similarly, the maximal ideal $\mathbf{m}_2$ is principal and is generated by $\frac{x+y}{y}$. Thus $\overline{R} = R[\frac{x}{y}]$ is a PID, and hence is integrally closed. To better understand the structure of R and $\overline{R}$, it is instructive to extend the homomorphism

$$\varphi : S \longrightarrow \frac{S}{(X^2 - Y^2 - Y^3)S} = R.$$

Let $X_1 := X/Y$ and $S' := S[X_1]$. Then S' is a regular integral domain and the map φ can be extended to a map $\psi : S' \to R[\frac{x}{y}]$ such that $\psi(X_1) = \frac{x}{y}$. The kernel of ψ is a prime ideal of S' that contains $X^2 - Y^2 - Y^3$. Since $X = YX_1$, and Y^2 is not in ker ψ, we see that ker $\psi = (X_1^2 - 1 - Y)S'$. Thus

$$\psi : S' \longrightarrow \frac{S'}{(X_1^2 - 1 - Y)S'} = R[\frac{x}{y}] = \overline{R}.$$

Notice that $X_1^2 - 1 - Y$ is contained in exactly two maximal ideals of S', namely

$$\mathbf{n}_1 := (X_1 - 1,\ Y)S' \quad \text{and} \quad \mathbf{n}_2 := (X_1 + 1,\ Y)S'.$$

The rings $S_1 := S'_{\mathbf{n}_1}$ and $S_2 := S'_{\mathbf{n}_2}$ are two-dimensional RLRs that are local quadratic transformations [2] of S, and the map ψ localizes to define maps

$$\psi_{\mathbf{n}_1} : S_1 \to \frac{S_1}{(X_1^2 - 1 - Y)S_1} = \overline{R}_{\mathbf{m}_1} \text{ and } \psi_{\mathbf{n}_2} : S_2 \to \frac{S_2}{(X_1^2 - 1 - Y)S_2} = \overline{R}_{\mathbf{m}_2}.$$

Thus the integral closure $\overline{R}$ of R is a homomorphic image of a regular domain of dimension two with precisely two maximal ideals.

Remark 4.13. Examples given by Akizuki [**9**] and Schmidt [**140**], provide one-dimensional Noetherian local domains R such that the integral closure $\overline{R}$ is not finitely generated as an R-module; equivalently, the completion $\widehat{R}$ of R has nonzero nilpotents; see [**117**, (32.2) and Ex. 1, page 122] and the paper of Katz [**86**, Corollary 5].

If R is a normal one-dimensional Noetherian local domain, then R is a rank-one discrete valuation domain (DVR) and it is well-known that the completion of R is again a DVR. Thus R is analytically irreducible. Zariski showed that the normal Noetherian local domains that occur in algebraic geometry are analytically normal; see [**165**, pages 313-320] and Section 3.4. In particular, the normal local domains occurring in algebraic geometry are analytically irreducible.

This motivated the question of whether there exists a normal Noetherian local domain for which the completion is not a domain. Nagata produced such examples in [**115**]. He also pinpointed sufficient conditions for a normal Noetherian local domain to be analytically irreducible [**117**, (37.8)].

In Example 4.14, we present a construction of Nagata [**115**], [**117**, Example 7, pages 209-211] of a two-dimensional regular local domain A with completion $\widehat{A} = k[[x, y]]$, where k is a field with $\operatorname{char} k \neq 2$. Nagata proves that A is Noetherian, but is not excellent. Nagata also constructs a related two-dimensional normal

[2] Chapter 14 contains more information about local quadratic transformations; see Definitions 14.1.

Noetherian local domain D that is analytically reducible.[3] Although Nagata constructs A as a nested union of subrings, we give in Example 4.14 a description of A as an intersection.

EXAMPLE 4.14. (Nagata) [**117**, Example 7, pages 209-211] Let x and y be algebraically independent over a field k, where $\operatorname{char} k \neq 2$, and let R be the localized polynomial ring $R = k[x,y]_{(x,y)}$. Then the completion of R is $\widehat{R} = k[[x,y]]$. Let $\tau \in xk[[x]]$ be an element that is transcendental over $k(x,y)$, e.g., if $k = \mathbb{Q}$ we may take $\tau = e^x - 1$. Let $\rho := y + \tau$ and $f := \rho^2 = (y+\tau)^2$. Now define

$$A := \ k(x,y,f) \cap k[[x,y]] \quad \text{and} \quad D := \ \frac{A[z]}{(z^2 - f)A[z]} \ ,$$

where z is an indeterminate. It is clear that the intersection ring A is a Krull domain having a unique maximal ideal. Nagata proves that f is a prime element of A and that A is a two-dimensional regular local domain with completion $\widehat{A} = k[[x,y]]$; see Proposition 6.13. Nagata also shows that D is a normal Noetherian local domain. We discuss and establish other properties of the integral domains A and D in Remarks 4.15. We show the ring A is Noetherian in Section 6.3.

REMARKS 4.15. (1) The integral domain D in Example 4.14 is analytically reducible. This is because the element f factors as a square in the completion $\widehat{A}$ of A. Thus

$$\widehat{D} \ = \ \frac{k[[x,y,z]]}{(z-(y+\tau))(z+(y+\tau))},$$

which is not an integral domain. As recorded in [**55**, page 670], David Shannon has observed that there exists a two-dimensional regular domain that birationally dominates D and is not essentially finitely generated over D. This behavior of D differs from the situation described in Remark 4.4.

(2) The two-dimensional regular local domain A of Example 4.14 is not a Nagata ring and therefore is not excellent.[4] To see that A is not a Nagata ring, notice that A has a principal prime ideal generated by f that factors as a square in $\widehat{A} = k[[x,y]]$; namely f is the square of the prime element ρ of $\widehat{A}$. Therefore the one-dimensional local domain A/fA has the property that its completion $\widehat{A}/f\widehat{A}$ has a nonzero nilpotent element. This implies that the integral closure of the one-dimensional Noetherian domain A/fA is not finitely generated over A/fA by Remark 3.16.2.i. Hence A is not a Nagata ring. Moreover, the map $A \hookrightarrow \widehat{A} = k[[x,y]]$ is not a regular morphism; see Section 3.4.

The existence of examples such as the normal Noetherian local domain D of Example 4.14 naturally motivated the question: Is a Nagata domain necessarily excellent? Rotthaus shows in [**131**] that the answer is "no" as described below.

In Example 4.16, we present the construction of Rotthaus. In [**131**] the ring A is constructed as a direct limit. We show in Christel's Example 4.16 that A can also be described as an intersection. For this we use that A is Noetherian implies that its completion $\widehat{A}$ is a faithfully flat extension, and then we apply Remark 2.31.9.

---

[3]These concepts are defined in Sections 3.4 and 3.1.

[4]For the definition of a Nagata ring, see Definition 2.11 of Chapter 2; for the definition of excellence, see Definition 3.37 of Chapter 3. More details about these concepts are given in Section 13.2 and Section **??** of Chapter 13.

EXAMPLE 4.16. (Christel) Let x, y, z be algebraically independent over a field k, where $\operatorname{char} k = 0$, and let R be the localized polynomial ring $R = k[x, y, z]_{(x,y,z)}$. Let $\sigma = \sum_{i=1}^{\infty} a_i x^i \in k[[x]]$ and $\tau = \sum_{i=1}^{\infty} b_i x^i \in k[[x]]$ be power series such that x, σ, τ are algebraically independent over k, for example, if $k = \mathbb{Q}$, we may take $\sigma = e^x - 1$ and $\tau = e^{x^2} - 1$. Let $u := y + \sigma$ and $v := z + \tau$. Define

$$A \ := \ k(x, y, z, uv) \ \cap \ (k[y, z]_{(y,z)}][[x]]).$$

We demonstrate some properties of the ring A in Remark 4.17.

REMARK 4.17. The integral domain A of Example 4.16 is a Nagata domain that is not excellent. Rotthaus shows in [**131**] that A is Noetherian and that the completion $\widehat{A}$ of A is $k[[x, y, z]]$, so A is a 3-dimensional regular local domain. Moreover she shows the formal fibers of A are reduced, but are not regular. Since u, v are part of a regular system of parameters of $\widehat{A}$, it is clear that $(u, v)\widehat{A}$ is a prime ideal of height two. It is shown in [**131**] that $(u, v)\widehat{A} \cap A = uvA$. Thus uvA is a prime ideal and $\widehat{A}_{(u,v)\widehat{A}}/uv\widehat{A}_{(u,v)\widehat{A}}$ is a non-regular formal fiber of A. Therefore A is not excellent.

Since A contains a field of characteristic zero, to see that A is a Nagata domain it suffices to show for each prime ideal P of A that the integral closure of A/P is a finite A/P-module; see Theorem 2.3. Since the formal fibers of A are reduced, the integral closure of A/P is a finite A/P-module; see Remark 3.16.1.

4.4. The Prototype

In this section we present a standard example, called the *Prototype*. The construction of the Prototype illustrates the *Inclusion Construction*. Moreover, the Prototype enables us to construct and verify more intricate and sophisticated examples.

SETTING 4.18. Let x be an indeterminate over a field k, and let s be a positive integer. By Fact 3.7, there exist elements $\tau_1, \ldots, \tau_s \in xk[[x]]$ that are algebraically independent over $k(x)$. In order to construct the prototype, we first construct a discrete valuation domain C_s such that

- $k[x] \subset C_s$,
- the maximal ideal of C_s is xC_s,
- the (x)-adic completion of C_s is $k[[x]]$,
- C_s has field of fractions $k(x, \tau_1, \ldots, \tau_s)$, and
- the transcendence degree of C_s over k is $s + 1$.

REMARK 4.19. If there exists a DVR C_s satisfying the properties in Setting 4.18, then $C_s = k(x, \tau_1, \ldots, \tau_s) \cap k[[x]]$, by Remark 3.2.4. Hence C_s is uniquely determined by its field of fractions.

There are two methods to obtain such an integral domain C_s, given below as Construction 4.20 and Construction 4.21.

CONSTRUCTION 4.20. The intersection method. In this case C_s is denoted A. This method is used in Example 4.7. We show that the intersection integral domain $A = k(x, \tau_1, \ldots, \tau_s) \cap k[[x]]$ satisfies the properties in Setting 4.18.

By Exercise 3 of Chapter 3, the integral domain A is a DVR with field of fractions $k(x, \tau_1, \ldots, \tau_s)$. Furthermore, we have $x^n k[[x]] \cap A = x^n A$, for every

positive integer n, and

$$\frac{k[x]}{x^n k[x]} \subseteq \frac{A}{x^n A} \subseteq \frac{k[[x]]}{x^n k[[x]]} = \frac{k[x]}{x^n k[x]}.$$

Thus the inclusions above are equalities, xA is the maximal ideal of A, and the (x)-adic completion of A is $\widehat{A} = k[[x]]$.

CONSTRUCTION 4.21. The approximation method: In this case, we denote the ring C_s by B.

This method is relevant for the construction of many examples later in the book. The ring B is defined as a nested union of subrings B_n of the field $k(x, \tau_1, \ldots, \tau_s)$. In order to define B we consider the last parts or the *endpieces* τ_{in} of τ_i. Suppose that for all $1 \leq i \leq s$ the power series τ_i is given by:

$$\tau_i := \sum_{j=1}^{\infty} a_{ij} x^j \in xk[[x]],$$

where $a_{ij} \in k$. The n^{th} endpiece of τ_i is the power series:

$$\tau_{in} := \frac{1}{x^n}(\tau_i - \sum_{j=1}^{n} a_{ij} x^j) = \sum_{j=n+1}^{\infty} a_{ij} x^{j-n} \in xk[[x]]. \tag{4.21.0}$$

For each $n \in \mathbb{N}$ and each $i \in \{1, \ldots, s\}$, we have an *endpiece recursion relation*:

$$\tau_{in} = \tau_{in+1} x + a_{in+1} x \in k(x, \tau_1, \ldots, \tau_s) \cap k[[x]]. \tag{4.21.1}$$

We define

$$B_n := k[x, \tau_{1n}, \ldots, \tau_{sn}]_{(x, \tau_{1n}, \ldots, \tau_{sn})}. \tag{4.21.2}$$

Each of the rings B_n is a localized polynomial ring in $s+1$ variables over the field k. Because of the recursion relation in Equation 4.21.1, we have that $B_n \subset B_{n+1}$ for each $n \in \mathbb{N}$. We define B to be the directed union:

$$B = \bigcup_{n \in \mathbb{N}} B_n = \varinjlim_n B_n.$$

We show that B has the five properties listed in Setting 4.18. We first describe a different construction of B. For each $n \in \mathbb{N}$ define:

$$U_n := k[x, \tau_{1n}, \ldots, \tau_{sn}]. \tag{4.21.3}$$

Notice that U_n is a polynomial ring in $s+1$ variables over the field k. By the recursion relation in Equation 4.21.1, we have $U_n \subset U_{n+1}$. Consider the directed union of polynomial rings:

$$U := \bigcup_{n \in \mathbb{N}} U_n = \varinjlim_n U_n.$$

By the recursion relation in Equation 4.21.1, each $\tau_{in} \in xU_{n+1}$; this implies that $xB \cap U_n$ is a maximal ideal of U_n, and it follows that $xB \cap U$ is a maximal ideal of U. Since each B_n is a localization of U_n, the ring B is a localization of the ring U at the maximal ideal $xB \cap U$. We show in Theorem 5.14 that B can also be expressed as $B = (1 + xU)^{-1}U$.

PROPOSITION 4.22. *With notation as in Construction 4.21, for each $\gamma \in U$ and each $t \in \mathbb{N}$, there exist elements $g_t \in k[x]$ and $\delta_t \in U$ such that:*

$$\gamma \;=\; g_t \;+\; x^t\delta_t.$$

PROOF. We have $\gamma \in U_n$ for some $n \in \mathbb{N}$. Thus we can write γ as a polynomial in $\tau_{1n}, \ldots, \tau_{sn}$ with coefficients in $k[x]$:

$$\gamma \;=\; \sum a_{(j)}\tau_{1n}^{j_1}\ldots\tau_{sn}^{j_s},$$

where $a_{(j)} \in k[x]$ and (j) represents the tuple $(j_1, \ldots, j_s)$. Using the recursion relation in Equation 4.21.1, for all $1 \le i \le s$, we have

$$\tau_{in} \;=\; x^t\tau_{in+t} + r_i$$

where $r_i \in k[x]$. By substituting $x^t\tau_{in+t} + r_i$ for τ_{in} we can write γ as an element of U_{n+t} as follows:

$$\gamma \;=\; \sum a_{(j)}(x^t\tau_{1n+t} + r_1)^{j_1}\ldots(x^t\tau_{sn+t} + r_s)^{j_t} \;=\; g_t \;+\; x^t\delta_t,$$

where $g_t \in k[x]$ and $\delta_t \in U_{n+t}$. □

PROPOSITION 4.23. *The ring B is a DVR with maximal ideal xB, and we have $x^tk[[x]] \cap B \;=\; x^tB$, for every $t \in \mathbb{N}$.*

PROOF. Let $\gamma \in B$ with $\gamma \in x^tk[[x]]$. First note that $\gamma = \gamma_0\epsilon$ where ϵ is a unit of B and $\gamma_0 \in U$. By Proposition 4.22,

$$\gamma_0 \;=\; g_{t+1} \;+\; x^{t+1}\delta_{t+1},$$

where $g_{t+1} \in k[x]$ and $\delta_{t+1} \in U$. By assumption, $\gamma \in x^tk[[x]]$; thus $g_{t+1} \in x^tk[[x]]$. Since the embedding $k[x]_{(x)} \hookrightarrow k[[x]]$ is faithfully flat, we have $g_{t+1} \in x^tk[x]_{(x)}$, and therefore $\gamma \in x^tB$. This shows that $x^tk[[x]] \cap B \;=\; x^tB$, for every $t \in \mathbb{N}$.

Since $\bigcap_{t\in\mathbb{N}}(x^t)k[[x]] = (0)$, every nonzero element $\gamma \in B$ can be written as $\gamma = x^t\epsilon$ where $\epsilon \in B$ is a unit. It follows that the ideals of B are linearly ordered and B is a DVR with maximal ideal xB. □

The ring B also satisfies the five conditions of Setting 4.18. Obviously, B dominates $k[x]_{(x)}$ and is dominated by $k[[x]]$. By Proposition 4.23, B is a DVR with maximal ideal xB, and by construction $k(x, \tau_1, \ldots, \tau_s)$ is the field of fractions of B. By Proposition 4.23, we have $x^tk[[x]] \cap B = x^tB$, for every $t \in \mathbb{N}$. Therefore we have:

$$\frac{k[x]}{x^tk[x]} \;\subseteq\; \frac{B}{x^tB} \;\subseteq\; \frac{k[[x]]}{x^tk[[x]]} \;=\; \frac{k[x]}{x^tk[x]}.$$

Thus the inclusions above are equalities, and so $\widehat{B} = k[[y]]$.

NOTE 4.24. By Remark 4.19, we have $C_s = A = B$, where A is the DVR described as an intersection in Construction 4.20 and B is the DVR described as a directed union in Construction 4.21.

We extend this example to higher dimensions by adjoining additional variables.

LOCAL PROTOTYPE EXAMPLE 4.25. Assume as in Setting 4.18 that x is an indeterminate over a field k, that s is a positive integer, and that $\tau_1, \ldots, \tau_s \in xk[[x]]$ are algebraically independent over $k(x)$. Let C_s be the DVR of Constructions 4.20 and 4.21 with maximal ideal xC_s. Let r be a positive integer and let $y_1, \ldots, y_r$ be additional indeterminates over C_s.

We construct a regular local ring D such that

(1) $k[x, y_1, \dots y_r] \subset D$,
(2) the maximal ideal of D is $(x, y_1, \dots, y_r)D$,
(3) the (x)-adic completion of D is $k[y_1, \dots, y_r]_{(y_1,\dots,y_r)}[[x]]$,
(4) The completion of D with respect to its maximal ideal is $\widehat{D} = k[[x, y_1, \dots, y_r]]$.
(5) D has field of fractions $k(x, \tau_1, \dots, \tau_s, y_1, \dots, y_r)$, and
(6) the transcendence degree of D over k is $s + r + 1$.

PROPOSITION 4.26. *With the notation of Setting 4.18 and Constructions 4.20 and 4.21, we define* $D := C_s[y_1, \dots, y_r]_{(x,y_1,\dots,y_r)}$. *Then we have:*

(1) *D satisfies properties 1-6 of Local Prototype Example 4.25.*
(2) $D = k(x, \tau_1, \dots, \tau_s, y_1, \dots, y_r) \cap k[y_1, \dots, y_r]_{(y_1,\dots,y_r)}[[x]]$ *and* $D = k(x, \tau_1, \dots, \tau_s, y_1, \dots, y_r) \cap k[[x, y_1, \dots, y_r]]$.
(3) $D = \bigcup_{i=1}^{\infty} k[x, \tau_{1n}, \dots, \tau_{sn}, y_1, \dots, y_r]_{(x,\tau_{1n},\dots,\tau_{sn},y_1,\dots,y_r)}$ *a directed union of localized polynomial rings, where each τ_{in} is the n^{th} endpiece of τ_i, as in Equation 4.21.1.*

PROOF. We first observe that D as defined is a regular local ring with maximal ideal $\mathbf{m} = (x, y_1, \dots, y_r)D$, that the $\mathbf{m}$-adic completion of D is $k[[x, y_1, \dots, y_r]]$, and that the (x)-adic completion of D is $k[y_1, \dots, y_r]_{(y_1,\dots,y_r)}[[x]]$. Therefore D satisfies the six properties of Local Prototype Example 4.25. Since completions of Noetherian local rings are faithfully flat, we have that D satisfies part 2 of Proposition 4.26; see Remark 3.2.4.

In order to establish that D is the directed union of localized polynomial rings of the third part of Proposition 4.26, we define for each $n \in \mathbb{N}$:

$$W_n \ := \ k[x, y_1, \dots, y_r, \tau_{1n}, \dots, \tau_{sn}] = U_n \ \otimes_k \ k[y_1, \dots, y_r]$$

and

$$D_n \ := \ B_n[y_1, \dots, y_r]_{(\mathbf{m}_n, y_1, \dots, y_r)},$$

where $\mathbf{m}_n = (x, \tau_{1n}, \dots, \tau_{sn})B_n$ is the maximal ideal of B_n. Thus W_n is a polynomial ring in $s + r + 1$ variables over the field k, and D_n is a localization of W_n at the maximal ideal of W_n generated by these $s + r + 1$ variables.

We have the inclusions $W_n \subset W_{n+1} \subset k[y_1, \dots, y_r][[x]]$, and

$$D_n \ \subset \ D_{n+1} \ \subset \ k[y_1, \dots, y_r]_{(y_1,\dots,y_r)}[[x]].$$

We define

$$W \ := \ \bigcup_{n\in\mathbb{N}} W_n \quad \text{and} \quad D' \ := \ \bigcup_{n\in\mathbb{N}} D_n.$$

Since direct limits commute with tensor products, we have:

$$W \ = \ U[y_1, \dots, y_r].$$

It follows that

$$D' \ = \ W_{(x,y_1,\dots,y_r)} \ = \ C_s[y_1, \dots, y_r]_{(x,y_1,\dots,y_r)} = D,$$

as desired for the proposition. □

A regular local ring D as described in Local Prototype Example 4.25 exists for each positive integer s and each nonnegative integer r. The basic technique we have used for constructing D is called the *Inclusion Construction*. We present a more detailed description of the Inclusion Construction in Chapter 5, where we also present other examples.

DEFINITION 4.27. With the notation of Local Prototype Example 4.25, the regular local ring $D = C_s[y_1, \ldots, y_r]_{(x,y_1,\ldots,y_r)}$ is called the *Local Prototype* or the *Local Prototype Domain for the Inclusion Construction.* The *Intersection Form of the Prototype* is

$$D = k(x, \tau_1, \ldots, \tau_s, y_1, \ldots, y_r) \cap k[y_1, \ldots, y_r]_{(y_1,\ldots,y_r)}[[x]]. \tag{4.27.1}$$

REMARKS 4.28. With the notation of Local Prototype Example 4.25, let R be the localized polynomial ring $R := k[x, y_1, \ldots, y_r]_{(x,y_1,\ldots,y_r)}$, and let R^* denote the (x)-adic completion of R. Thus $R^* = k[y_1, \ldots, y_r]_{(y_1,\ldots,y_r)}[[x]]$, and the Local Prototype Example D of Definition 4.27 is given by $D = R[y_1, \ldots, y_r]_{(\mathbf{m},y_1,\ldots,y_r)}[[x]]$, where $\mathbf{m}$ is the maximal ideal of R.

(1) Equation 4.27.1 implies that

$$D \;=\; \mathcal{Q}(R)(\tau_1, \ldots, \tau_s) \;\cap\; R^*, \tag{4.28.11}$$

where $\mathcal{Q}(R)$ denotes the field of fractions of R. The ring D is an example of Inclusion Construction 5.3.

(2) We call the ring D a "Prototype" because of its use in the construction of other examples. In many of these examples the insider constructed integral domain E dominates R and is dominated by the local integral domain D so that we have:

$$R \;=\; k[x, y_1, \ldots, y_r]_{(x,y_1,\ldots,y_r)} \;\hookrightarrow\; E \;\hookrightarrow\; D \;\hookrightarrow\; k[[x, y_1, \ldots, y_r]].$$

Exercises

(1) Prove that the intersection domain A of Example 4.7 is a DVR with field of fractions L and (y)-adic completion $A^* = \mathbb{Q}[[y]]$.

Comment. Exercise 2 of Chapter 2 implies that A is a DVR. With the additional hypothesis of Example 4.7, it is true that the (y)-adic completion of A is $\mathbb{Q}[[y]]$.

(2) Let R be an integral domain with field of fractions K.
   (i) Let F be a subfield of K and let $S := F \cap R$. For each principal ideal aS of S, prove that $aS = aR \cap S$.
   (ii) Assume that S is a subring of R with the same field of fractions K. Prove that $aS = aR \cap S$ for each $a \in S \iff S = R$.

(3) Let R be a local domain with maximal ideal $\mathbf{m}$ and field of fractions K. Let F be a subfield of K and let $S := F \cap R$. Prove that S is local with maximal ideal $\mathbf{m} \cap S$, and thus conclude that R dominates S. Give an example where R is not Noetherian, but S is Noetherian.

Remark. It can happen that R is Noetherian while S is not Noetherian; see Chapter 15.

(4) Assume the notation of Theorem 4.8. Thus y is an indeterminate over the DVR C and $D = C[[y]] \cap L$, where L is a subfield of the field of fractions of $C[[y]]$ with $C[y] \subset L$. Let x be a generator of the maximal ideal of C and let $R := C[y]_{(x,y)C[y]}$. Observe that R is a two-dimensional RLR with maximal ideal $(x, y)R$ and that $C[[y]]$ is a two-dimensional RLR with maximal ideal $(x, y)C[[y]]$ that dominates R. Let $\mathbf{m} := (x, y)C[[y]] \cap D$.

(i) Using Exercise 2, prove that

$$C \;\cong\; \frac{R}{yR} \;\hookrightarrow\; \frac{D}{yD} \;\hookrightarrow\; \frac{C[[y]]}{yC[[y]]} \;\cong\; C.$$

(ii) Deduce that $C \cong \frac{D}{yD}$, and that $\mathbf{m} = (x, y)D$.

(iii) Let $k := \frac{C}{xC}$ denote the residue field of C. Prove that $\frac{D}{xD}$ is a DVR and that

$$k[y] \;\hookrightarrow\; \frac{R}{xR} \;\hookrightarrow\; \frac{D}{xD} \;\hookrightarrow\; \frac{C[[y]]}{xC[[y]]} \;\cong\; k[[y]].$$

(iv) For each positive integer n, prove that

$$\frac{R}{(x,y)^n R} \;\cong\; \frac{D}{(x,y)^n D} \;\cong\; \frac{C[[y]]}{(x,y)^n C[[y]]}.$$

Deduce that $\widehat{R} = \widehat{D} = \widehat{C}[[y]]$, where $\widehat{C}$ is the completion of C.

(v) Let P be a prime ideal of D such that $x \notin P$. Prove that there exists $b \in P$ such that $b(D/xD) = y^r(D/xD)$ for some positive integer r, and deduce that $P \subset (b, x)D$.

(vi) For $a \in P$, observe that $a = c_1 b + a_1 x$, where c_1 and a_1 are in D. Since $x \notin P$, deduce that $a_1 \in P$ and hence $a_1 = c_2 b + a_2 x$, where c_2 and a_2 are in D. Conclude that $P \subset (b, x^2)D$. Continuing this process, deduce that

$$bD \;\subseteq P \;\subseteq\; \bigcap_{n=1}^{\infty} (b, x^n)D.$$

(vii) Extending the ideals to $C[[y]]$, observe that

$$bC[[y]] \;\subseteq\; PC[[y]] \;\subseteq\; \bigcap_{n=1}^{\infty} (b, x^n)C[[y]] \;=\; bC[[y]],$$

where the last equality is because the ideal $bC[[y]]$ is closed in the topology defined by the ideals generated by the powers of x on the Noetherian local ring $C[[y]]$. Deduce that $P = bD$.

(viii) Conclude by Theorem 2.19 that D is Noetherian and hence a two-dimensional regular local domain with completion $\widehat{D} = \widehat{C}[[y]]$.

(5) Let k be a field and let $f \in k[[x, y]]$ be a formal power series of order $r \geq 2$. Let $f = \sum_{n=r}^{\infty} f_n$, where $f_n \in k[x, y]$ is a homogeneous form of degree n. If the leading form f_r factors in $k[x, y]$ as $f_r = \alpha \cdot \beta$, where α and β are coprime homogeneous polynomials in $k[x, y]$ of positive degree, prove that f factors in $k[[x, y]]$ as $f = g \cdot h$, where g has leading form α and h has leading form β.

Suggestion. Let $G = \bigoplus_{n \geq 0} G_n$ represent the polynomial ring $k[x, y]$ as a graded ring obtained by defining $\deg x = \deg y = 1$. Notice that G_n has dimension $n + 1$ as a vector space over k. Let $\deg \alpha = a$ and $\deg \beta = b$. Then $a + b = r$ and for each integer $n \geq r + 1$, we have $\dim(\alpha \cdot G_{n-a}) = n - a + 1$ and $\dim(\beta \cdot G_{n-b}) = n - b + 1$. Since α and β are coprime, we have

$$(\alpha \cdot G_{n-a}) \;\cap\; (\beta \cdot G_{n-b}) \;=\; f_r \cdot G_{n-r}.$$

Conclude that $\alpha \cdot G_{n-a} \;+\; \beta \cdot G_{n-b}$ is a subspace of G_n of dimension $n + 1$ and hence that $G_n \;=\; \alpha \cdot G_{n-a} \;+\; \beta \cdot G_{n-b}$. Let $g_a := \alpha$ and $h_b := \beta$. Since

$f_{r+1} \in G_{r+1} = \alpha \cdot G_{r+1-a} + \beta \cdot G_{r+1-b} = g_a \cdot G_{b+1} + h_b \cdot G_{a+1}$, there exist forms $h_{b+1} \in G_{b+1}$ and $g_{a+1} \in G_{a+1}$ such that $f_{r+1} = g_a \cdot h_{b+1} + h_b \cdot g_{a+1}$. Since $G_{r+2} = g_a \cdot G_{b+2} + h_b \cdot G_{a+2}$, there exist forms $h_{b+2} \in G_{b+2}$ and $g_{a+2} \in G_{a+2}$ such that $f_{r+2} - g_{a+1} \cdot h_{b+1} = g_a \cdot h_{b+2} + h_b \cdot g_{a+2}$. Proceeding by induction, assume for a positive integer s that there exist forms $g_a, g_{a+1}, \ldots, g_{a+s}$ and $h_b, h_{b+1}, \ldots, h_{b+s}$ such that the power series $f - (g_a + \cdots + g_{a+s})(h_b + \cdots + h_{b+s})$ has order greater than or equal to $r + s + 1$. Using that

$$G_{r+s+1} = g_a \cdot G_{b+s+1} + h_b \cdot G_{a+s+1},$$

deduce the existence of forms $g_{a+s+1} \in G_{a+s+1}$ and $h_{b+s+1} \in G_{b+s+1}$ such that the power series $f - (g_a + \cdots + g_{a+s+1})(h_b + \cdots + h_{b+s+1})$ has order greater than or equal to $r + s + 2$.

(6) Let k be a field of characteristic zero. Prove that both

$$xy + z^3 \quad \text{and} \quad xyz + x^4 + y^4 + z^4$$

are irreducible in the formal power series ring $k[[x, y, z]]$. Thus there does not appear to be any natural generalization to the case of three variables of the result in the previous exercise.

CHAPTER 5

The Inclusion Construction

We discuss a technique that yields the examples of Chapter 4 and also leads to more examples. This technique, Inclusion Construction 5.3, is a version of Basic Construction Equation 1.3 from Chapter 1. As defined in Section 5.1, Construction 5.3 gives an "Intersection Domain" $A := R^* \cap L$, where R^* is an ideal-adic completion of an integral domain R and L is a subfield of the total quotient ring of R^* that contains the field of fractions of R.

The approximation methods in Section 5.2 of this chapter yield a subring B of the constructed domain A of Inclusion Construction 5.3. This subring B is useful for describing A. We present the "Approximation Domain" B as a directed union of localized polynomial rings over R.

Section 5.3 contains basic properties of Inclusion Construction 5.3. For example, Construction Properties Theorem 5.14 states that each of the domains A and B have ideal-adic completion R^* with the setting and hypotheses of Setting 5.1. By Theorem 5.17, under certain circumstances, if R is a UFD, then B is also a UFD.

5.1. The Inclusion Construction and a picture

We establish the following setting for Inclusion Construction 5.3:

SETTING 5.1. Let R be an integral domain with field of fractions K and let $z \in R$ be a nonzero nonunit. Assume that

- R is separated in the (z)-adic topology, that is, $\bigcap_{n\in\mathbb{N}} z^n R = (0)$,
- the (z)-adic completion R^* of R is a Noetherian ring, and
- z is a regular element of R^*.

In many of our applications, the ring R is a Noetherian integral domain. Often the ring R is a polynomial ring in one or more variables over a field.

REMARKS 5.2. (1) If z is a nonzero nonunit of a Noetherian integral domain R, then the three conditions of Setting 5.1 hold by Krull's Theorem 2.16.2, by Remarks 3.2, parts 5 and 2, and by Remark 2.31.6.

(2) Moreover, if R is Noetherian, Remark 3.3 implies that R^* has the form

$$R^* \;=\; \frac{R[[y]]}{(y-z)R[[y]]},$$

where y is an indeterminate over R. It is natural to ask for conditions that imply R^* is an integral domain, or equivalently, that imply $(y-z)R[[y]]$ is a prime ideal. The element $y - z$ obviously generates a prime ideal of the polynomial ring $R[y]$. Our assumption that z is a nonunit of R implies that $(y-z)R[[y]]$ is a proper ideal. We consider in Exercise 1 of this chapter examples where $(y - z)R[[y]]$ is a prime ideal and examples where it is not a prime ideal.

With R, z and R^* as in Setting 5.1 we describe an "Intersection Domain" A associated with Inclusion Construction 5.3. The integral domain A is transcendental over R and is contained in a power series extension of R.

INCLUSION CONSTRUCTION 5.3. Let $\tau_1, \ldots, \tau_s \in zR^*$ be algebraically independent elements over R and be such that $K(\tau_1, \ldots, \tau_s) \subseteq \mathcal{Q}(R^*)$.[1] Thus every nonzero element of $R[\tau_1, \ldots, \tau_s]$ is a regular element of R^*. We define A to be the *Intersection Domain* $A := K(\tau_1, \ldots, \tau_s) \cap R^*$, inside $\mathcal{Q}(R^*)$. Thus A is a subring of R^* and is a transcendental extension of R.

Diagram 5.3 below shows how A is situated.

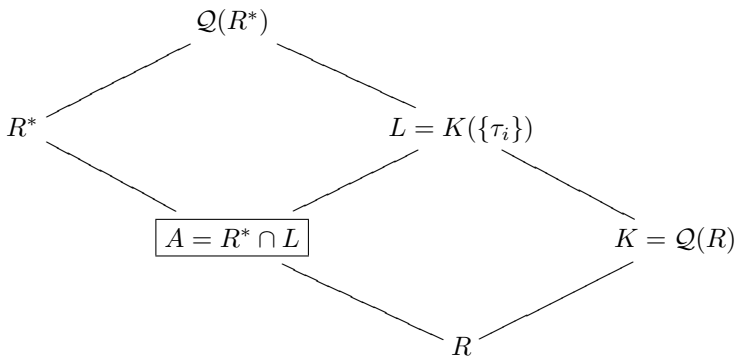


Diagram 5.3.1. $A := L \cap R^*$

The first difficulty we face with Construction 5.3 is identifying precisely what we have constructed—because, while the form of the example as an intersection as given in Construction 5.3 is wonderfully concise, sometimes it is difficult to fathom. For this reason, we construct in Section 5.2 an "Approximation Domain" B that is useful for describing A.

5.2. Approximations for the Inclusion Construction

In this section we give an explicit description of the Approximation Domain B for Inclusion Construction 5.3. We use the last parts, the *endpieces*, of the power series $\tau_1, \ldots, \tau_s$. First we describe the endpieces for a general element γ of R^*.

ENDPIECE NOTATION 5.4. Let R, z and R^* be as in Setting 5.1. Each $\gamma \in zR^*$ has an expansion as a power series in z over R,

$$\gamma \ := \ \sum_{i=1}^{\infty} c_i z^i, \quad \text{where} \quad c_i \in R.$$

[1]Since we are interested in the polynomial ring $R[\tau_1, \ldots, \tau_s]$, there is no loss of generality in the assumption that the $\tau_i \in zR^*$ rather than $\tau_i \in R^*$. If we add elements of R to the τ_i, then the rings $R[\tau_1, \ldots, \tau_s]$ and $K(\tau_1, \ldots, \tau_s)$ are unchanged.

For each nonnegative integer n we define the n^{th} *endpiece* γ_n of γ with respect to this expansion:

$$\gamma_n \ := \ \sum_{i=n+1}^{\infty} c_i z^{i-n}. \tag{5.4.1}$$

It follows that, for each nonnegative integer n, we have a basic relation that we often use. For easy reference we call it "Endpiece Recursion Relation 5.5".

Endpiece Recursion Relation 5.5. With R, z and R^* as in Setting 5.1, and $\gamma = \sum_{n=1}^{\infty} c_n z^n$, where each $c_n \in R$, we have the following *Endpiece Recursion Relation* for γ:

$$\gamma_n \ = \ c_{n+1}z \ + \ z\gamma_{n+1}. \tag{5.4.2}$$

We have the following additional Endpiece Recursion Relations:

$$\begin{aligned}
&\gamma_n \ = \ c_{n+1}z \ + \ z\gamma_{n+1} \ ; \quad \gamma_{n+1} \ = \ c_{n+2}z \ + \ z\gamma_{n+2} \ ; \\
&\gamma_n \ = \ c_{n+2}z + c_{n+1}z^2 \ + \ z^2\gamma_{n+2} \ ; \ \cdots . \\
&\gamma_n \ = \ c_{n+r}z \ + + \ \cdots \ + \ c_{n+1}z^r \ + z^r\gamma_{n+r} \ \implies \\
&\gamma_n \ = \ az \ + z^r\gamma_{n+r} \qquad \text{and} \qquad \gamma_{n+1} \ = \ bz \ + z^{r-1}\gamma_{n+r} \ ,
\end{aligned} \tag{5.5.1}$$

for some $a \in (c_{n+1}, \ldots, c_{n+r})R$ and $b \in (c_{n+2}, \ldots, c_{n+r})R$.

We now assume that elements $\tau_1, \ldots \tau_s \in zR^*$ are algebraically independent over the field of fractions $\mathcal{Q}(R)$ of R and have the property that every nonzero element of the polynomial ring $R[\tau_1, \ldots, \tau_s]$ is a regular element of R^*. Thus $\mathcal{Q}(R[\tau_1, \ldots, \tau_s])$ is contained in the total quotient ring $\mathcal{Q}(R^*)$. As in Inclusion Construction 5.3, we define the *Intersection Domain* $A := R^* \cap \mathcal{Q}(R[\tau_1, \ldots, \tau_s])$ inside $\mathcal{Q}(R^*)$. We set

$$U_0 \ := \ R[\tau_1, \ldots, \tau_s] \ \subseteq \ A \ := \ R^* \cap \mathcal{Q}(R[\tau_1, \ldots, \tau_s]).$$

Thus U_0 is a polynomial ring in s variables over R. Each $\tau_i \in zR^*$ has a representation $\tau_i := \sum_{j=1}^{\infty} r_{ij} z^j$, where the $r_{ij} \in R$. For each positive integer n, we associate with this representation of τ_i the n^{th} endpiece,

$$\tau_{in} \ := \ \sum_{j=n+1}^{\infty} r_{ij} z^{j-n}. \tag{5.4.3}$$

We define

$$U_n \ := \ R[\tau_{1n}, \ldots, \tau_{sn}] \quad \text{and} \quad B_n \ := \ (1 + zU_n)^{-1}U_n \tag{5.4.4}$$

For each $n \in \mathbb{N}$, the ring U_n is a polynomial ring in s variables over R, and z is in every maximal ideal of B_n, so $z \in \mathcal{J}(B_n)$, the Jacobson radical of B_n; see Section 2.1. Using Endpiece Recursion Relation 5.5, we have a birational inclusion of polynomial rings $U_n \subset U_{n+1}$, for each $n \in \mathbb{N}$. We also have $U_{n+1} \subset U_n[1/z]$. By Remark 3.2.1, the element z is in $\mathcal{J}(R^*)$. Hence the localization B_n of U_n is

also a subring of A and $B_n \subset B_{n+1}$. We define rings U and B associated to the construction:

$$U := \bigcup_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty} R[\tau_{1n}, \ldots, \tau_{sn}] \quad \text{and} \quad B := \bigcup_{n=1}^{\infty} B_n. \tag{5.4.5}$$

REMARKS 5.6. (1) The ring U is a directed union of polynomial rings over R, and the ring B, the *Approximation Domain* for the construction, is a localization of U. We have

$$B = (1 + zU)^{-1}U \quad \text{and} \quad B \subseteq A := R^* \cap \mathcal{Q}(R[\tau_1, \ldots, \tau_s]). \tag{5.4.6}$$

Thus z is in the Jacobson radical of B.

(2) By Endpiece Recursion Relation 5.5 and Definitions 5.4.4 and 5.4.5, we have

$$R[\tau_1, \ldots, \tau_n][1/z] = U_0[1/z] = U_1[1/z] = \cdots = U[1/z]. \tag{5.4.7}$$

DEFINITION 5.7. With Setting 5.1, the ring $A = R^* \cap \mathcal{Q}(R[\tau_1, \ldots, \tau_s])$ is called the *Intersection Domain associated to* $\tau_1, \ldots, \tau_s$. The ring $B = \bigcup_{n=1}^{\infty} B_n$ is called the *Approximation Domain associated to* $\tau_1, \ldots, \tau_s$.

REMARK 5.8. With the notation and setting of (5.4), the representation

$$\tau_i = \sum_{j=1}^{\infty} r_{ij} z^j$$

of τ_i as a power series in z with coefficients in R is not unique. Indeed, since $z \in R$, it is always possible to modify the coefficients r_{ij} in this representation. It follows that the endpiece τ_{in} is also not unique. However, as we observe in Proposition 5.9 the rings U and U_n are uniquely determined by the τ_i.

PROPOSITION 5.9. *Assume the notation and setting of (5.4). Then the ring U and the rings U_n are independent of the representation of the τ_i as power series in z with coefficients in R. Hence also the ring B and the rings B_n are independent of the representation of the τ_i as power series in z with coefficients in R.*

PROOF. For $1 \leq i \leq s$, assume that τ_i and $\omega_i = \tau_i$ have representations

$$\tau_i := \sum_{j=1}^{\infty} a_{ij} z^j \qquad \text{and} \qquad \omega_i := \sum_{j=1}^{\infty} b_{ij} z^j,$$

where each $a_{ij}, b_{ij} \in R$. We define the n^{th}-endpieces τ_{in} and ω_{in} as in (5.4):

$$\tau_{in} = \sum_{j=n+1}^{\infty} a_{ij} z^{j-n} \qquad \text{and} \qquad \omega_{in} = \sum_{j=n+1}^{\infty} b_{ij} z^{j-n}.$$

Then we have

$$\tau_i = \sum_{j=1}^{\infty} a_{ij} z^j = \sum_{j=1}^{n} a_{ij} z^j + z^n \tau_{in} = \sum_{j=1}^{\infty} b_{ij} z^j = \sum_{j=1}^{n} b_{ij} z^j + z^n \omega_{in} = \omega_i.$$

Therefore, for $1 \leq i \leq s$ and each positive integer n,

$$z^n \tau_{in} - z^n \omega_{in} = \sum_{j=1}^{n} b_{ij} z^j - \sum_{j=1}^{n} a_{ij} z^j, \quad \text{and so} \quad \tau_{in} - \omega_{in} = \frac{\sum_{j=1}^{n} (b_{ij} - a_{ij}) z^j}{z^n}.$$

Thus $\sum_{j=1}^{n}(b_{ij}-a_{ij})z^j \in R$ is divisible by z^n in R^* . Since z^nR is closed in the (z)-adic topology on R, we have $z^nR = R \cap z^nR^*$. It follows that z^n divides the sum $\sum_{j=1}^{n}(b_{ij}-a_{ij})z^j$ in R. Therefore $\tau_{in}-\omega_{in} \in R$. Thus the rings U_n and $U = \bigcup_{n=1}^{\infty} U_n$ are independent of the representation of the τ_i. Since $B_n = (1+zU_n)^{-1}U_n$ and $B = \bigcup_{n=1}^{\infty} B_n$, the rings B_n and the ring B are also independent of the representation of the τ_i. □

It is important to identify conditions in order that the Approximation Domain B equals the Intersection Domain A of Inclusion Construction 5.3. In Definition 5.10, we introduce the term "limit-intersecting" for this situation.

Definition 5.10. Using the notation in (5.4.3) and (5.4.5), we say Inclusion Construction 5.3 is *limit-intersecting* over R with respect to the τ_i if $B = A$. In this case, we refer to the sequence of elements $\tau_1, \dots \tau_s \in zR^*$ as *limit-intersecting* over R, or briefly, as *limit-intersecting* for A.

We observe that with the ring $R = k[x]$, the elements $\tau_1, \dots, \tau_s$ are limit-intersecting for the DVR of Constructions 4.20 and 4.21, since these Constructions do yield the same thing. With the ring $R = k[x, y_1, \dots, y_m]_{(x,y_1,\dots,y_m)}$, the elements $\tau_1, \dots, \tau_s$ are limit-intersecting for Local Prototype Example 4.25.

Remark 5.11. The limit-intersecting property depends on the choice of the elements $\tau_1, \dots, \tau_s$ in the completion we use. For example, if R is the polynomial ring $\mathbb{Q}[x, y]$, then the (x)-adic completion $R^* = \mathbb{Q}[y][[x]]$. Let $s = 1$, and let $\tau_1 = \tau := e^x - 1 \in xR^*$. Then τ is algebraically independent over $\mathbb{Q}(x, y)$. Let $U_0 = R[\tau]$. Local Prototype Example 4.25 shows that τ is limit-intersecting. On the other hand, the element $y\tau$ is not limit-intersecting. If $U_0' := R[y\tau,]$, then $\mathcal{Q}(U_0) = \mathcal{Q}(U_0')$ and the Intersection Domain

$$A \;=\; \mathcal{Q}(U_0) \;\cap\; R^* \;=\; \mathcal{Q}(U_0') \;\cap\; R^*$$

is the same for τ and $y\tau$. However the Approximation Domain B' associated to U_0' does not contain τ. Indeed, $\tau \notin R[y\tau][1/x]$. Hence B' is properly contained in the Approximation Domain B associated to U_0. We have $B' \subsetneq B = A$ and the limit-intersecting property fails for the element $y\tau$.

5.3. Basic properties of the constructed domains

In order to prove basic properties of the integral domains A and B of Construction 5.3 and Equation 5.4.5, we use the following two lemmas.

Lemma 5.12. *Let S be a subring of a ring T and let $z \in S$ be a regular element of T. The following conditions are equivalent.*

1. *Both* (i) $zS = zT \cap S$ *and* (ii) $S/zS = T/zT$ *hold.*
2. *For each positive integer n we have:* $z^nS \;=\; z^nT \cap S, \quad S/z^nS \;=\; T/z^nT \quad \text{and} \quad T \;=\; S + z^nT.$
3. *The rings S and T have the same (z)-adic completion.*
4. *Both* (i) $S = S[1/z] \cap T$ *and* (ii) $T[1/z] = S[1/z] + T$ *hold.*

Proof. To see that item 1 implies item 2, observe that

$$z^nT \cap S = z^nT \cap zS = z(z^{n-1}T \cap S),$$

so the equality $z^{n-1}S = z^{n-1}T \cap S$ implies the equality $z^nS = z^nT \cap S$. Moreover $S/zS = T/zT$ implies $T = S + zT = S + z(S + zT) = \cdots = S + z^nT$, so $S/z^nS = T/z^nT$ for every $n \in \mathbb{N}$. Therefore (1) implies (2).

It is clear that item 2 is equivalent to item 3.

To see that item 2 implies (4i), let $s/z^n \in S[1/z] \cap T$ with $s \in S$ and $n \geq 0$. Item 2 implies that $s \in z^nT \cap S = z^nS$ and therefore $s/z^n \in S$. To see (4ii), let $\frac{t}{z^n} \in T[1/z]$ with $t \in T$ and $n \geq 0$. Item 2 implies that $t = s + z^nt_1$ for some $s \in S$ and $t_1 \in T$. Therefore $\frac{t}{z^n} = \frac{s}{x^n} + t_1$. Thus (2) implies (4).

It remains to show that item 4 implies item 1. To see that (4) implies (1i), let $t \in T$ and $s \in S$ be such that $zt = s$. Then $t = s/z \in S[1/z] \cap T = S$, by (4i). Thus $zt \in zS$. To see that (4) implies (1ii), let $t \in T$. Then $\frac{t}{z} = \frac{s}{z^n} + t'$, for some $n \in \mathbb{N}$, $s \in S$ and $t' \in T$ by (4ii). Thus $t = \frac{s}{z^{n-1}} + t'z$. Hence by (4ii)

$$t - t'z = \frac{s}{z^{n-1}} \in S[1/z] \cap T = S. \qquad \square$$

The following lemma is a generalization of Proposition 4.22 of Chapter 4.

LEMMA 5.13. *Assume R, z and R^* are as in Setting 5.1, the elements $\tau_1, \ldots, \tau_s$ of zR^* are algebraically independent over K and the rings U_n, U, B_n, B, and A are as in Construction 5.3 and Equations 5.4.4 and 5.4.5.*

(1) *For every $\eta \in U$ and every $t \in \mathbb{N}$, there exist elements $g_t \in R$ and $\delta_t \in U$ such that $\eta = g_t + z^t\delta_t$.*
(2) *For each $t \in \mathbb{N}$, $z^tR^* \cap U = z^tU$.*

PROOF. Since R^* is the (z)-adic completion of R, we have $z^nR^* \cap R = z^nR$. For item 1, suppose that $\eta \in U_n$ for some $n \in \mathbb{N}$. Then η can be written as:

$$\eta = \sum_{(j)\in\mathbb{N}^s} r_{(j)}\tau_{1n}^{j_1} \ldots \tau_{sn}^{j_n},$$

where $r_{(j)} \in R$, each (j) represents a tuple $(j_1, \ldots, j_s)$, and only finitely many of the $r_{(j)}$ are different from zero. By the Endpiece Recursion Relations 5.5.1, for τ_{jn} we have for each $j \in \{1, \ldots, s\}$:

$$\tau_{jn} = z^t\tau_{jn+t} + h_j$$

where $h_j \in R$. Using these expressions for the τ_{jn}, we obtain:

$$\eta = \sum_{(j)\in\mathbb{N}^s} r_{(j)}(z^t\tau_{1n+t} + h_1)^{j_1} \ldots (z^t\tau_{sn+t} + h_s)^{j_n} = g_t + z^t\delta_t$$

where $g_t \in R$ and $\delta_t \in U_{n+t}$.

For item 2, assume that $\eta \in z^tR^* \cap U$. Then $\eta = g_t + z^t\delta_t$, where $g_t \in R$ and $\delta_t \in U$. Therefore $g_t \in z^tR^* \cap R$. Since $z^tR^* \cap R = z^tR$, we have $\eta \in z^tU$. $\square$

We record in Construction Properties Theorem 5.14 several basic properties of the integral domains associated with Inclusion Construction 5.3.

CONSTRUCTION PROPERTIES THEOREM 5.14. (Inclusion Version) *Assume the notation of Setting 5.1. Thus R is an integral domain with field of fractions K, and $z \in R$ is a nonzero nonunit such that $\bigcap_{n\in\mathbb{N}} z^nR = (0)$, the (z)-adic completion R^* of R is a Noetherian ring, and z is a regular element of R^*. Let $\underline{\tau} = \{\tau_1, \ldots, \tau_s\}$ be a set of elements of zR^* that are algebraically independent over K; thus $R[\underline{\tau}]$ is a polynomial ring in s variables over R. As in Inclusion Construction 5.3, define*

$A = A_{inc} := K(\underline{\tau}) \cap R^*$. Let U_n, B_n, B and U be defined as in Equations 5.4.4 and 5.4.5. Then:

(1) $z^n R^* \cap A = z^n A, \quad z^n R^* \cap B = z^n B \quad$ and $\quad z^n R^* \cap U = z^n U$, *for each* $n \in N$.
(2) $R/z^n R = U/z^n U = B/z^n B = A/z^n A = R^*/z^n R^*$, *for each* $n \in N$.
(3) *The (z)-adic completions of the rings U, B and A are all equal to R^*, that is, $R^* = U^* = B^* = A^*$.*
(4) $R[\underline{\tau}][1/z] = U[1/z], \quad U = R[\underline{\tau}][1/z] \cap B = R[\underline{\tau}]1/z] \cap A,$ *and* $B[1/z]$ *is a localization of* $R[\underline{\tau}]$. *The integral domains $R[\underline{\tau}], U, B$ and A all have the same field of fractions, namely $K(\underline{\tau})$.*
(5) *The definitions in Equation 5.4.5 of B and U are independent of the representations given in Notation 5.4 for the τ_i as power series in R^*.*
(6) *If R^* is local with maximal ideal $\mathbf{m}^*$, then $\mathbf{m} := \mathbf{m}^* \cap R$ is a maximal ideal of R and B is local with maximal ideal $\mathbf{m}^* \cap B$. We also have*

$$B = (1 + zU)^{-1}U = \bigcup_{n=1}^{\infty} (U_n)_{(\mathbf{m}, \tau_{1n}, \ldots, \tau_{sn})U_n},$$

where $U_n = R[\tau_{1n}, \ldots, \tau_{sn}]$, as defined in Equation 5.4.4, and the τ_{in} are the n^{th} endpoints of the τ_i, using Endpiece Notation 5.4.

PROOF. For item 1, $z^n R^* \cap U = z^n U$ by Lemma 5.13, and $z^n R^* \cap A = z^n A$ by Exercise 2 at the end of this chapter. If $\eta \in z^n R^* \cap B$, then $\eta = \eta_0 \epsilon$, where $\eta_0 \in U$ and ϵ a unit in B. Since z is in the Jacobson radical of R^*, ϵ is also a unit in R^* and therefore $\eta_0 \in z^n R^* \cap U = z^n U$. Thus $\eta \in z^n B$.

To prove item 2, observe that from item 1, we have embeddings:

$$R/z^n R \hookrightarrow U/z^n U \hookrightarrow B/z^n B \hookrightarrow A/z^n A \hookrightarrow R^*/z^n R^*.$$

Since $R/z^n R \hookrightarrow R^*/z^n R^*$ is an isomorphism, all equalities follow.

Item 3 follows from item 2.

To prove item 4, we have $U[1/z] = R[\underline{\tau}][1/z]$ by Remark 5.6.2. By item 3 and Lemma 5.12.4, we have $U = U[1/z] \cap B = R[\underline{\tau}][1/z] \cap B = R[\underline{\tau}][1/z] \cap A$. By Remark 5.6.1, B is a localization of U. Since $U[1/z] = R[\underline{\tau}][1/z]$, it follows that $B[1/z]$ is a localization of $R[\underline{\tau}]$.

Item 5 is Proposition 5.9.

For item 6, notice that $z \in \mathbf{m}$. By Remark 5.6.1, $z \in \mathcal{J}(B)$, that is, z is in every maximal ideal of B. By item 2, we have $B/zB = R^*/zR^*$. Since R^* is local with maximal ideal $\mathbf{m}^*$, it follows that B is local with maximal ideal $\mathbf{m}^* \cap B$. The first equality of the displayed equation of item 6 is by Remark 5.6.1.

We show that B is also the directed union of the localized polynomial rings $C_n := (U_n)_{P_n}$, where $P_n := (\mathbf{m}, \tau_{1n}, \ldots, \tau_{sn})U_n$ and $U_n = R[\tau_{1n}, \ldots, \tau_{sn}]$. Note that P_n is a maximal ideal of U_n with $\mathbf{m}^* \cap U_n = P_n$, We have $C_n \subseteq C_{n+1}$. Also $P_n \cap (1 + zU_n) = \emptyset$ implies that $B_n \subseteq C_n$. We show that $C_n \subseteq B$: Let $\frac{a}{d} \in C_n$, where $a \in U_n$ and $d \in U_n \setminus P_n$. Then $a \in B$ and $d \in B \setminus (\mathbf{m}^* \cap B)$. Since B is local with maximal ideal $\mathbf{m}^* \cap B$, d is a unit in B. Hence $a/d \in B$. This completes the proof of Theorem 5.14. □

REMARK 5.15. Let R, z and R^* be as in Setting 5.1, and let $\tau_1, \ldots, \tau_s \in zR^*$ be algebraically independent elements over R as in Construction 5.3. In items 1 and 3 below we apply part 6 of Construction Properties Theorem 5.14 to the case

that the base ring R is a localized polynomial ring over a field and the completion is taken with respect to one of the variables. For one variable, the idea is quite simple, as shown in items 1 and 2 below.

(1) In the special case where $R = k[x]_{(x)}$, the (x)-adic completion of R is $R^* = k[[x]]$ and $U_n = k[x]_{(x)}[\tau_{1n}, \ldots, \tau_{sn}]$. Then $B = \bigcup k[x]_{(x)}[\tau_{1n}, \ldots, \tau_{sn}]_{P_n}$, where $P_n := (x, \tau_{1n}, \ldots, \tau_{sn})k[x]_{(x)}[\tau_{1n}, \ldots, \tau_{sn}]$, by Theorem 5.14.6. It follows that also $B = \bigcup k[x, \tau_{1n}, \ldots, \tau_{sn}]_{(x,\tau_{1n},\ldots,\tau_{sn})}$.

(2) If $R = k[x]$, then $R^* = k[[x]]$ is the (x)-adic completion of R and $U_n = k[x, \tau_{1n}, \ldots, \tau_{sn}]$. By part 6 of Theorem 5.14, $B = \bigcup k[x, \tau_{1n}, \ldots, \tau_{sn}]_{(x,\tau_{1n},\ldots,\tau_{sn})}$. That is, the ring B is the same for $R = k[x]$ as for $R = k[x]_{(x)}$.

(3) Let R be the localized polynomial ring $k[x, y_1, \ldots, y_r]_{(x,y_1,\ldots,y_r)}$ over a field k with variables $x, y_1, \ldots, y_r$, and let $\mathbf{m} := (x, y_1, \ldots, y_r)R$. Let $\tau_1, \ldots, \tau_s$ be elements of $xk[[x]]$ that are algebraically independent over $k(x)$. Then $R^* = k[y_1, \ldots, y_r]_{(y_1,\ldots,y_r)}[[x]]$ is the (x)-adic completion of R. By part 6 of Construction Properties Theorem 5.14.6,

$$B \;=\; \bigcup_{n=0}^{\infty} R[\tau_{1n}, \ldots, \tau_{sn}]_{(\mathbf{m},\tau_{1n},\ldots,\tau_{sn})}.$$

Since R is the localization of $k[x, y_1, \ldots, y_r]$ at the maximal ideal generated by $x, y_1, \ldots, y_r$, we have

$$R[\tau_{1n}, \ldots, \tau_{sn}]_{(\mathbf{m},\tau_{1n},\ldots,\tau_{sn})} = k[x, y_1, \ldots, y_r, \tau_{1n}, \ldots, \tau_{sn}]_{(x,y_1,\ldots,y_r,\tau_{1n},\ldots,\tau_{sn})}.$$

By Proposition 4.26, the ring B is then the ring D of Local Prototype Example 4.25.

Proposition 5.16 concerns the extension to R^* of a prime ideal of either A or B that does not contain z, and provides information about the maps from $\operatorname{Spec} R^*$ to $\operatorname{Spec} A$ and to $\operatorname{Spec} B$. We use Proposition 5.16 in Chapters 12 and 15.

PROPOSITION 5.16. *With the notation of Construction Properties Theorem 5.14:*

(1) *z is in the Jacobson radical of each of the rings B, A and R^*. Thus if $P \in \operatorname{Spec} B$ or $P \in \operatorname{Spec} A$, then $PR^* \neq R^*$.*
(2) *Let q be a prime ideal of R. Then*
   (a) *qU is a prime ideal in U.*
   (b) *Either $qB = B$ or qB is a prime ideal of B.*
   (c) *If $qB \neq B$, then $qB \cap U = qU$ and $U_{qU} = B_{qB}$.*
   (d) *If $z \notin q$, then $qU \cap U_n = qU_n$ and $U_{qU} = (U_n)_{qU_n}$.*
   (e) *If $z \notin q$ and $qB \neq B$, then $qB \cap B_n = qB_n$ and*

$$(U_n)_{qU_n} \;=\; U_{qU} \;=\; B_{qB} \;=\; (B_n)_{qB_n}\ .$$

(3) *Let I be an ideal of B or of A and let $t \in \mathbb{N}$. Then $z^t \in IR^* \iff z^t \in I$.*
(4) *Let $P \in \operatorname{Spec} B$ or $P \in \operatorname{Spec} A$ with $z \notin P$. Then z is a nonzerodivisor on R^*/PR^*. Thus $z \notin Q$ for each associated prime of R^*/PR^*. Since z is in the Jacobson radical of R^*, it follows that PR^* is contained in a nonmaximal prime ideal of R^*.*
(5) *If R is local, then R^*, A and B are local. Let $\mathbf{m}_R$, $\mathbf{m}_{R^*}, \mathbf{m}_A$ and $\mathbf{m}_B$ denote the maximal ideals of R, R^*, A and B, respectively. In this case*
   (a) *$\mathbf{m}_B = \mathbf{m}_R B$, $\mathbf{m}_A = \mathbf{m}_R A$ and each prime ideal P of B such that $\operatorname{ht}(\mathbf{m}_B/P) = 1$ is contracted from R^*.*

(b) *Let I be an ideal of B. Then IR^* is primary for $\mathbf{m}_{R^*}$ $\iff$ I is primary for $\mathbf{m}_B$. In this case, $IR^* \cap B = I$ and $B/I \cong R^*/IR^*$.*

PROOF. For item 1, since $B_n = (1 + zU_n)^{-1}U_n$, it follows that $1 + zb$ is a unit of B_n for each $b \in B_n$. Therefore z is in the Jacobson radical of B_n for each n and thus z is in the Jacobson radical of B. By Remark 3.2.1, z is in the Jacobson radical of R^*. Hence $1 + az$ is a unit of R^* for every $a \in R^*$. Since $A = R^* \cap \mathcal{Q}(A)$ an element of A is a unit of A if and only if it is a unit of R^*. Thus z is in the Jacobson radical of A.

For item 2, since each U_n is a polynomial ring over R, the ideal qU_n is a prime ideal of U_n and thus $qU = \bigcup_{n=0}^{\infty} qU_n$ is a prime ideal of U. Since B is a localization of U, either $qB = B$, or qB is a prime ideal of B such that $qB \cap U = qU$ and $U_{qU} = B_{qB}$.

For part d of item 2, since $U_n[1/z] = U[1/z]$ and the ideals qU_n and qU are prime ideals in U_n and U that do not contain z, the localizations $(U_n)_{qU_n}$ and U_{qU} are both further localizations of $U[1/z]$. Moreover, they both equal $U[1/z]_{qU[1/z]}$. Thus we have $U_{qU} = (U_n)_{qU_n}$. Since $U_n \subset U$, we also have $qU \cap U_n = qU_n$. Since B is a localization of U, the assertions in part e follow as in the proof of part d.

To see item 3, let I be an ideal of B. The proof for A is identical. We observe that there exist elements $b_1, \ldots, b_s \in I$ such that $IR^* = (b_1, \ldots, b_s)R^*$. If $z^t \in IR^*$, there exist $\alpha_i \in R^*$ such that

$$z^t \;=\; \alpha_1 b_1 + \cdots + \alpha_s b_s.$$

We have $\alpha_i = a_i + z^{t+1}\lambda_i$ for each i, where $a_i \in B$ and $\lambda_i \in R^*$. Thus

$$z^t[1 - z(b_1\lambda_1 + \cdots + b_s\lambda_s)] \;=\; a_1 b_1 + \cdots + a_s b_s \in B \cap z^t B^* = z^t B.$$

Therefore $\gamma := 1 - z(b_1\lambda_1 + \cdots + b_s\lambda_s) \in B$. Thus $z(b_1\lambda_1 + \cdots + b_s\lambda_s) \in B \cap zR^* = zB$, and so $b_1\lambda_1 + \cdots + b_s\lambda_s \in B$. By item 1, the element z is in the Jacobson radical of B. Hence γ is invertible in B. Since $\gamma z^t \in (b_1, \cdots, b_s)B$, it follows that $z^t \in I$. If $z^t \in I$, then $z^t \in IR^*$. This proves item 3.

For item 4, assume that $P \in \operatorname{Spec} B$. The proof for $P \in \operatorname{Spec} A$ is identical. We have that

$$P \cap zB = zP \quad \text{and so} \quad \frac{P}{zP} = \frac{P}{P \cap zB} \cong \frac{P + zB}{zB}$$

By Construction Properties Theorem 5.14.3, B/zB is Noetherian. Hence the B-module P/zP is finitely generated. Let $g_1, \ldots, g_t \in P$ be such that $P = (g_1, \ldots, g_t)B + zP$. Then also $PR^* = (g_1, \ldots, g_t)R^* + zPR^* = (g_1, \ldots, g_t)R^*$, the last equality by Nakayama's Lemma.

Let $\widehat{f} \in R^*$ be such that $z\widehat{f} \in PR^*$. We show that $\widehat{f} \in PR^*$.

Since $\widehat{f} \in R^*$, we have $\widehat{f} := \sum_{i=0}^{\infty} c_i z^i$, where each $c_i \in R$. For each $m > 1$, let $f_m := \sum_{i=0}^{m} c_i z^i$, the first $m + 1$ terms of this expansion of $\widehat{f}$. Then $f_m \in R \subseteq B$ and there exists an element $\widehat{h_1} \in R^*$ so that.

$$\widehat{f} = f_m + z^{m+1}\widehat{h_1}.$$

Since $z\widehat{f} \in PR^*$, we have

$$z\widehat{f} = \widehat{a_1}g_1 + \cdots + \widehat{a_t}g_t,$$

where $\widehat{a_i} \in R^*$. The $\widehat{a_i}$ have power series expansions in z over R, and thus there exist elements $a_{im} \in R$ such that $\widehat{a_i} - a_{im} \in z^{m+1}R^*$. Thus

$$z\widehat{f} = a_{1m}g_1 + \cdots + a_{tm}g_t + z^{m+1}\widehat{h_2},$$

where $\widehat{h_2} \in R^*$, and

$$zf_m = a_{1m}g_1 + \cdots + a_{tm}g_t + z^{m+1}\widehat{h_3},$$

where $\widehat{h_3} = \widehat{h_2} - z\widehat{h_1} \in R^*$. Since the g_i are in B, we have $z^{m+1}\widehat{h_3} \in z^{m+1}R^* \cap B = z^{m+1}B$, the last equality by Construction Properties Theorem 5.14.1. Therefore $\widehat{h_3} \in B$. Rearranging the last displayed equation above, we obtain

$$z(f_m - z^m\widehat{h_3}) = a_{1m}g_1 + \cdots + a_{tm}g_t \in P.$$

Since $z \notin P$, we have $f_m - z^m\widehat{h_3} \in P$. It follows that $\widehat{f} \in P + z^m R^* \subseteq PR^* + z^m R^*$, for each $m > 1$. Hence we have that $\widehat{f} \in PR^*$, as desired.

For item 5, if R is local, then B is local, A is local, $\mathbf{m}_B = \mathbf{m}_R B$ and $\mathbf{m}_A = \mathbf{m}_R A$ since $R/zR = B/zB = A/zA = R^*/zR^*$ and z is in the Jacobson radical of B and of A. If $z \notin P$, then item 4 implies that no power of z is in PR^*. Hence PR^* is contained in a prime ideal Q of R^* that does not meet the multiplicatively closed set $\{z^n\}_{n=1}^{\infty}$. Thus $P \subseteq Q \cap B \subsetneq \mathbf{m}_B$. Since $\operatorname{ht}(\mathbf{m}_B/P) = 1$, we have $P = Q \cap B$, so P is contracted from R^*. If $z \in P$, then $B/zB = R^*/zR^*$ implies that PR^* is a prime ideal of R^* and $P = PR^* \cap B$.

For the second part of item 5, let I be an ideal of B. By item 3, for each $t \in \mathbb{N}$, we have $z^t \in IR^* \iff z^t \in I$. If either IR^* is $\mathbf{m}_{R^*}$-primary or I is $\mathbf{m}_B$-primary, then $z^t \in I$ for some $t \in \mathbb{N}$. By Theorem 5.14.3, $B/z^tB = R^*/z^tR^*$. Hence the $\mathbf{m}_B$-primary ideals containing z^t are in one-to-one inclusion preserving correspondence with the $\mathbf{m}_{R^*}$-primary ideals that contain z^t. This completes the proof of item 5. □

In many of the examples constructed in this book, the ring R is a polynomial ring (or a localized polynomial ring) in finitely many variables over a field; such rings are UFDs. We observe in Theorem 5.17 that the constructed ring B is a UFD if R is a UFD and z is a prime element.

THEOREM 5.17. *With the notation of Construction Properties Theorem 5.14:*

(1) *If R is a UFD and z is a prime element of R, then zU and zB are principal prime ideals of U and B respectively, and U and B are UFDs.*
(2) *If R is a regular Noetherian UFD, then $B[1/z]$ is also a regular Noetherian UFD.*

PROOF. By Proposition 5.16.2, parts a and b, zU and zB are prime ideals. Since R is a UFD and $R[\underline{\tau}]$ is a polynomial ring over R, we see that $R[\underline{\tau}]$ is a UFD. By Theorem 5.14.2, the rings $U[1/z]$ and $B[1/z]$ are localizations of $R[\underline{\tau}]$ and thus are UFDs; moreover $B[1/z]$ is regular if R is regular. It suffices to prove U is a UFD for the remaining assertion, since B is a localization of U. By Theorem 5.14.4, the (z)-adic completion of U is R^*. By Proposition 5.16.1, z is in the Jacobson radical of R^*. Since R^* is Noetherian, $\bigcap_{n=1}^{\infty} z^n R^* = (0)$. Thus $\bigcap_{n=1}^{\infty} z^n U = (0)$. It follows that U_{zB} is a DVR [**117**, (31.5)].

By Fact 2.22, we have $U = U[1/z] \cap U_{zB}$. Therefore U is a Krull domain. Since $U[1/z]$ is a UFD and U is a Krull domain, Theorem 2.21 implies that U is a UFD. Then also B is a UFD and the proof is complete. □

Exercises

(1) Let z be a nonzero nonunit of a Noetherian integral domain R, let y be an indeterminate, and let $R^* = \frac{R[[y]]}{(z-y)R[[y]]}$ be the (z)-adic completion of R.

(i) If $z = ab$, where $a, b \in R$ are nonunits such that $aR + bR = R$, prove that there exists a factorization

$$z - y = (a + a_1 y + \cdots) \cdot (b + b_1 y + \cdots) = (\sum_{i=0}^{\infty} a_i y^i) \cdot (\sum_{i=0}^{\infty} b_i y^i),$$

where the $a_i, b_i \in R$, $a_0 = a$ and $b_0 = b$.

(ii) If R is a principal ideal domain (PID), prove that R^* is an integral domain if and only if zR has prime radical.

(2) Let A be an integral domain with field of fractions F. Let C be an extension ring of A such that every nonzero element of A is a regular element of C. If $A = C \cap F$, prove that $zA = zC \cap F$, for every $z \in A$.

(3) Prove item 3 of Remark 5.15, that is, with R a polynomial ring $k[x]$ over a field k and $R^* = k[[x]]$ the (x)-adic completion of R, show that with the notation of Construction 5.3 we have $B = \bigcup C_n$, where $C_n = (U_n)_{P_n}$ and $P_n := (x, \tau_{1n}, \ldots, \tau_{sn})U_n$. Thus B is a DVR that is the directed union of a birational family of localized polynomial rings in $n + 1$ indeterminates.

CHAPTER 6

Flatness and the Noetherian property

In this chapter we prove that the Noetherian property for the ring B of Inclusion Construction 5.3 is equivalent to the flatness of a certain map.

In particular, we prove the following theorem.

THEOREM 6.1. *Let R be a Noetherian integral domain with field of fractions K. Let z be a nonzero nonunit of R and let R^* denote the (z)-adic completion of R. Let $\tau_1, \ldots, \tau_s \in zR^*$ be algebraically independent elements over K such that the field $K(\tau_1, \ldots, \tau_s)$ is a subring of the total quotient ring of R^*. As in Equations 5.4.4, 5.4.5 and 5.4.6, define*

$$U_n := R[\tau_{1n}, \ldots, \tau_{sn}], \quad U := \bigcup_{n=1}^{\infty} U_n, \quad \text{and} \quad B := (1 + zU)^{-1}U.$$

Then B is Noetherian if and only the extension $R[\tau_1, \ldots \tau_s] \hookrightarrow R^*[1/z]$ is flat.

Theorem 6.1 is implied by Noetherian Flatness Theorem 6.3, proved in Section 6.1. In Section 6.1, we also prove a crucial lemma relating flatness and the Noetherian property.

Motivated by Noetherian Flatness Theorem 6.3 (inclusion Version), we study the embedding $U_0 \to R^*[1/z]$ in Section 6.2, and we seek necessary and sufficient conditions that this embedding be flat. For this, we use the idea of the "Insider Construction", which combines Local Prototype 4.27 with Inclusion Construction 5.3. In Sections 6.3 and 6.4, we apply Noetherian Flatness Theorem 6.3 and the Insider Construction to show that Nagata's Example 4.14 and Christel's Example 4.16 are Noetherian.

6.1. The Noetherian Flatness Theorem

We use Lemma 6.2 in the proof of Noetherian Flatness Theorem 6.3. We thank Roger Wiegand for observing Lemma 6.2 and its proof.

LEMMA 6.2. *Let S be a subring of a ring T and let $z \in S$ be a regular element of T. Assume that $zS = zT \cap S$ and $S/zS = T/zT$. Then*

1. *$T[1/z]$ is flat over $S \iff T$ is flat over S.*
2. *If T is flat over S, then $D := (1 + zS)^{-1}T$ is faithfully flat over $C := (1 + zS)^{-1}S$.*
3. *If T is Noetherian and T is flat over S, then $C = (1 + zS)^{-1}S$ is Noetherian.*
4. *If T and $S[1/z]$ are both Noetherian and T is flat over S, then S is Noetherian.*

PROOF. For item 1, if T is flat over S, then by transitivity of flatness, Remark 2.31.13, the ring $T[1/z]$ is flat over S. For the converse, Lemma 5.12 implies that $S = S[1/z] \cap T$ and $T[1/z] = S[1/z] + T$. Thus we have an exact sequence

$$0 \to S = S[1/z] \cap T \xrightarrow{\alpha} S[1/z] \oplus T \xrightarrow{\beta} T[1/z] = S[1/z] + T \to 0,$$

where $\alpha(b) = (b, -b)$ for all $b \in S$ and $\beta(c, d) = c + d$ for all $c \in S[1/z],\ d \in T$. Since the two end terms are flat S-modules, the middle term $S[1/z] \oplus T$ is also S-flat by Remark 2.31.12. By Definition 2.30, a direct summand of a flat S-module is S-flat. Hence T is S-flat .

For item 2, since the map $S \to T$ is flat, the embedding

$$C \ = \ (1 + zS)^{-1}S \ \hookrightarrow \ (1 + zS)^{-1}T \ = \ D$$

is flat. Since zC is in the Jacobson radical of C and $C/zC = S/zS = T/zT = D/zD$, each maximal ideal of C is contained in a maximal ideal of D, and so D is faithfully flat over C. This establishes item 2.

If T is Noetherian, then D is Noetherian. Since D is faithfully flat over C, it follows that C is Noetherian by Remark 2.31.8, and thus item 3 holds.

For item 4, let J be an ideal of S. By item 3, C is Noetherian, and by hypothesis $S[1/z]$ is Noetherian. Thus there exists a finitely generated ideal $J_0 \subseteq J$ such that $J_0S[1/z] = JS[1/z]$ and $J_0C = JC$. To show $J_0 = J$, it suffices to show for each maximal ideal $\mathbf{m}$ of S that $J_0S_{\mathbf{m}} = JS_{\mathbf{m}}$. If $z \notin \mathbf{m}$, then $S_{\mathbf{m}}$ is a localization of $S[1/z]$, and so $J_0S_{\mathbf{m}} = JS_{\mathbf{m}}$, while if $z \in \mathbf{m}$, then $S_{\mathbf{m}}$ is a localization of C, and so $JS_{\mathbf{m}} = J_0S_{\mathbf{m}}$. Therefore $J = J_0$ is finitely generated. It follows that S is Noetherian. □

NOETHERIAN FLATNESS THEOREM 6.3. (Inclusion Version) *As in Setting 5.1, assume that R is an integral domain with field of fractions K, $z \in R$ is a nonzero nonunit, $\bigcap_{n\in\mathbb{N}} z^nR = (0)$, the (z)-adic completion R^* of R is a Noetherian ring, and z is a regular element of R^*. Let $\tau_1, \ldots, \tau_s \in zR^*$ be algebraically independent elements over K such that the field $K(\tau_1, \ldots, \tau_s)$ is a subring of the total quotient ring of R^*. As in Equations 5.4.4, 5.4.5 and 5.4.6 of Notation 5.4, define*

$$U_n := R[\tau_{1n}, \ldots, \tau_{sn}], \qquad U \ := \ \bigcup_{n=1}^{\infty} U_n, \qquad B_n = (1 + zU_n)^{-1}U_n,$$

$$A \ := \ K(\tau_1, \ldots, \tau_s) \ \cap \ R^*, \qquad \textit{and} \qquad B \ := \ \bigcup_{n=1}^{\infty} B_n = (1 + zU)^{-1}U.$$

Then:

(1) *The following statements are equivalent:*
   (a) *The extension $U_0 := R[\tau_1, \ldots \tau_s] \hookrightarrow R^*[1/z]$ is flat.*
   (b) *The ring B is Noetherian.*
   (c) *The extension $B \hookrightarrow R^*$ is faithfully flat.*
   (d) *The ring A is Noetherian and $A = B$.*
   (e) *The ring A is Noetherian, and A is a localization of a subring of $U_0[1/z] = U[1/z]$.*

(2) *The equivalent conditions of item 1 imply the map $R \hookrightarrow R^*$ is flat.*

(3) *If z is an element of the Jacobson radical $\mathcal{J}(R)$ of R, e.g. if R is a local domain, the equivalent conditions of item 1 imply that R is Noetherian.*

(4) *If R is assumed to be Noetherian, then items a-e are equivalent to the ring U being Noetherian.*

PROOF. For item 1, (a) $\implies$ (b), if $U_0 = R[\tau_1, \ldots, \tau_s] \hookrightarrow R^*[1/z]$ is flat, then $U[1/z] = U_0[1/z] = R[\tau_1, \ldots, \tau_s][1/z] \hookrightarrow R^*[1/z]$ is flat, and so $U \hookrightarrow R^*[1/z]$ is flat. Since B is a localization of U formed by inverting elements of $(1 + zU)$, it follows that $B \hookrightarrow R^*[1/z]$ is flat. By Lemma 6.2.3 with $S = U$ and $T = R^*$, the ring B is Noetherian.

For (b) $\implies$ (c), since B is Noetherian, the extension $B^* = R^*$ is flat over B by Remark 3.2.2. By Proposition 5.16.1, the element $z \in \mathcal{J}(B)$. Thus $B^* = R^*$ is faithfully flat over B by Remark 3.2.4.

For (c) $\implies$ (d), assume $B^* = R^*$ is faithfully flat over B. Then

$$B = \mathcal{Q}(B) \cap R^* = \mathcal{Q}(A) \cap R^* = K \cap R^* = A,$$

by Remark 2.31.9, and so $A = B$ is Noetherian.

For (d) $\implies$ (e), since $B = A$, the ring A is a localization of U, and U is a subring of $R[\tau_1, \ldots, \tau_s][1/z] = U_0[1/z]$.

For (e) $\Longrightarrow$ (a), since A is a localization of a subring D of $R[\tau_1, \ldots, \tau_s][1/z]$, we have $A := \Gamma^{-1}D$, where Γ is a multiplicatively closed subset of D. Now

$$R[\tau_1, \ldots, \tau_s] \ \subseteq \ A = \Gamma^{-1}D \ \subseteq \ \Gamma^{-1}R[\tau_1, \ldots, \tau_s][1/z] \ \subseteq \ \Gamma^{-1}A[1/z] = A[1/z],$$

and so $A[1/z]$ is a localization of $R[\tau_1, \ldots, \tau_s]$. That is, to obtain $A[1/z]$ we localize $R[\tau_1, \ldots, \tau_s]$ by the elements of Γ and then localize by the powers of z. Since A is Noetherian, $A \hookrightarrow A^* = R^*$ is flat by Remark 3.2.2. Thus $A[1/z] \hookrightarrow R^*[1/z]$ is flat. Since $A[1/z]$ is a localization of $R[\tau_1, \ldots, \tau_s]$, it follows that $R[\tau_1, \ldots, \tau_s] \hookrightarrow R^*[1/z]$ is flat. This completes the proof of item 1.

For item 2, since U_0 is flat over R, condition a of item 1 implies that $R^*[1/z]$ is flat over R. By Lemma 6.2.1 with $S = R$ and $T = R^*$, if $R \hookrightarrow R^*[1/z]$ is flat, then $R \hookrightarrow R^*$ is flat.

For item 3, assume the equivalent conditions of item 1 hold and $z \in \mathcal{J}(R)$. The extension $R \hookrightarrow R^*$ is flat by item 2. If P is a maximal ideal of R, then $z \in P$ and $R/zR = R^*/zR^*$. Hence $PR^* \neq R^*$. Therefore $R \hookrightarrow R^*$ is faithfully flat. By Remark 2.31.8, R is Noetherian.

For item 4, assume the equivalent conditions of item 1 hold and R is Noetherian; then $U_0[1/z] = U[1/z]$ is Noetherian. The composite embedding

$$U \hookrightarrow B = A \hookrightarrow B^* = A^* = R^*$$

is flat because B is a localization of U and $B^* = R^*$ is faithfully flat over B. Thus by Lemma 6.2, parts 1 and 4, with $S = U$ and $T = R^*$, it follows that U is Noetherian. If U is Noetherian, then the localization B of U is Noetherian, and so condition b holds. □

COROLLARY 6.4. *Assume notation as as in Noetherian Flatness Theorem 6.3. If* $\dim R^* = 1$*, then the equivalent conditions of item 1 of Theorem 6.3 hold.*

PROOF. We show the map $\psi : R[\tau_1, \ldots, \tau_s] \hookrightarrow R^*[1/z]$ is flat. Since $\dim R^* = 1$ and z is a regular element in R^* with $z \in \mathcal{J}(R^*)$, we have $\dim R^*[1/z] = 0$. Hence $R^*[1/z]$ is the total quotient ring of R^*, and the map ψ factors as the composition of the inclusion maps $R[\tau_1, \ldots, \tau_s] \hookrightarrow K(\tau_1, \ldots, \tau_s) \hookrightarrow R^*[1/z]$. Modules over a field are free and hence flat, and compositions of flat maps are flat by Remarks 2.31, parts 2 and 13. Hence the map ψ is flat. □

REMARK 6.5. Let R, $z \in R$ and $\tau_1, \ldots, \tau_s$ be as in Theorem 6.3. Assume that the equivalent conditions of item 1 of Theorem 6.3 hold. Let R' be a localization of R such that z is a nonunit of R'. The approximation domain B' associated to R' and the τ_i is a localization of the approximation domain B associated to R and the τ_i. Thus B' is Noetherian, and so the equivalent conditions of item 1 of Theorem 6.3 hold for the construction over R'.

In part 3 of Corollary 6.6, we record a simplified flatness property for Local Prototypes.

COROLLARY 6.6. *Let $R = k[x, y_1, \ldots, y_r]_{(x,y_1,\ldots,y_r)}$, where $x, y_1, \ldots, y_r$ are variables over a field k, let $R^* = k[y_1, \ldots, y_r]_{(y_1,\ldots,y_r)}[[x]]$ denote the (x)-adic completion of R and let $\tau_1, \ldots, \tau_s$ be elements of R^* that are algebraically independent over R. Then:*

1. *The ring B of Noetherian Flatness Theorem 6.3 equals the following directed union:*
$$B = \bigcup_{n=0}^{\infty} k[x, y_1, \ldots, y_r, \tau_{1n}, \ldots, \tau_{sn}]_{(x,y_1,\ldots,y_r,\tau_{1n},\ldots,\tau_{sn})},$$
where τ_{in} is the n^{th}-endpiece of τ_i for each i with $1 \leq i \leq s$.
2. *The conditions of item 1 of Theorem 6.3 are equivalent to the flatness of the map*
$$\psi' : U_0' := k[x, y_1, \ldots, y_r, \tau_1, \ldots, \tau_s] \hookrightarrow R^*[1/x].$$
3. *If $\tau_1, \ldots, \tau_s$ are elements of $xk[[x]]$, then*
   (a) *$D = V[y_1, \ldots, y_r]_{(x,y_1,\ldots,y_r)}$, where $V = k(x, \tau_1, \ldots, \tau_s) \cap k[[x]]$, is the Local Prototype of Definition 4.27,*
   (b) *The maps $\psi : R[\tau_1, \ldots, \tau_s] \hookrightarrow R^*[1/x]$ and $\psi' : U_0' \hookrightarrow R^*[1/x]$ are flat.*

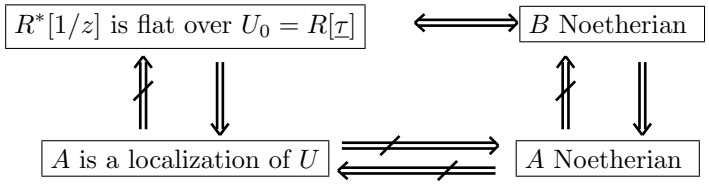
PROOF. Item 1 is Remark 5.15.3. For item 2, the map ψ' is the composition
$$U_0' \hookrightarrow U_0 \stackrel{\psi}{\hookrightarrow} R^*[1/x],$$
and $U_0' \hookrightarrow U_0$ is flat by 2.31.4. Thus flatness of $U_0 \hookrightarrow R^*[1/x]$ implies ψ' is flat, by Remark 2.31.13. If ψ' is flat, then $U_0 \hookrightarrow R^*[1/x]$ is flat by Remark 2.31.1, and so item 2 holds.

For item 3, by Remark 4.28.1, the ring $D = \mathcal{Q}(R)(\tau_1, \ldots, \tau_s) \cap R^*$. That is, D is the intersection domain of Inclusion Construction 5.3 for R with respect to the τ_i. Thus Noetherian Flatness Theorem 6.3 applies. By Proposition 4.26, D equals its approximation domain, given in item 1. Since D is an RLR, the equivalent conditions of item 1 of Theorem 6.3 hold, and so the map ψ is flat. Equivalently, by item 2, the map ψ' is flat. □

REMARK 6.7. The original proof given for Noetherian Flatness Theorem 6.3 (Inclusion Version) in [**66**] is an adaptation of a proof given by Heitmann in [**81**, page 126]. Heitmann considers the case where there is one transcendental element τ and defines the corresponding extension U to be a *simple PS-extension of R for z*. Heitmann proves in this case that a certain monomorphism condition on a sequence of maps is equivalent to U being Noetherian [**81**, Theorem 1.4].

REMARK 6.8. Examples where $A = B$ and A is not Noetherian show that it is possible for A to be a localization of U and yet for A, and therefore also U, to fail to be Noetherian; see Example 16.1 and Theorem 16.5. Thus the equivalent conditions of Noetherian Flatness Theorem 6.3 are not implied by the property that A is a localization of U.

The following diagram displays the situation concerning possible implications among certain statements for Inclusion Construction 5.3 and the approximations in Section 5.2:



REMARK 6.9. It is sometimes difficult to determine whether or not the map $R[\underline{\tau}] := R[\tau_1, \ldots, \tau_s] \hookrightarrow R^*[1/z]$ of Inclusion Construction 5.3 is flat. One helpful fact is given in Remark 2.31.10: If there exists a prime ideal P of $R^*[1/z]$ such that $\operatorname{ht} P < \operatorname{ht}(P \cap R[\underline{\tau}]$, then $R[\underline{\tau}] \hookrightarrow R^*[1/z]$ is not flat, and hence U is not Noetherian.

6.2. Introduction to the Insider Construction

In this section we introduce a technique using Inclusion Construction 5.3 to construct a variety of examples that are contained inside a Local Prototype domain—a localized polynomial ring with coefficients in a DVR as defined in Definition 4.27.[1] We call this technique the "Insider Inclusion Construction", or more briefly, the "Insider Construction". The integral domains constructed in this way are called "Insider Examples", because they are inside a Local Prototype domain.

We present in this chapter several examples using the Insider Construction, including two classical examples of Nagata and Rotthaus. We show how the Insider Construction simplifies the verification of properties of examples constructed using Inclusion Construction 5.3.

For the examples considered in this chapter, we use Setting 6.10:

SETTING 6.10. Let k be a field, let $s \in \mathbb{N}$ and $r \in \mathbb{N}_0$, let $x, y_1, \ldots y_r$ be variables over k, and let $R = k[x, y_1, \ldots, y_r]_{(x,y_1,\ldots,y_r)}$ be the localized polynomial ring in these variables. Let the elements $\tau_1, \ldots, \tau_s \in xk[[x]]$ be algebraically independent over $k(x)$. As in Corollary 6.6, the ring R is the base ring of a Local Prototype domain

$$D = V[x, y_1, \ldots, y_r]_{(x,y_1,\ldots,y_r)} \cap R^*,$$

where R^* is the (x)-adic completion of R, $V = k(x, y_1, \ldots, y_r) \cap k[[x]]$, and the map

$$\psi : R[\tau_1, \ldots, \tau_s] \hookrightarrow R^*[1/x]$$

is flat. We construct two "insider" integral domains A and B inside the Local Prototype D, where A is an intersection domain as in Construction 5.3, and B is an integral domain that "approximates" A as in Section 5.2.

[1] This technique is studied in more generality and detail in Insider Construction 10.1.

Let $f \in R[\tau_1, \ldots, \tau_s] \subseteq R^*$ be transcendental over $\mathcal{Q}(R) = k(x, y_1, \ldots, y_r)$. As in Insider Construction 5.3, let $A = \mathcal{Q}(R)(f) \cap R^*$. Define endpieces f_n as in Equation 5.4.1 of Notation 5.4, and define the approximation domain B associated with f, as in Equation 5.4.5 and Noetherian Flatness Theorem 6.3. By Corollary 6.6.1, B is a directed union

$$B = \bigcup_{n=1}^{\infty} R[f_n]_{(\mathbf{m}, f_n)},$$

where $\mathbf{m}$ is the maximal ideal of R. Let $S = R[f]$ and let $T = R[\tau_1, \ldots, \tau_s]$.

As we describe in Theorem 6.11, the condition that the insider approximation domain B is Noetherian is related to flatness of the extension $S \hookrightarrow T$, an extension of polynomial subrings of R^*. We apply Theorem 6.11 to conclude that Nagata's Example 4.14 and Christel's Example 4.16 are Noetherian.

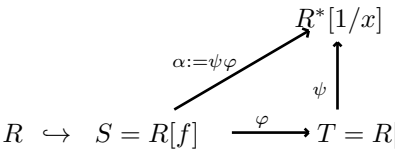
THEOREM 6.11. *In the notation of Setting 6.10, if the extension*

$$S := R[f] \overset{\varphi}{\hookrightarrow} T := R[\tau_1, \ldots, \tau_s]$$

is flat, then B is Noetherian and A equals B. Hence A is Noetherian.

PROOF. By Corollary 6.6 the map $\psi : T \hookrightarrow R^*[1/x]$ is flat. By hypothesis, $\varphi : S := R[f] \hookrightarrow R[\tau_1, \ldots, \tau_s]$ is flat.

(6.12.1)



Since the composition of flat maps is again flat (Remark 2.31.13), we conclude that $\alpha : S \hookrightarrow R^*[1/x]$ is flat. By Noetherian Flatness Theorem 6.3, we have that $A = B$, as desired. □

This idea is the basis for Insider Construction 10.1. The same argument goes through for several elements $f_1, \ldots, f_t \in T$ that are algebraically independent over $\mathcal{Q}(R)$. Moreover, non-flatness of the extension φ sometimes implies non-flatness of the extension $U_0 \hookrightarrow R^*[1/z]$; see Theorem 10.3. In Corollary 7.6 we show $\varphi : S \hookrightarrow T$ is flat if and only if $\operatorname{ht} Q \geq \operatorname{ht}(Q \cap S)$ for every $Q \in \operatorname{Spec} T$.

6.3. Nagata's example

In Proposition 6.13 we use Theorem 6.11 to prove that Nagata's Example 4.14 is Noetherian.

SETTING 6.12. Let k be a field, let x and y be indeterminates over k, and set

$$R := k[x, y]_{(x,y)} \qquad \text{and} \qquad R^* := k[y]_{(y)}[[x]].$$

The power series ring R^* is the xR-adic completion of R. Let $\tau \in xk[[x]]$ be a transcendental element over $k(x)$. Since R^* is an integral domain, every nonzero element of the polynomial ring $R[\tau]$ is a regular element of R^*. Thus the field $k(x, y, \tau)$ is a subfield of $\mathcal{Q}(R^*)$. The Local Prototype domain D corresponding to τ is $D := k(x, y, \tau) \cap R^*$, as in Definition 4.27. By Proposition 4.26, D is a two-dimensional regular local domain and is a directed union of localized polynomial rings in three variables over the field k.

Let f be a polynomial in $R[\tau]$ that is algebraically independent over $\mathcal{Q}(R)$, for example, $f = (y+\tau)^2$, as in Nagata's example. Let $A := \mathcal{Q}(R[f]) \cap R^*$ be the *intersection domain* corresponding to f. Since $R[f] \subseteq R[\tau]$, we have $k(x,y,f) = \mathcal{Q}(R[f]) \subseteq \mathcal{Q}(R^*)$. The intersection domain A is a subring of the Local Prototype domain D.

By Corollary 6.6.1, the natural approximation domain B associated to A is

$$B \;=\; \bigcup_{\mathbf{n}\in\mathbb{N}} k[x,y,f_n]_{(x,y,f_n)}, \tag{6.12.0}$$

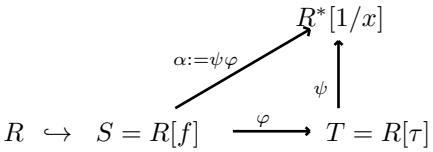
where the f_n are the n^{th} endpieces of f.

By Corollary 6.6.3b, the extension $T := R[\tau] \overset{\psi}{\hookrightarrow} R^*[1/x]$ is flat, where ψ is the inclusion map. Let $S := R[f] \subseteq R[\tau]$ and let φ be the embedding

$$\varphi : S := R[f] \overset{\varphi}{\hookrightarrow} T = R[\tau]. \tag{6.12.e}$$

Put $\alpha := \psi \circ \varphi : S \to R^*[1/x]$. Then we have the following commutative diagram:

(6.12.1)



The proof in Proposition 6.13 of the Noetherian property for Nagata's Example 4.14 is different from the proof given in [**117**, Example 7, pp.209-211].

PROPOSITION 6.13. *With the notation of Setting 6.12, let $f := (y+\tau)^2$. In Nagata Example 4.14, the ring $B = A$ and B is Noetherian with completion $k[[x,y]]$. By Theorem 3.23, B is a two-dimensional regular local domain.*

PROOF. The ring $T = R[\tau]$ is a free S-module with free basis $\langle 1, y+\tau\rangle$. By Remark 2.31.2, the map φ is flat. By Theorem 6.11, B is Noetherian and $B = A$. □

REMARKS 6.14. (1) In Nagata's original example [**117**, Example 7,pp. 209-211], the field k has characteristic different from 2. This assumption is not necessary for showing that the domain B of Proposition 6.13 is a two-dimensional regular local domain.

(2) Whether or not the ring B is Noetherian depends upon the polynomial f. In Example 6.18.2, the ring B is constructed in a similar way to the ring B of Proposition 6.13, but the ring B of Example 6.18.2 is not Noetherian.

6.4. Christel's Example

In this section we present more examples using the techniques of Section 6.3, usually with two elements σ and τ that are algebraically independent elements over the power series ring $k[[x]]$ where k is a field. To describe these examples, we modify Setting 6.12 as follows.

SETTING 6.15. Let k be a field, let x, y, z be indeterminates over k, and set

$$R := \; k[x,y,z]_{(x,y,z)} \qquad \text{and} \qquad R^* := \; k[y,z]_{(y,z)}[[x]].$$

The power series ring R^* is the xR-adic completion of R. Let σ and τ in $xk[[x]]$ be algebraically independent over $k(x)$. We use the Local Prototype Domain D corresponding to σ, τ as in Definition 4.27, that is,

$$D := k(x, y, z, \sigma, \tau) \cap k[y, z]_{(y,z)}[[x]].$$

In the examples of this section we define f to be an element of $R[\sigma, \tau]$ such that f is transcendental over $K = \mathcal{Q}(R)$. The *intersection domain* of Inclusion Construction 5.3 corresponding to f is

$$A = K(f) \cap R^* = k(x, y, z, f) \cap k[y, z]_{(y,z)}[[x]].$$

Thus A is an "insider" intersection domain contained in the Local Prototype Domain D. As in Setting 6.12 for the Nagata Example, the approximation domain B associated to A is a directed union of localized polynomial rings over k in four variables.

REMARK 6.16. With Setting 6.15, let $T := R[\sigma, \tau]$ and let $S := R[f]$, where f is a polynomial in $R[\sigma, \tau]$ that is algebraically independent over $\mathcal{Q}(R)$. Let $\varphi : S \hookrightarrow T$ denote the inclusion map from S to T. Then, since σ and τ are algebraically independent over R, the element f in $R[\sigma, \tau]$ has a unique expression

$$f = c_{00} + c_{10}\sigma + c_{01}\tau + \cdots + \ c_{ij}\sigma^i\tau^j + \cdots + c_{mn}\sigma^m\tau^n,$$

where the $c_{ij} \in R$. The c_{ij} with at least one of i or j nonzero are the *nonconstant coefficients* of f. The ideal $L := (c_{10}, c_{01}, \ldots, c_{mn})R$ is the *ideal generated by the nonconstant coefficients* of f. We show in Theorem 7.23 of Chapter 7 that

$$\varphi \text{ is flat} \iff LR = R. \tag{6.16.b}$$

We use Theorem 6.11 to show the Noetherian property for the following example of Rotthaus [**131**], Example 4.16 of Chapters 4.

EXAMPLE 6.17. (Christel) This is the first example of a Nagata ring that is not excellent. With Setting 6.15, let $f := (y + \sigma)(z + \tau)$ and consider the intesection domain $A = k(x, y, z, f) \cap R^*$ contained in Local Prototype $D = k(x, y, z, \sigma, \tau) \cap R^*$. The nonconstant coefficients of $f = yz + \sigma z + \tau y + \sigma\tau$ as a polynomial in $R[\sigma, \tau]$ are $\{1, z, y\}$. They do generate the unit ideal of R, and so, since we assume Remark 6.16.4 for now, we have φ is flat. Thus, by Theorem 6.11, the associated nested union domain B is Noetherian and is equal to A.

6.5. Further implications of the Noetherian Flatness Theorem

Noetherian Flatness Theorem 6.3 also yields examples that are not Noetherian even if he approximation domain B is equal to the intersection domain A.

EXAMPLES 6.18. (1) With Setting 6.15, let $f := y\sigma + z\tau$. We show in Examples 10.9 that the map $R[f] \hookrightarrow R[\sigma, \tau]$ is not flat and that $A = B$, i.e., A is "limit-intersecting" as in Definition 5.10, but is not Noetherian. Thus we have a situation where the intersection domain equals the approximation domain, but is not Noetherian.

(2) The following is a related simpler example: Again with the notation of Setting 6.15, let $f := y\tau + z\tau^2 \in R[\tau] \subseteq D = k(x, y, z, \tau) \cap R^*$, the Prototype. Then

the constructed approximation domain B (using f) is not Noetherian by Theorem 10.7. Moreover, B is equal to the intersection domain $A := R^* \cap k(x, y, z, f)$ by Corollary 10.5.

In dimension two (the two variable case), an immediate consequence of Valabrega's Theorem 4.8 is the following.

THEOREM 6.19. (Valabrega) *Let x and y be indeterminates over a field k and let $R = k[x, y]_{(x,y)}$. Then $\widehat{R} = k[[x, y]]$ is the completion of R. If L is a field between the field of fractions of R and the field of fractions of $k[y]_{(y)}[[x]]$, then $A = L \cap \widehat{R}$ is a two-dimensional regular local domain with completion $\widehat{R}$.*

Example 6.18 shows that the dimension three analog to Valabrega's result fails. With $R = k[x, y, z]_{(x,y,z)}$ the field $L = k(x, y, z, f)$ is between $k(x, y, z)$ and the fraction field of $k[y, z]\,[[x]]$, but $L \cap \widehat{R} = L \cap R^*$ is not Noetherian.

EXAMPLE 6.20. The following example is given in Section 23.4. With the notation of Setting 6.15, let $f = (y + \sigma)^2$ and $g = (y + \sigma)(z + \tau)$. It is shown in Chapter 23 that the intersection domain $A := R^* \cap k(x, y, z, f, g)$ properly contains its associated approximation domain B and that both A and B are non-Noetherian.

We use Ratliff's Equidimension Theorem 3.18 to show that the universally catenary property is preserved by Inclusion Construction 5.3, if the constructed domain is Noetherian.

THEOREM 6.21. *Assume the notation of Noetherian Flatness Theorem 6.3, and assume that $(R, \mathbf{m})$ is a universally catenary Noetherian local domain. Then:*

(1) *If A is Noetherian, then A is a universally catenary Noetherian local domain.*
(2) *If B is Noetherian, then $B = A$ and B is a universally catenary local domains.*

PROOF. By Construction Properties Theorem 5.14.4, $R^* = B^* = A^*$. By Proposition 5.16.5, A and B are local and their maximal ideals are $\mathbf{m}A$ and $\mathbf{m}B$, respectively. The $\mathbf{m}$-, $\mathbf{m}A$- and $\mathbf{m}B$-adic completions of R, A and B, respectively, all equal the $\mathbf{m}R^*$-adic completion of R^*, and so $\widehat{R} = \widehat{A} = \widehat{B}$. Ratliff's Equidimension Theorem 3.18 states that a Noetherian local domain is universally catenary if and only if its completion is equidimensional. By assumption R is universally catenary, and so $\widehat{R}$ is equidimensional by Ratliff's Theorem 3.18. Thus, if A is Noetherian, then A is also universally catenary. If B is Noetherian, then $B = A$, by Noetherian Flatness Theorem 6.3, and so B is universally catenary. □

Exercise

(1) For the strictly descending chain of one-dimensional local domains

$$A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$$

that are birational extensions of $R = k[x, y]$ given in Example 17.18, describe the integral domain $D := \bigcap_{n=1}^{\infty} A_n$.

Suggestion: Since $\mathbf{n}_n \cap R = (x, y)R$, we have $R_{(x,y)R} \subset A_n$ for each $n \in \mathbb{N}$. By Exercise 4 of Chapter 5, the ring A_n may be described as

$$A_n = \{\ a/b \mid a, b \in R,\ b \neq 0 \ \text{ and } a \in I^n + bR^*\ \}.$$

Show that $a \in I^n + bR^*$ for all $n \in \mathbb{N}$ if and only if $a/b \in R_{(x,y)R}$.

CHAPTER 7

The flat locus of an extension of polynomial rings

Let R be a Noetherian ring, let n be a positive integer and let $z_1, \ldots, z_n$ be indeterminates over R. In this chapter we examine the flat locus of an extension φ of polynomial rings of the form

$$S := R[f_1, \ldots, f_m] \overset{\varphi}{\hookrightarrow} R[z_1, \ldots, z_n] =: T, \tag{7.01}$$

where the f_j are polynomials in $R[z_1, \ldots, z_n]$ that are algebraically independent over R. We are motivated to examine the flat locus of the extension φ by the flatness condition of Theorem 6.11 in the Insider Inclusion Construction of Section 6.2.

We discuss in Section 7.1 a general result on flatness. Then in Section 7.2 we consider the Jacobian ideal of the map $\varphi : S \hookrightarrow T$ of (7.01) and describe the nonsmooth and nonflat loci of this map. In Section 7.3 we discuss applications to polynomial extensions. Related results are given in the papers of Picavet [**126**] and Wang [**155**].

7.1. Flatness criteria

Recall that a Noetherian local ring $(R, \mathbf{m})$ of dimension d is *Cohen-Macaulay* if there exist elements $x_1, \ldots, x_d$ in $\mathbf{m}$ that form a regular sequence as defined in Chapter 2; see [**103**, pages 134, 136].

The following definition is useful in connection with what is called the "local flatness criterion" [**103**, page 173].

DEFINITION 7.1. Let I be an ideal of a ring A.

(1) An A-module N is *separated* for the I-adic topology if $\bigcap_{n=1}^{\infty} I^n N = (0)$.
(2) An A-module M is said to be *I-adically ideal-separated* if $\mathfrak{a} \otimes M$ is separated for the I-adic topology for every finitely generated ideal $\mathfrak{a}$ of A.

REMARK 7.2. In Theorem 7.3, we use the following result on flatness. Let I be an ideal of a Noetherian ring A and let M be an I-adically ideal-separated A-module. By [**103**, part (1) $\iff$ (3) of Theorem 22.3], we have M is A-flat $\iff$ the following two conditions hold: (a) $I \otimes_A M \cong IM$, and (b) M/IM is (A/I)-flat.

Theorem 7.3 is a general result on flatness involving the Cohen-Macaulay property and a trio of Noetherian local rings.

THEOREM 7.3. *Let $(R, \mathbf{m}), (S, \mathbf{n})$ and (T, ℓ) be Noetherian local rings, and assume there exist local maps:*

$$R \longrightarrow S \longrightarrow T,$$

such that

(i) *$R \to T$ is flat and $T/\mathbf{m}T$ is a Cohen-Macaulay ring, and*

(ii) *$R \to S$ is flat and $S/\mathbf{m}S$ is a regular local ring.*

Then the following statements are equivalent:

(1) *$S \to T$ is flat.*
(2) *For each prime ideal $\mathbf{w}$ of T, we have $ht(\mathbf{w}) \geq ht(\mathbf{w} \cap S)$.*
(3) *For each prime ideal $\mathbf{w}$ of T such that $\mathbf{w}$ is minimal over $\mathbf{n}T$, we have* $\operatorname{ht}(\mathbf{w}) \geq \operatorname{ht}(\mathbf{n})$.

PROOF. The implication (2) $\Longrightarrow$ (3) is obvious and the implication (1) $\Longrightarrow$ (2) is clear by Remark 2.31.10. To prove (3) $\Longrightarrow$ (1), we observe that T is an $\mathbf{m}S$-adically ideal-separated S-module, since T is a Noetherian local ring; see Definition 7.1 and Krull's Intersection Theorem 2.16. Hence, by Remark 7.2 with $A = S$, $I = \mathbf{m}S$ and $M = T$, it suffices to show:

(a) $\mathbf{m}S \otimes_S T \cong \mathbf{m}T$.
(b) The map $S/\mathbf{m}S \longrightarrow T/\mathbf{m}T$ is faithfully flat.

Proof of (a): Since $R \hookrightarrow S$ is flat, we have $\mathbf{m}S \cong \mathbf{m}R \otimes_R S$. Therefore

$$\mathbf{m}S \otimes_S T \cong (\mathbf{m} \otimes_R S) \otimes_S T \cong \mathbf{m} \otimes_R T \cong \mathbf{m}T,$$

where the last isomorphism follows because the map $R \to T$ is flat.

Proof of (b): By assumption, $T/\mathbf{m}T$ is Cohen-Macaulay and $S/\mathbf{m}S$ is a regular local ring. We also have $T/\mathbf{n}T = (T/\mathbf{m}T) \otimes_{S/\mathbf{m}S} (S/\mathbf{n})$. By [**103**, Theorem 23.1], if

$$\dim(T/\mathbf{m}T) \;=\; \dim(S/\mathbf{m}S) + \dim(T/\mathbf{n}T), \tag{7.3.c}$$

then $S/\mathbf{m}S \to T/\mathbf{m}T$ is flat. Thus to prove (3) $\Longrightarrow$ (1), it suffices to establish Equation 7.3.c.

In order to prove Equation 7.3.c, we may reduce to the case where $\mathbf{m} = 0$. Thus we may assume that R is a field, S is an RLR and T is a Cohen-Macaulay local ring. Let $\mathbf{w} \in \operatorname{Spec} T$ be such that $\mathbf{n}T \subseteq \mathbf{w}$. Since the map $S \to T$ is a local homomorphism, we have $\mathbf{w} \cap S = \mathbf{n}$. By [**103**, Theorem 15.1i] we have

$$\operatorname{ht}(\mathbf{w}) \;\leq\; \operatorname{ht}(\mathbf{n}) \;+\; \dim(T_{\mathbf{w}}/\mathbf{n}T_{\mathbf{w}}).$$

If $\mathbf{w}$ is minimal over $\mathbf{n}T$, then $\dim(T_{\mathbf{w}}/\mathbf{n}T_{\mathbf{w}}) = 0$, and hence $\operatorname{ht}\mathbf{w} \leq \operatorname{ht}\mathbf{n}$. By condition 3, $\operatorname{ht}(\mathbf{w}) \geq \operatorname{ht}(\mathbf{n})$, and therefore $\operatorname{ht}(\mathbf{w}) = \operatorname{ht}(\mathbf{n})$, for every minimal prime divisor $\mathbf{w}$ of $\mathbf{n}T$. Thus $\operatorname{ht}(\mathbf{n}) = \operatorname{ht}(\mathbf{n}T)$.

Since T is Cohen-Macaulay and hence is catenary, we have

$$\dim(T/\mathbf{n}T) = \dim(T) - \operatorname{ht}(\mathbf{n}T) = \dim(T) - \operatorname{ht}(\mathbf{n})$$

Thus $\dim T = \dim S + \dim(T/\mathbf{n}T)$, as desired. □

In Theorem 7.4 we present a result closely related to Theorem 7.3 with a Cohen-Macaulay hypothesis on all the fibers of $R \to T$ and a regularity hypothesis on all the fibers of $R \to S$. A ring homomorphism $f : A \to B$ of Noetherian rings has *Cohen-Macaulay fibers* with respect to f if, for every $P \in \operatorname{Spec} A$, the ring $B \otimes_A k(P)$ is Cohen-Macaulay, where $k(P)$ is the field of fractions of A/P. For more information about the fibers of a map, see Discussion 3.22 and Definition 3.28.

THEOREM 7.4. *Let $(R, \mathbf{m}), (S, \mathbf{n})$ and (T, ℓ) be Noetherian local rings, and assume there exist local maps:*

$$R \longrightarrow S \longrightarrow T,$$

such that

(i) *$R \to T$ is flat with Cohen-Macaulay fibers, and*
(ii) *$R \to S$ is flat with regular fibers.*

Then the following statements are equivalent:

(1) *$S \to T$ is flat with Cohen-Macaulay fibers.*
(2) *$S \to T$ is flat.*
(3) *For each prime ideal $\mathbf{w}$ of T, we have $ht(\mathbf{w}) \geq ht(\mathbf{w} \cap S)$.*
(4) *For each prime ideal $\mathbf{w}$ of T such that $\mathbf{w}$ is minimal over $\mathbf{n}T$, we have $ht(\mathbf{w}) \geq ht(\mathbf{n})$.*

PROOF. The implications (1) $\Longrightarrow$ (2) and (3) $\Longrightarrow$ (4) are obvious and the implication (2) $\Longrightarrow$ (3) is clear by Remark 2.31.10. By Theorem 7.3, item 4 implies that $S \to T$ is flat.

To show Cohen-Macaulay fibers for $S \to T$, it suffices to show, for each prime ideal Q of T, if $P := Q \cap S$ then T_Q/PT_Q is Cohen-Macaulay. Let $Q \cap R = q$. By passing to $R/q \subseteq S/qS \subseteq T/qT$, we may assume $Q \cap R = (0)$. Let $\operatorname{ht} P = n$. Since $R \to S_P$ has regular fibers and $P \cap R = (0)$, S_P is an RLR, and the ideal PS_P is generated by n elements. Moreover, faithful flatness of the map $S_P \to T_Q$ implies that the ideal PT_Q has height n by Remark 2.31.10. Since T_Q is Cohen-Macaulay, a set of n generators of PS_P forms a regular sequence in T_Q. Hence T_Q/PT_Q is Cohen-Macaulay [**103**, Theorems 17.4 and 17.3]. □

Since flatness is a local property by Remark 2.31.4, the following two corollaries are immediate from Theorem 7.4; see also [**126**, Théorème 3.15].

COROLLARY 7.5. *Let T be a Noetherian ring and let $R \subseteq S$ be Noetherian subrings of T. Assume that $R \to T$ is flat with Cohen-Macaulay fibers and that $R \to S$ is flat with regular fibers. Then $S \to T$ is flat if and only if, for each prime ideal P of T, we have $\operatorname{ht}(P) \geq \operatorname{ht}(P \cap S)$.*

As a special case of Corollary 7.5, we have:

COROLLARY 7.6. *Let R be a Noetherian ring and let $z_1, \ldots, z_n$ be indeterminates over R. Assume that $f_1, \ldots, f_m \in R[z_1, \ldots, z_n]$ are algebraically independent over R. Then*

(1) *$\varphi : S := R[f_1, \ldots, f_m] \hookrightarrow T := R[z_1, \ldots, z_n]$ is flat if and only if, for each prime ideal P of T, we have $\operatorname{ht}(P) \geq \operatorname{ht}(P \cap S)$.*
(2) *For $Q \in \operatorname{Spec} T$, $\varphi_Q : S \to T_Q$ is flat if and only if for each prime ideal $P \subseteq Q$ of T, we have $\operatorname{ht}(P) \geq \operatorname{ht}(P \cap S)$.*

PROOF. Since S and T are polynomial rings over R, the maps $R \to S$ and $R \to T$ are flat with regular fibers. Hence both assertions follow from Corollary 7.5. □

7.2. The Jacobian ideal and the smooth and flat loci

We use the following definitions as in Swan [**148**].

DEFINITION 7.7. Let R be a ring. An R-algebra A is said to be *quasi-smooth* over R if for every R-algebra B and ideal N of B with $N^2 = 0$, every R-algebra homomorphism $g : A \to B/N$ lifts to an R-algebra homomorphims $f : A \to B$. In the commutative diagram below, let the maps $\theta : R \to A$ and $\psi : R \to B$ be the canonical ring homomorphisms that define A and B as R-algebras and let the map $\pi : B \to B/N$ be the canonical quotient ring map

(7.7.1)
$$\begin{array}{ccc} R & \xrightarrow{\theta} & A \\ {\scriptstyle\psi}\downarrow & \overset{\exists f}{\swarrow} & \downarrow{\scriptstyle g} \\ B & \xrightarrow{\pi} & B/N \end{array}$$

If A is quasi-smooth over R, then there exists an R-algebra homomorphism f from A to B such that $\pi \circ f = g$. If A is finitely presented and quasi-smooth over R, then A is said to be *smooth* over R. If A is essentially finitely presented and quasi-smooth over R, then A is said to be *essentially smooth* over R; see Chapter 2 for the definitions of finitely presented and essentially finitely presented.

The terminology for smoothness varies. Matsumura [**103**, p. 193] uses the term 0-*smooth* for what Swan calls "quasi-smooth". Others such as Tanimoto [**150**], [**151**] use *smooth* for "quasi-smooth".

Recall from Definition 3.31 that a homomorphism $f : R \to \Lambda$ of Noetherian rings is said to be regular if f is flat and has geometrically regular fibers. To avoid any possible confusion in the case where R is a field, Swan in [**148**] calls such a homomorphism f geometrically regular.

Swan's article [**148**] gives a detailed presentation of D. Popescu's proof that a regular morphism of Noetherian rings is a filtered colimit of smooth morphisms. From Popescu's result, it follows that for extensions of finite type the concepts of regular and smooth are equivalent.

THEOREM 7.8. [**148**, Corollary 1.2] *Let $f : R \to \Lambda$ be a homomorphism of Noetherian rings with Λ a finitely generated R-algebra. Then the following are equivalent:*

(1) *f is regular.*
(2) *f is smooth, that is, Λ is a smooth R-algebra.*

PROOF. This follows by taking $\Lambda = A$ in [**148**, Corollary 1.2]. □

However, even if R is a field and the R-algebra Λ is a Noetherian ring, the map $f : R \hookrightarrow \Lambda$ may be a regular morphism but not be quasi-smooth. Tanimoto shows in [**150**, Lemma 2.1] that, for a field k and an indeterminate x over k, the regular morphism $k \longrightarrow k[[x]]$ is quasi-smooth as in Definition 7.7 if and only if k has characteristic $p > 0$ and $[k : k^p] < \infty$.

DEFINITIONS 7.9. Let A be an R-algebra over a ring R; say $A = R[Z]/I$, where $Z = \{z_\gamma\}_{\gamma\in\Gamma}$ is a set of indeterminates over R indexed by a possibly infinite index set Γ and I is an ideal of the polynomial ring $R[Z]$.

(1) We define $F := \bigoplus_{\gamma\in\Gamma} A dz_\gamma$ to be the free A-module on a basis $\{dz_\gamma\}_{\gamma\in\Gamma}$; this basis is to be in $1-1$ correspondence with the set $\{z_\gamma\}_{\gamma\in\Gamma}$. Define $D : I \to F$ by $D(f) = \sum_{\gamma\in\Gamma} \frac{\partial f}{\partial z_\gamma} dz_\gamma$, for every $f \in I$, where $\frac{\partial}{\partial z_\gamma}$ is the usual partial derivative function on $R[Z]$, with elements of $R[Z \setminus \{z_\gamma\}]$ considered to be "constants". The map D is a *derivation* in the sense that D is an R-module homomorphism and

$$D(fg) \;=\; gD(f) \;+ fD(g), \qquad \text{for every} \qquad f, g \in I.$$

We have $D(I^2) = (0)$, since $D(I^2) \subseteq IF = (0)$. Hence D induces a map d, called the *differential morphism* on I/I^2, such that

$$d : I/I^2 \;\to\; F = \bigoplus_{\gamma\in\Gamma} A dz_\gamma \qquad \text{and} \qquad d(f + I^2) \;=\; \sum_{\gamma\in\Gamma} \frac{\partial f}{\partial z_\gamma} dz_\gamma.$$

The differential morphism d is an A-linear map, since, for each $a \in A$, each $f \in I$, and each z_γ, we have $\frac{\partial(af)}{\partial z_\gamma} = a\frac{\partial f}{\partial z_\gamma} + f\frac{\partial a}{\partial z_\gamma}$, and $f\frac{\partial a}{\partial z_\gamma}$ is in $IF = (0)$. See [**103**, p.190-2] for more discussion about derivations and differentials.

(2) If $Z = \{z_1, \ldots, z_n\}$ is a finite set, that is, $n \in \mathbb{N}$, and if $g_1, \ldots, g_s$ are elements of I, we define the *Jacobian matrix* of the g_i with respect to the z_j to be the $s \times n$ matrix

$$\mathfrak{J}(g_1, \ldots, g_s; z_1, \ldots, z_n) \;:=\; \left(\frac{\partial g_i}{\partial z_j}\right)_{1\le i\le s,\ 1\le j\le n}.$$

Define the *Delta ideal* of the g_i, $\Delta(g_1, \ldots, g_s)$, to be the ideal of A generated by the $s \times s$ minors of the Jacobian matrix $\mathfrak{J}(g_1, \ldots, g_s; z_1, \ldots, z_n)$.[1] If $s = 0$, we set $\Delta(\emptyset) = A$.

(3) Assume that A is a finitely presented R-algebra. Then we may assume that $Z = \{z_1, \ldots, z_n\}$ is a finite set, and there exist $f_1, \ldots, f_m$ in $R[Z]$ such that $I = (f_1, \ldots, f_m)R[Z]$.

Define the *Elkik ideal* $\widetilde{H}$ of the ring A to be

$$\text{(7.9.a)} \qquad \widetilde{H} \;:=\; \widetilde{H}_{A/R} := \sqrt{\Big(\sum_{g_1,\ldots,g_s} (\Delta(g_1, \ldots, g_s) \cdot \big[(g_1, \ldots, g_s) :_{R[Z]} I\big] + I)A\Big)},$$

where $\sqrt{\ }$ denotes the radical of the enclosed ideal, and the sum is taken over all choices of s polynomials $g_, \ldots, g_s$ from the ideal I for all $s \in \mathbb{N}$; see Elkik[**39**, p. 555] and Swan [**148**, Section 4]. Swan mentions that the Elkik ideal provides a "very explicit definition" for the non-smooth locus of A; see the Elkik-Swan Theorem 7.11.

We define a simpler ideal H that is similar to $\widetilde{H}$ as follows:

$$\text{(7.9.b)} \quad H \;:=\; H_{A/R} \;:=\; \sqrt{\Big(\sum_{g_1,\ldots,g_s} (\Delta(g_1, \ldots, g_s) \cdot \big[(g_1, \ldots, g_s) :_{R[Z]} I\big] + I)A\Big)},$$

where the sum is taken over all subsets $\{g_1, \ldots, g_s\}$, for all $1 \le s \le m$, of the given finite set $\{f_1, \ldots, f_m\}$ of generators of I. It is clear that $H \subseteq \widetilde{H}$. We show in Theorem 7.11 that $H = \widetilde{H}$.

---

[1] Formally $\Delta(g_1, \ldots, g_s)$ is an ideal of $R[Z]$ but when we "multiply" it by A if becomes an ideal of A.

The following theorem from Swan's article [**148**] connects quasi-smoothness of an R-algebra A to the differential morphism d of Definition 7.9.1 being a split monomorphism.

THEOREM 7.10. [**148**, Parts of Theorem 3.4] *Let R be a ring and let A be an R-algebra $A := R[Z]/I$, where $Z = \{z_\gamma\}_{\gamma\in\Gamma}$ is a possibly infinite set of indeterminates over R, and I is an ideal of the polynomial ring $R[Z]$. Then the following two statements are equivalent:*

(1) *$R \to A$ is quasi-smooth.*
(2) *The differential morphism $d : I/I^2 \to \bigoplus_{\gamma\in\Gamma} A dz_\gamma$ is a split monomorphism.*

Theorem 7.11 is a modification of [**148**, Theorem 4.1], with the Elkik ideal $\widetilde{H}$ replaced by the simpler ideal H of Equation 7.9.b. This proof shows that H defines the non-smooth locus of A and that the Elkik ideal equals H. For the proof we adapt Swan's elegant argument. We call this theorem the *Elkik-Swan* Theorem

THEOREM 7.11. The Elkik-Swan Theorem. [**148**, Theorem 4.1] *Let A be a finitely presented algebra over a ring R. Write $A = R[Z]/I$, where $Z = \{z_1, \dots, z_n\}$ and $I = (f_1, \dots, f_m)R[Z]$ are as in Definition 7.9. Let H be the ideal of A defined in Definition 7.9.a, and let P be a prime ideal of A. Then*

(1) *A_P is essentially smooth over R if and only if H is not contained in P.*
(2) *H is the intersection of all $P \in \operatorname{Spec} A$ such that A_P is not essentially smooth over R.*
(3) *H is independent of the choice of presentation.*
(4) *The Elkik ideal describes the nonsmooth locus of A and can be computed using the formula given in Definition 7.9.a for the ideal H.*

PROOF. Let $P \in \operatorname{Spec} A$ and assume that H is not contained in P. Then some summand in the expression for H is not contained in P. By relabeling the set $\{f_1, \dots, f_m\}$, we let $\{f_1, \dots, f_r\}$ denote the subset associated with the summand not contained in P, where $r \le m$. Thus $(\Delta(f_1, \dots, f_r)\cdot[(f_1, \dots, f_r) :_{R[Z]} I])A$ is not contained in P. Let Q be the pre-image of P in $R[Z]$. Then $[(f_1, \dots, f_r) :_{R[Z]} I]$ is not contained in Q. Therefore $(f_1, \dots, f_r)R[Z]_Q = IR[Z]_Q = I_Q$, and so the images of $f_1, \dots, f_r$ generate $(I/I^2)_P = I_Q/I_Q^2$. Also $(\Delta(f_1, \dots, f_r)A$ is not contained in P. Hence the image of some $r \times r$ minor of $\mathfrak{J}(f_1, \dots, f_r; z_1, \dots, z_n)$ is not contained in P. By relabeling the z's, we may assume that the image in A of $\det(\frac{\partial f_i}{\partial z_j})_{1\le i,j\le r}$ is not contained in P. We consider the composition of the following maps when localizing the algebras at P:

$$(A_P)^r \xrightarrow{f} (I/I^2)_P \xrightarrow{d_P} \bigoplus A_P dz_j \xrightarrow{p} (A_P)^r,$$

where $p : \bigoplus_{i=1}^n A_P dz_i \to A_P^r$ is the projection on the first r summands and f is the linear map given by $f(a_1, \dots, a_r) = \sum a_i f_i$ for all r-tuples $(a_1, \dots, a_r) \in (A_P)^r$. Then the composition is given by the invertible $r \times r$ matrix $(\frac{\partial f_i}{\partial z_j})_{1\le i,j\le r}$. Thus the left hand map is an isomorphism and d_P is a split monomorphism. Therefore, by Theorem 7.10, A_P is a smooth R-algebra.

Conversely, if A_P is a smooth R-algebra, d_P is a split monomorphism by Theorem 7.10. Thus $(I/I^2)_P$ is free, say of rank r. By relabeling, we assume that $f_1, \dots, f_r$ map to a basis of $(I/I^2)_P = I_Q/I_Q^2$. By Nakayama's lemma, these elements generate I_Q, and so $[(f_1, \dots, f_r) :_{R[Z]} I]$ is not contained in Q.

We identify $(A_P)^r$ and $(I/I^2)_P$ by the isomorphism $f(a_1, \dots, a_r) = \sum a_i f_i$. Then the map $d_p : (I/I^2)_P \longrightarrow \oplus A_P dz_i$ can be identified with the linear map

$$d_P : (A_P)^r \longrightarrow \oplus A_P dz_i$$

given by the Jacobian matrix $(\frac{\partial f_i}{\partial z_j})_{1\le i\le r; 1\le j\le n}$. Since d_p is split, the induced map

$$\overline{d_P} : \ (A_P/PA_P)^r \longrightarrow \oplus (A_P/PA_P) dz_i$$

remains injective. Thus some $r \times r$-minor of $(\frac{\partial f_i}{\partial z_j})_{1\le i\le r; 1\le j\le n}$ is invertible in A_P.

Since H is a radical ideal, and every prime ideal $P \in \operatorname{Spec} A$ containing H is such that A_P is not essentially smooth, we see that H equals the intersection given in the second statement of Theorem 7.11. Since every presentation ideal I and generating set $f_1, \dots, f_m$ of I yield that H equals same intersection of prime ideals, the ideal H is independent of presentation.

For the "Moreover" statement, Swan's Theorem in [**148**, Theorem 4.1] shows that $\widetilde{H}$ is the same intersection as H. Thus H equals the Elkik ideal. □

We return to the extension φ of polynomial rings from Equation 7.01

$$S \ := \ R[f_1, \dots, f_m] \ \overset{\varphi}{\hookrightarrow} \ R[z_1, \dots, z_n] \ =: \ T,$$

where the f_j are polynomials in $R[z_1, \dots, z_n]$ that are algebraically independent over R.

Definitions and Remarks 7.12. (1) The *Jacobian ideal* J of the extension $S \hookrightarrow T$ is the ideal of T generated by the $m \times m$ minors of the $m \times n$ matrix $\mathfrak{J}$ defined as follows:[2]

$$\mathfrak{J} := \left(\frac{\partial f_i}{\partial z_j}\right)_{1\le i\le m, \ 1\le j\le n}.$$

(2) For the extension $\varphi : S \hookrightarrow T$, the *nonflat locus* of φ is the set $\mathcal{F}$, where

$$\mathcal{F} := \{Q \in \operatorname{Spec}(T) \mid \text{the map} \quad \varphi_Q : S \to T_Q \text{ is not flat } \}.$$

We also define the set $\mathcal{F}_{\min}$ and the ideal F of T as follows:

$$\mathcal{F}_{\min} := \{ \text{ minimal elements of } \mathcal{F}\} \ \text{ and } \ F := \bigcap \{Q \mid Q \in \mathcal{F}\}.$$

By [**103**, Theorem 24.3], the set $\mathcal{F}$ is closed in the Zariski topology on $\operatorname{Spec} T$. Hence

$$\mathcal{F} \ = \ \mathcal{V}(F) \ := \ \{P \in \operatorname{Spec} T \mid F \subseteq P \ \}.$$

Thus the set $\mathcal{F}_{\min}$ is a finite set and is equal to the set $\operatorname{Min}(F)$ of minimal primes of the ideal F of T.

Since a flat homomorphism satisfies the going-down theorem by Remark 3.2.9, Corollary 7.6 implies that

(i) $\mathcal{F}_{\min} \subseteq \{Q \in \operatorname{Spec} T \mid \operatorname{ht} Q < \operatorname{ht}(Q \cap S)\}$, and

(ii) If $Q \in \mathcal{F}_{\min}$, then every prime ideal $P \subsetneq Q$ satisfies $\operatorname{ht} P \ge \operatorname{ht}(P \cap S)$.

Example and Remarks 7.13. (1) Let k be a field, let x and y be indeterminates over k and set $f = x$, $g = (x-1)y$. Then $k[f, g] \overset{\varphi}{\hookrightarrow} k[x, y]$ is not flat.

Proof. For the prime ideal $P := (x-1) \in \operatorname{Spec}(k[x, y])$, we see that $\operatorname{ht}(P) = 1$, but $\operatorname{ht}(P \cap k[f, g]) = 2$; thus the extension is not flat by Corollary 7.6. □

[2]For related information on the Jacobian ideal of an algebra over a ring, see [**149**, Section 4.4, p. 65].

(2) The Jacobian ideal J of f and g in (1) is given by:

$$J \;=\; (\det \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\delta g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix})k[x,y] \;=\; (\det \begin{pmatrix} 1 & 0 \\ y & x-1 \end{pmatrix})k[x,y] \;=\; (x-1)k[x,y].$$

(3) In the example of item 1, the nonflat locus is equal to the set of prime ideals Q of $k[x,y]$ that contain the Jacobian ideal $(x-1)k[x,y]$, thus $J = F$.

(4) One can also describe the example of item 1 by taking the base ring R to be the polynomial ring $k[x]$ rather than the field k. Then both $T = R[y]$ and $S = R[g]$ are polynomial rings in one variable over R with $g = (x-1)y$. The Jacobian ideal J is the ideal of T generated by $\frac{\partial g}{\partial y} = x - 1$, so is the same as in item 1.

We record in Theorem 7.14 connections between the Jacobian ideal of the morphism $\varphi : S \hookrightarrow T$ of Equation 7.01 and the smoothness or flatness of localizations of φ.

THEOREM 7.14. *Let R be a Noetherian ring, let $z_1, \ldots, z_n$ be indeterminates over R, and let $f_1, \ldots, f_m \in R[z_1, \ldots, z_n]$ be algebraically independent over R. Consider the embedding $\varphi : S := R[f_1, \ldots, f_m] \hookrightarrow T := R[z_1, \ldots, z_n]$. Let J denote the Jacobian ideal of φ, and let F and $\mathcal{F}_{min}$ be as in* (7.12)*. Then*

(1) *$Q \in \operatorname{Spec} T$ does not contain $J \iff \varphi_Q : S \to T_Q$ is essentially smooth. Thus J defines the nonsmooth locus of φ.*
(2) *If $Q \in \operatorname{Spec} T$ does not contain J, then $\varphi_Q : S \to T_Q$ is flat. Thus $J \subseteq F$.*
(3) *$\mathcal{F}_{min} \subseteq \{Q \in \operatorname{Spec} T \mid J \subseteq Q$ and $\operatorname{ht}(Q \cap S) > \operatorname{ht} Q\}$.*
(4) *$\mathcal{F}_{min} \subseteq \{Q \in \operatorname{Spec} T \mid J \subseteq Q,\ \operatorname{ht} Q < \dim S$ and $\operatorname{ht}(Q \cap S) > \operatorname{ht} Q\}$.*
(5) *φ is flat $\iff$ for every $Q \in \operatorname{Spec}(T)$ such that $J \subseteq Q$ and $\operatorname{ht}(Q) < \dim S$, we have $\operatorname{ht}(Q \cap S) \leq \operatorname{ht}(Q)$.*
(6) *If $\operatorname{ht} J \geq \dim S$, then φ is flat.*

PROOF. For item 1, we show that, for our definition of the Jacobian ideal J given in Definition 7.12, the radical of J is the Elkik ideal of an extension given in the Elkik-Swan Theorem 7.11. Using Theorem 7.11, we can work with the simpler description H of the Elkik ideal given in Equation 7.9.b.

Let $u_1, \ldots, u_m$ be indeterminates over $R[z_1, \ldots, z_n]$ and identify

$$R[z_1, \ldots, z_n] \quad \text{with} \quad \frac{R[u_1, \ldots, u_m][z_1, \ldots, z_n]}{(\{u_i - f_i\}_{i=1,\ldots,m})}.$$

Since $u_1, \ldots, u_m$ are algebraically independent, the ideal J generated by the minors of $\mathfrak{J}$ is the Jacobian ideal of the extension φ by means of this identification. We make this more explicit as follows.

Let $B := R[u_1, \ldots, u_m, z_1, \ldots, z_n]$ and $I = (\{f_i - u_i\}_{i=1,\ldots,m})B$. Consider the following commutative diagram

$$\begin{array}{ccc} S := R[f_1, \ldots, f_m] & \longrightarrow & T := R[z_1, \ldots, z_n] \\ \cong\big\downarrow & & \cong\big\downarrow \\ S_1 := R[u_1, \ldots, u_m] & \longrightarrow & T_1 := B/I \end{array}$$

To see that $J \subseteq H$, defined in Equation 7.9.b, we observe that u_i is a constant with respect to z_j, we have $\left(\frac{\partial(f_i - u_i)}{\partial z_j}\right) = \left(\frac{\partial f_i}{\partial z_j}\right)$. Thus $J \subseteq H$.

To show that $H \subseteq \sqrt{(J)}$, let $g_1, \ldots, g_s \in \{f_1 - u_1, \ldots, f_m - u_m\}$. Notice that $f_1 - u_1, \ldots, f_m - u_m$ is a regular sequence in B. Thus, if $s < m$, we have $[(g_1, \ldots, g_s) :_B I] = (g_1, \ldots, g_s)B$. Thus the $m \times m$-minors of $\left(\frac{\partial f_i}{\partial z_j}\right)$ generate H up to radical, and so $H = \sqrt{J}$. Hence by the Elkik-Swan Theorem 7.11, for every prime ideal Q of T, T_Q is essentially smooth over S if and only if Q does not contain J.

For item 2, suppose $Q \in \operatorname{Spec} T$ and $J \nsubseteq Q$. Choose $h \in J \setminus Q$ and consider the extension $\varphi_h : S \hookrightarrow T[1/h]$. By item 1, φ_h is smooth. Since a smooth map is flat [**148**, page 2], φ_h is flat. Thus $\varphi_Q : S \hookrightarrow T_Q$ is flat. In view of Corollary 7.6 and Definition 7.12.2, item 3 follows from item 2.

If $\operatorname{ht} Q \geq \dim S$, then $\operatorname{ht}(Q \cap S) \leq \dim S \leq \operatorname{ht} Q$. Hence the set

$$\begin{aligned} &\{Q \in \operatorname{Spec} T \mid J \subseteq Q \text{ and } \operatorname{ht}(Q \cap S) > \operatorname{ht} Q\} \\ = \ &\{Q \in \operatorname{Spec} T \mid J \subseteq Q,\ \operatorname{ht} Q < \dim S \text{ and } \operatorname{ht}(Q \cap S) > \operatorname{ht} Q\}. \end{aligned}$$

Thus item 3 is equivalent to item 4.

The ($\Longrightarrow$) direction of item 5 is clear [**103**, Theorem 9.5]. For the ($\Longleftarrow$) direction of item 5 and for item 6, it suffices to show $\mathcal{F}_{\min}$ is empty, and this holds by item 4. □

REMARKS 7.15. (1) For φ as in Theorem 7.14, it would be interesting to identify the set $\mathcal{F}_{\min} = \operatorname{Min}(F)$. In particular we are interested in conditions for $J = F$ and/or conditions for $J \subsetneq F$. Example 7.13 is an example where $J = F$, whereas Examples 7.18 contains several examples where $J \subsetneq F$.
(2) If R is a Noetherian integral domain, then the zero ideal is not in $\mathcal{F}_{\min}$ and so $F \neq \{0\}$.
(3) In view of Theorem 7.14.3, we can describe $\mathcal{F}_{\min}$ precisely as

$$\mathcal{F}_{\min} = \{Q \in \operatorname{Spec} T \mid J \subseteq Q, \operatorname{ht}(Q \cap S) > \operatorname{ht} Q \text{ and } \forall P \subsetneq Q, \operatorname{ht}(P \cap S) \leq \operatorname{ht}(P)\}.$$

(4) Item 3 of Theorem 7.14 implies that for each prime ideal Q of $\mathcal{F}_{\min}$ there exist prime ideals P_1 and P_2 of S with $P_1 \subsetneq P_2$ such that Q is minimal over both P_1T and P_2T.

Corollary 7.16 is immediate from Theorem 7.14.

COROLLARY 7.16. *Let k be a field, let $z_1, \ldots, z_n$ be indeterminates over k and let $f, g \in k[z_1, \ldots, z_n]$ be algebraically independent over k. Consider the embedding $\varphi : S := k[f, g] \hookrightarrow T := k[z_1, \ldots, z_n]$. Assume that the associated Jacobian ideal J is nonzero.*[3] *Then*

(1) $\mathcal{F}_{min} \subseteq \{$*minimal primes Q of J with* $\operatorname{ht}(Q \cap S) > \operatorname{ht} Q = 1$ $\}$.
(2) *φ is flat* $\iff$ *for every height-one prime ideal $Q \in \operatorname{Spec} T$ such that $J \subseteq Q$ we have* $\operatorname{ht}(Q \cap S) \leq 1$.
(3) *If* $\operatorname{ht} J \geq 2$, *then φ is flat.*

REMARK 7.17. In the case where k is algebraically closed, another argument can be used for Corollary 7.16.3: Each height-one prime ideal $Q \in \operatorname{Spec} T$ has the form $Q = hT$ for some polynomial $h \in T$. If φ is not flat, then there exists a prime ideal Q of T of height one, such that $\operatorname{ht}(Q \cap S) = 2$. Then $Q \cap S$ has the form $(f - a, g - b)S$, where $a, b \in k$. Thus $f - a = f_1 h$ and $g - b = g_1 h$

[3]This is automatic if the field k has characteristic zero.

for some polynomials $f_1, g_1 \in T$. Now the Jacobian ideal J of f, g is the same as the Jacobian ideal of $f - a, g - b$, and an easy computation shows that $J \subseteq hT$. Therefore $\operatorname{ht} J \leq 1$.

EXAMPLES 7.18. Let k be a field of characteristic different from 2 and let x, y, z be indeterminates over k.

(1) With $f = x$ and $g = xy^2 - y$, consider $S := k[f, g] \overset{\varphi}{\hookrightarrow} T := k[x, y]$. Then $J = (2xy - 1)T$. Since $\operatorname{ht}((2xy - 1)T \cap S) = 1$, φ is flat by Corollary 7.16.2. But φ is not smooth, since J defines the nonsmooth locus and $J \neq T$; see Theorem 7.14.1. Here we have $J \subsetneq F = T$.

(2) With $f = x$ and $g = yz$, consider $S := k[f, g] \overset{\varphi}{\hookrightarrow} T := k[x, y, z]$. Then $J = (y, z)T$. Since $\operatorname{ht} J \geq 2$, φ is flat by Corollary 7.16.3. Again φ is not smooth since $J \neq T$.

(3) The examples given in items 1 and 2 may also be described by taking $R = k[x]$. In item 1, we then have $S := R[xy^2 - y] \hookrightarrow R[y] =: T$. The Jacobian $J = (2xy - 1)T$ is the same but is computed now as just a derivative. In item 2, we have $S := R[yz] \hookrightarrow R[y, z] =: T$. The Jacobian $J = (y, z)T$ is now computed by taking the partial derivatives $\frac{\partial(yz)}{\partial y}$ and $\frac{\partial(yz)}{\partial z}$.

(4) Let $R = k[x]$ and $S = R[xyz] \hookrightarrow R[y, z] =: T$. Then $J = (xz, xy)T$. Thus J has two minimal primes xT and $(y, z)T$. Notice that $xT \cap S = (x, xyz)S$ is a prime ideal of S of height two, while $(y, z)T \cap S$ has height one. Therefore $J \subsetneq F = xT$.

(5) Let $R = k[x]$ and $S = R[xy + xz] \hookrightarrow R[y, z] =: T$. Then $J = xT$. The map φ is not flat, since $xT \cap S = (x, xy + xz)S$.

(6) Let $R = k[x]$ and $S = R[xy + z^2] \hookrightarrow R[y, z] =: T$. Then $J = (y, z)T$. Hence $S \hookrightarrow T$ is flat but not smooth.

(7) Let $R = k[x]$ and $S = R[xy + z] \hookrightarrow R[y, z] =: T$. Then $J = T$. Hence $S \hookrightarrow T$ is a smooth map.

COROLLARY 7.19. *With the notation of Theorem 7.14, we have*

(1) *If $Q \in \mathcal{F}_{min}$, then Q is a nonmaximal prime of T.*
(2) $\mathcal{F}_{min} \subseteq \{Q \in \operatorname{Spec} T : J \subseteq Q, \dim(T/Q) \geq 1$ *and* $\operatorname{ht}(Q \cap S) > \operatorname{ht} Q\}$.
(3) *φ is flat* $\iff \operatorname{ht}(Q \cap S) \leq \operatorname{ht}(Q)$ *for every nonmaximal $Q \in \operatorname{Spec}(T)$ with $J \subseteq Q$.*
(4) *If $\dim R = d$ and $\operatorname{ht} J \geq d + m$, then φ is flat.*

PROOF. For item 1, suppose $Q \in \mathcal{F}_{\min}$ is a maximal ideal of T. Then $\operatorname{ht} Q < \operatorname{ht}(Q \cap S)$ by Theorem 7.14.3. By localizing at $R \setminus (R \cap Q)$, we may assume that R is local with maximal ideal $Q \cap R := \mathbf{m}$. Since Q is maximal, T/Q is a field finitely generated over $R/\mathbf{m}$. By the Hilbert Nullstellensatz [**103**, Theorem 5.3], T/Q is algebraic over $R/\mathbf{m}$ and $\operatorname{ht} Q = \operatorname{ht}(\mathbf{m}) + n$. It follows that $Q \cap S = P$ is maximal in S and $\operatorname{ht} P = \operatorname{ht}(\mathbf{m}) + m$. The algebraic independence hypothesis for the f_i implies that $m \leq n$, and therefore that $\operatorname{ht} P \leq \operatorname{ht} Q$. This contradiction proves item 1. Item 2 follows from Theorem 7.14.3 and item 1.

Item 3 follows from Theorem 7.14.5 and item 1, and item 4 follows from Theorem 7.14.6. □

As an immediate corollary to Theorem 7.14 and Corollary 7.19, we have:

COROLLARY 7.20. *Let R be a Noetherian ring, let $z_1, \ldots, z_n$ be indeterminates over R and let $f_1, \ldots, f_m \in R[z_1, \ldots, z_n]$ be algebraically independent over R. Consider the embedding $\varphi : S := R[f_1, \ldots, f_m] \hookrightarrow T := R[z_1, \ldots, z_n]$, let J be the Jacobian ideal of φ and let F be the radical ideal that describes the nonflat locus of φ as in Definition 7.12.2. Then $J \subseteq F$ and either $F = T$, that is, φ is flat, or $\dim(T/Q) \geq 1$, for each $Q \in \operatorname{Spec}(T)$ that is minimal over F.*

7.3. Applications to polynomial extensions

Proposition 7.21 concerns the behavior of the extension $\varphi : S \hookrightarrow T$ with respect to prime ideals of R.

PROPOSITION 7.21. *Let R be a commutative ring, let $z_1, \ldots, z_n$ be indeterminates over R, and let $f_1, \ldots, f_m \in R[z_1, \ldots, z_n]$ be algebraically independent over R. Consider the embedding $\varphi : S := R[f_1, \ldots, f_m] \hookrightarrow T := R[z_1, \ldots, z_n]$.*

1. *If $\mathbf{p} \in \operatorname{Spec} R$ and $\varphi_{\mathbf{p}T} : S \to T_{\mathbf{p}T}$ is flat, then $\mathbf{p}S = \mathbf{p}T \cap S$ and the images $\overline{f_i}$ of the f_i in $T/\mathbf{p}T \cong (R/\mathbf{p})[z_1, \ldots, z_n]$ are algebraically independent over $R/\mathbf{p}$.*
2. *If φ is flat, then for each $\mathbf{p} \in \operatorname{Spec}(R)$ we have $\mathbf{p}S = \mathbf{p}T \cap S$ and the images $\overline{f_i}$ of the f_i in $T/\mathbf{p}T \cong (R/\mathbf{p})[z_1, \ldots, z_n]$ are algebraically independent over $R/\mathbf{p}$.*

PROOF. Item 2 follows from item 1, so it suffices to prove item 1. Assume that $T_{\mathbf{p}T}$ is flat over S. Then $\mathbf{p}T \neq T$ and it follows from [**103**, Theorem 9.5] that $\mathbf{p}T \cap S = \mathbf{p}S$. If the $\overline{f_i}$ were algebraically dependent over $R/\mathbf{p}$, then there exist indeterminates $t_1, \ldots, t_m$ and a polynomial $G \in R[t_1, \ldots, t_m] \setminus \mathbf{p}R[t_1, \ldots, t_m]$ such that $G(f_1, \ldots, f_m) \in \mathbf{p}T$. This implies $G(f_1, \ldots, f_m) \in \mathbf{p}T \cap S$. But $f_1, \ldots, f_m$ are algebraically independent over R and $G(t_1, \ldots, t_m) \notin \mathbf{p}R[t_1, \ldots, t_m]$ implies $G(f_1, \ldots, f_m) \notin \mathbf{p}S = \mathbf{p}T \cap S$, a contradiction. □

PROPOSITION 7.22. *Let R be a Noetherian integral domain containing a field of characteristic zero. Let $z_1, \ldots, z_n$ be indeterminates over R and let $f_1, \ldots, f_m \in R[z_1, \ldots, z_n]$ be algebraically independent over R. Consider the embedding $\varphi : S := R[f_1, \ldots, f_m] \hookrightarrow T := R[z_1, \ldots, z_n]$. Let J be the associated Jacobian ideal and let F be the reduced ideal of T defining the nonflat locus of φ . Then*

1. *If $\mathbf{p} \in \operatorname{Spec} R$ and $J \subseteq \mathbf{p}T$, then $\varphi_{\mathbf{p}T} : S \to T_{\mathbf{p}T}$ is not flat. Thus we also have $F \subseteq \mathbf{p}T$.*
2. *If the embedding $\varphi : S \hookrightarrow T$ is flat, then for every $\mathbf{p} \in \operatorname{Spec} R$ we have $J \nsubseteq \mathbf{p}T$.*

PROOF. Item 2 follows from item 1, so it suffices to prove item 1. Let $\mathbf{p} \in \operatorname{Spec} R$ with $J \subseteq \mathbf{p}T$, and suppose $\varphi_{\mathbf{p}T}$ is flat. Let $\overline{f_i}$ denote the image of f_i in $T/\mathbf{p}T$. Consider

$$\overline{\varphi} : \overline{S} := (R/\mathbf{p})[\overline{f_1}, \ldots, \overline{f_m}] \to \overline{T} := (R/\mathbf{p})[z_1, \ldots, z_n].$$

By Proposition 7.21, $\overline{f_1}, \ldots, \overline{f_m}$ are algebraically independent over $\overline{R} := R/\mathbf{p}$. Since the Jacobian ideal commutes with homomorphic images, the Jacobian ideal of $\overline{\varphi}$ is zero. Thus for each $Q \in \operatorname{Spec} \overline{T}$ the map $\overline{\varphi}_Q : \overline{S} \to \overline{T}_Q$ is not smooth. But taking $Q = (0)$ gives $\overline{T}_Q$ is a field separable over the field of fractions of $\overline{S}$ and hence $\overline{\varphi}_Q$ is a smooth map. This contradiction completes the proof. □

Theorem 7.23 follows from [**126**, Proposition 2.1] in the case of one indeterminate z, so in the case where $T = R[z]$.

THEOREM 7.23. *Let R be a Noetherian integral domain, let $z_1, \ldots, z_n$ be indeterminates over R, and let $T = R[z_1, \ldots z_n]$. Suppose $f \in T \setminus R$. Then the following are equivalent:*

(1) *$R[f] \to T$ is flat.*
(2) *For each prime ideal $\mathbf{q}$ of R, we have $\mathbf{q}T \cap R[f] = \mathbf{q}R[f]$.*
(3) *For each maximal ideal $\mathbf{q}$ of R, we have $\mathbf{q}T \cap R[f] = \mathbf{q}R[f]$.*
(4) *The nonconstant coefficients of f generate the unit ideal of R.*
(5) *$R[f] \to T$ is faithfully flat.*

PROOF. Since $f \in T \setminus R$ and R is an integral domain, the ring $R[f]$ is a polynomial ring in the indeterminate f over R. Thus the map $R \to R[f]$ is flat with regular fibers. Hence Corollary 7.6 implies that $R[f] \hookrightarrow T$ is flat $\iff$ for each $Q \in \operatorname{Spec} T$ we have $\operatorname{ht} Q \geq \operatorname{ht}(Q \cap R[f])$. Let $\mathbf{q} := Q \cap R$. We have $\operatorname{ht} \mathbf{q} = \operatorname{ht}(\mathbf{q}R[f]) = \operatorname{ht}(\mathbf{q}T)$. Thus $R[f] \hookrightarrow T$ is flat implies for each $\mathbf{q} \in \operatorname{Spec} R$ that $\mathbf{q}T \cap R[f] = \mathbf{q}R[f]$. Moreover, if $P := Q \cap R[f]$ properly contains $\mathbf{q}R[f]$, then $\operatorname{ht} P = 1 + \operatorname{ht} \mathbf{q}$, while if Q properly contains $\mathbf{q}T$, then $\operatorname{ht} Q \geq 1 + \operatorname{ht} \mathbf{q}$. Therefore (1) $\iff$ (2) follows from Corollary 7.6. It is obvious that (2) $\implies$ (3).

(3) $\implies$ (4): Let $a \in R$ be the constant term of f. If the nonconstant coefficients of f are contained in a maximal ideal $\mathbf{q}$ of R, then $f - a \in \mathbf{q}T \cap R[f]$. Since R is an integral domain, the element $f - a$ is transcendental over R and $f - a \notin \mathbf{q}R[f]$ since $R[f]/\mathbf{q}R[f]$ is isomorphic to the polynomial ring $(R/\mathbf{q})[x]$. Therefore $\mathbf{q}T \cap R[f] \neq \mathbf{q}R[f]$ if the nonconstant coefficients of f are in $\mathbf{q}$.

(4) $\implies$ (2): Let $\mathbf{q} \in \operatorname{Spec} R$ and consider the map

$$R[f] \otimes_R \tfrac{R}{\mathbf{q}} = \tfrac{R[f]}{\mathbf{q}R[f]} \xrightarrow{\varphi} T \otimes_R \tfrac{R}{\mathbf{q}} = \tfrac{T}{\mathbf{q}T} \cong (\tfrac{R}{\mathbf{q}})[z_1, \ldots, z_n]. \tag{7.23.a}$$

Since the nonconstant coefficients of f generate the unit ideal of R, the image of f in $(R/\mathbf{q})[z_1, \ldots, z_n]$ has positive degree. This implies that φ is injective and $\mathbf{q}T \cap R[f] = \mathbf{q}R[f]$.

This completes a proof that items (1), (2), (3) and (4) are equivalent. To show that these equivalent statements imply (5), it suffices to show for $P \in \operatorname{Spec}(R[f])$ that $PT \neq T$. Let $\mathbf{q} = P \cap R$, and let $\kappa(\mathbf{q})$ denote the field of fractions of $R/\mathbf{q}$. Let $\overline{f}$ denote the image of f in $R[f]/\mathbf{q}R[f]$. Then $R[f]/\mathbf{q}R[f] \cong (R/\mathbf{q})[\overline{f}]$, a polynomial ring in one variable over $R/\mathbf{q}$, since the nonconstant coefficients of f are not in $\mathbf{q}$. Tensoring the map φ of equation 7.23.a with $\kappa(\mathbf{q})$ gives an embedding of the polynomial ring $\kappa(\mathbf{q})[\overline{f}]$ into $\kappa(\mathbf{q})[z_1, \ldots, z_n]$. The image of P in $\kappa(\mathbf{q})[\overline{f}]$ is either zero or a maximal ideal of $\kappa(\mathbf{q})[\overline{f}]$. In either case, its extension to $\kappa(\mathbf{q})[z_1, \ldots, z_n]$ is a proper ideal. Therefore $PT \neq T$. It is obvious that (5) $\implies$ (1), and so this completes the proof of Theorem 7.23. □

COROLLARY 7.24. *Let R be a Noetherian integral domain, let $z_1, \ldots, z_n$ be indeterminates over R, and let $T = R[z_1, \ldots z_n]$. Suppose $f \in T \setminus R$. Let L denote the ideal of R generated by the nonconstant coefficients of f. Then LT defines the nonflat locus of the map $R[f] \hookrightarrow T$.*

PROOF. Let $Q \in \operatorname{Spec} T$ and let $\mathbf{q} = Q \cap R$. Tensoring the map $R[f] \to T$ with $R_{\mathbf{q}}$, we see that $R[f] \hookrightarrow T_Q$ is flat if and only if $R_{\mathbf{q}}[f] \hookrightarrow T_Q$ is flat. Consider the extensions:

$$R_{\mathbf{q}}[f] \overset{\theta}{\hookrightarrow} R_{\mathbf{q}}[z_1, \ldots, z_n] := T_{\mathbf{q}} \overset{\psi}{\hookrightarrow} T_Q.$$

Since ψ is a localization the composite $\psi \circ \theta$ is flat if θ is flat.

Assume $L \nsubseteq Q$. Then $L \nsubseteq \mathbf{q}$, and so $LR_{\mathbf{q}} = R_{\mathbf{q}}$. By (4) $\implies$ (1) of Theorem 7.23, we have $R[f] \hookrightarrow T_Q$ is flat.

Assume $L \subseteq Q$. Then $L \subseteq \mathbf{q}$, and we have $f - a \in \mathbf{q}T_Q \cap R_{\mathbf{q}}[f]$, where a is the constant term of f. However, $R_{\mathbf{q}}[f] = R_{\mathbf{q}}[f - a]$ is a polynomial ring in one variable over $R_{\mathbf{q}}$ since $f - a$ is transcendental over $R_{\mathbf{q}}$. Therefore $f - a \notin \mathbf{q}R_{\mathbf{q}}[f]$. It follows that $\mathbf{q}R_{\mathbf{q}}[f] \neq \mathbf{q}T_{\mathbf{q}} \cap R_{\mathbf{q}}[f]$. By Theorem 7.23, $R_{\mathbf{q}}[f] \overset{\theta}{\hookrightarrow} T_{\mathbf{q}}$ is not flat. Hence $R_{\mathbf{q}}[f] \overset{\psi\circ\theta}{\hookrightarrow} T_Q$ is not flat. We conclude that L defines the nonflat locus of the map $R[f] \hookrightarrow T$. □

REMARK 7.25. A different proof that (4) $\implies$ (1) in Theorem 7.23 is as follows: Let v be another indeterminate and consider the commutative diagram

$$\begin{array}{ccc} R[v] & \longrightarrow & T[v] = R[z_1, \ldots, z_n, v] \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ R[f] & \xrightarrow{\varphi} & \frac{R[z_1,\ldots,z_n,v]}{(v - f(z_1,\ldots,z_n))}. \end{array}$$

where π maps $v \to f$ and π' is the canonical quotient homomorphism. By [**101**, Corollary 2, p. 152] or [**103**, Theorem 22.6 and its Corollary, p. 177], φ is flat if the coefficients of $f - v$ generate the unit ideal of $R[v]$. Moreover, the coefficients of $f - v$ as a polynomial in $z_1, \ldots, z_n$ with coefficients in $R[v]$ generate the unit ideal of $R[v]$ if and only if the nonconstant coefficients of f generate the unit ideal of R. For if $a \in R$ is the constant term of f and $a_1, \ldots, a_r$ are the nonconstant coefficients of f, then $(a_1, \ldots, a_r)R = R$ clearly implies that $(a - v, a_1, \ldots, a_r)R[v] = R[v]$. On the other hand, if $(a - v, a_1, \ldots, a_r)R[v] = R[v]$, then setting $v = a$ implies that $(a_1, \ldots, a_r)R = R$.

We observe in Proposition 7.26 that item 1 implies item 4 of Theorem 7.23 also holds for more than one polynomial f; see also [**126**, Theorem 3.8] for a related result concerning flatness.

PROPOSITION 7.26. *Let $z_1, \ldots, z_n$ be indeterminates over an integral domain R. Let $f_1, \ldots, f_m$ be polynomials in $R[z_1, \ldots, z_n] := T$ that are algebraically independent over $\mathcal{Q}(R)$. If the inclusion map $\varphi : S := R[f_1, \ldots f_m] \to T$ is flat, then the nonconstant coefficients of each of the f_i generate the unit ideal of R.*

PROOF. The algebraic independence of the f_i implies that the inclusion map $R[f_i] \hookrightarrow R[f_1, \ldots, f_m]$ is flat, for each i with $1 \leq i \leq m$. If $S \longrightarrow T$ is flat, then so is the composition $R[f_i] \longrightarrow S \longrightarrow T$, and the statement follows from Theorem 7.23. □

Exercises

(1) Let k be an algebraically closed field of characteristic zero and let T denote the polynomial ring $k[x]$. Let $f \in T$ be a polynomial of degree $d \geq 2$ and let $S := k[f]$.

(i) Prove that the map $S \hookrightarrow T$ is free and hence flat.

(ii) Prove that the prime ideals $Q \in \operatorname{Spec} T$ for which $S \to T_Q$ is not a regular map are precisely the primes Q such that the derivative $\frac{df}{dx} \in Q$.

(iii) Deduce that $S \hookrightarrow T$ is not smooth.

(2) With $S = k[x, xy^2 - y] \hookrightarrow T = k[x, y]$ and $J = (2xy-1)T$ as in Examples 7.18.1, prove that $\operatorname{ht}(J \cap S) = 1$.

Suggestion. Show that $J \cap S \cap k[x] = (0)$ and use that, for A an integral domain, prime ideals of the polynomial ring $A[y]$ that intersect A in (0) are in one-to-one correspondence with prime ideals of $K[y]$, where $K = \mathcal{Q}(A)$ is the field of fractions of A.

(3) Let $z_1, \ldots, z_n$ be indeterminates over a ring R, and let $T = R[z_1, \ldots z_n]$. Fix an element $f \in T \setminus R$. Modify the proof of (3) implies (4) of Theorem 7.23 to prove that $\mathbf{q}T \cap R[f] = \mathbf{q}R[f]$ for each maximal ideal $\mathbf{q}$ of R implies that the nonconstant coefficients of f generate the unit ideal of R without the assumption that the ring R is an integral domain.

Suggestion. Assume that the nonconstant coefficients of f are contained in a maximal ideal $\mathbf{q}$ of R. Observe that one may assume that f as a polynomial in $R[z_1, \ldots, z_n]$ has zero as its constant term and that the ring R is local with maximal ideal $\mathbf{q}$. Let M be a monomial in the support of f of minimal total degree and let $b \in R$ denote the coefficient of M for f. Then b is nonzero, but $f \in \mathbf{q}R[f]$ implies that $b \in \mathbf{q}$ and this implies, by Nakayama's lemma, that $b = 0$.

(4) Let k be a field and let $T = k[[u, v, w, z]]$ be the formal power series ring over k in the variables u, v, w, z. Define a k-algebra homomorphism φ of T into the formal power series ring $k[[x, y]]$ by defining

$$\varphi(u) \;=\; x^4, \quad \varphi(v) \;=\; x^3y, \quad \varphi(w) \;=\; xy^3, \quad \varphi(z) \;=\; y^4.$$

Let $P = \ker(\varphi)$ and let $I = (v^3 - u^2w,\ w^3 - z^2v)T$. Notice that $I \subset P$, and that the ring $\varphi(T) = k[[x^4, x^3y, xy^3, y^4]]$ is not Cohen-Macaulay. Let $S = T/I$, and let $R = k[[u, z]] \subset T$.

(a) Prove that $P \cap R = (0)$.
(b) Prove that the ring S is Cohen-Macaulay and a finite free R-module.
(c) Prove that PS is a minimal prime of S and S/PS is not flat over R.

Suggestion. To see that S is module finite over R, observe that

$$\frac{S}{(u, z)S} \;=\; \frac{T}{(u, z, v^3 - u^2w, w^3 - z^2v)T},$$

and the ideal $(u, z, v^3 - u^2w, w^3 - z^2v)T$ is primary for the maximal ideal of T. Hence by Theorem 3.9, S is a finite R-module.

(5) Let k be a field and let $A = k[x, xy] \subset k[x, y] = B$, where x and y are indeterminates. Let $R = k[x] + (1 - xy)B$.

(a) Prove that R is a proper subring of B that contains A.
(b) Prove that B is a flat R-module.
(c) Prove that B is contained in a finitely generated R-module.
(d) Prove that R is not a Noetherian ring.
(e) Prove that $P = (1-xy)B$ is a prime ideal of both R and B with $R/P \cong k[x]$ and $B/P \cong R[x, 1/x]$.
(f) Prove that the map $\operatorname{Spec} B \to \operatorname{Spec} R$ is one-to-one but not onto.

Question. What prime ideals of R are not finitely generated?

CHAPTER 8

Height-one primes and limit-intersecting elements

Let z be a nonzero nonunit of a normal Noetherian integral domain R and let R^* denote the (z)-adic completion of R. As in Construction 5.3, we consider in this chapter the structure of a subring A of R^* of the form $A := \mathcal{Q}(R)(\tau_1, \tau_2, \dots \tau_s) \cap R^*$, where $\tau_1, \tau_2, \dots, \tau_s \in zR^*$ are algebraically independent elements over R and every nonzero element of $R[\tau_1, \tau_2, \dots, \tau_s]$ is regular on R^*.

If the intersection ring A can be expressed as a directed union B of localized polynomial extension rings of R as in Section 5.2, then the computation of A is easier. Recall that $\tau_1, \tau_2, \dots, \tau_s$ are called *limit-intersecting* for A if the ring A is such a directed union; see Definition 5.10.

The main result of Section 8.1 is Weak Flatness Theorem 8.7. In this theorem we give criteria for $\tau_1, \tau_2, \dots, \tau_s$ to be limit-intersecting for A. In Section 8.2 with the setting of extensions of Krull domains, we continue to analyze the properties of height-one primes considered in Section 8.1.

Weak Flatness Theorem 8.7 is used in Examples 10.9 to obtain a family of examples where the approximating ring B is equal to the intersection ring A and is not Noetherian.

8.1. The limit-intersecting condition

In this section we prove the Weak Flatness Theorem. This theorem gives conditions in order that the intersection domain A be equal to the approximation domain B; that is, the construction is limit-intersecting. For this purpose, we consider the following properties of an extension of commutative rings:

Definitions 8.1. Let $S \hookrightarrow T$ be an extension of commutative rings.

(1) We say that the extension $S \hookrightarrow T$ is *weakly flat*, or that T is *weakly flat* over S, if every height-one prime ideal P of S with $PT \neq T$ satisfies $PT \cap S = P$.

(2) We say that the extension $S \hookrightarrow T$ is *height-one preserving*, or that T is a *height-one preserving* extension of S, if for every height-one prime ideal P of S with $PT \neq T$ there exists a height-one prime ideal Q of T with $PT \subseteq Q$.

(3) For $d \in \mathbb{N}$, we say that $\varphi : S \hookrightarrow T$ satisfies LF_d (*locally flat in height d*) if, for each $P \in \operatorname{Spec} T$ with $\operatorname{ht} P \leq d$, the composite map $S \to T \to T_P$ is flat.

REMARK 8.2. Let $\varphi : S \hookrightarrow T$ be an extension of commutative rings, and let $P \in \operatorname{Spec} T$. With $Q := P \cap S$, the composite map $S \to T \to T_P$ factors through S_Q, and the map $S \to T_P$ is flat if and only if the map $S_Q \to T_P$ is faithfully flat.

PROPOSITION 8.3. *Let $S \hookrightarrow T$ be an extension of commutative rings where S is a Krull domain.*

(1) *If every nonzero element of S is regular on T and each height-one prime ideal of S is contracted from T, then $S = T \cap \mathcal{Q}(S)$.*
(2) *If $S \hookrightarrow T$ is a birational extension and each height-one prime of S is contracted from T, then $S = T$.*
(3) *If T is a Krull domain and $T \cap \mathcal{Q}(S) = S$, then each height-one prime of S is the contraction of a height-one prime of T, and the extension $S \hookrightarrow T$ is height-one preserving and weakly flat.*

PROOF. Item 1 follows from item 2. For item 2, recall from Remark 2.8.1 that $S = \cap\{S_{\mathbf{p}} \,|\, \mathbf{p} \text{ is a height-one prime ideal of } S\}$. We show that $T \subseteq S_{\mathbf{p}}$, for each height-one prime ideal of S. Since $\mathbf{p}$ is contracted from T, there exists a prime ideal $\mathbf{q}$ of T such that $\mathbf{q} \cap S = \mathbf{p}$; see Exercise 9 of Chapter 2. Then $S_{\mathbf{p}} \subseteq T_{\mathbf{q}}$ and $T_{\mathbf{q}}$ birationally dominates $S_{\mathbf{p}}$. Since $S_{\mathbf{p}}$ is a DVR, we have $S_{\mathbf{p}} = T_{\mathbf{q}}$. Therefore $T \subseteq S_{\mathbf{p}}$, for each $\mathbf{p}$. It follows that $T = S$.

For item 3, since T is a Krull domain, Remark 2.8.1 implies that

$$T = \bigcap\{T_{\mathbf{q}} \,|\, \mathbf{q} \text{ is a height-one prime ideal of } T\}.$$

Hence

$$S = T \cap \mathcal{Q}(S) = \bigcap\{T_{\mathbf{q}} \cap \mathcal{Q}(S) \,|\, \mathbf{q} \text{ is a height-one prime ideal of } T\}.$$

Since each $T_{\mathbf{q}}$ is a DVR, Remark 2.1 implies that $T_{\mathbf{q}} \cap \mathcal{Q}(S)$ is either the field $\mathcal{Q}(S)$ or a DVR birational over S. By Remark 2.8.1, for each height-one prime $\mathbf{p}$ of S, the localization $S_{\mathbf{p}}$ is a DVR of the form $T_{\mathbf{q}} \cap \mathcal{Q}(S)$. It follows that each height-one prime ideal $\mathbf{p}$ of S is contracted from a height-one prime ideal $\mathbf{q}$ of T, and that T is height-one preserving and weakly flat over S. □

Corollary 8.4 demonstrates the relevance of the weak flatness property for an extension of a Krull domain.

COROLLARY 8.4. *Let $S \hookrightarrow T$ be an extension of commutative rings where S is a Krull domain such that every nonzero element of S is regular on T and $PT \neq T$ for every height-one prime ideal P of S.*

(i) *If $S \hookrightarrow T$ is weakly flat, then $S = \mathcal{Q}(S) \cap T$.*
(ii) *If T is Krull, then T is weakly flat over $S \iff S = \mathcal{Q}(S) \cap T$. Moreover, in this setting, these equivalent conditions imply that $S \hookrightarrow T$ is height-one preserving.*

PROOF. For item i, each height-one prime ideal of S is contracted from T. Thus by Proposition 8.3.1, $S = \mathcal{Q}(S) \cap T$.

For item ii, we apply Proposition 8.3.3. □

REMARKS 8.5. Let $S \hookrightarrow T$ be an extension of commutative rings.

(a) If $S \hookrightarrow T$ is flat, then $S \hookrightarrow T$ is weakly flat; see [**103**, Theorem 9.5].

(b) Let G be a multiplicative system in S consisting of units of T. Then $S \hookrightarrow G^{-1}S$ is flat and every height-one prime ideal of $G^{-1}S$ is the extension of a height-one prime ideal of S. Thus $S \hookrightarrow T$ is weakly flat $\iff G^{-1}S \hookrightarrow T$ is weakly flat.

REMARKS 8.6. Let $S \hookrightarrow T$ be an extension of Krull domains.

(a) If $S \hookrightarrow T$ is flat, then $S \hookrightarrow T$ is height-one preserving and satisfies PDE. See Definition 2.10 and [**17**, Chapitre 7, Proposition 15, page 19].
(b) Let G be a multiplicative system in S consisting of units of T. It follows as in Remarks 8.5.b that:
  (i) $S \hookrightarrow T$ is height-one preserving $\iff G^{-1}S \hookrightarrow T$ is height-one preserving.
  (ii) $S \hookrightarrow T$ satisfies PDE $\iff G^{-1}S \hookrightarrow T$ satisfies PDE.
(c) If each height-one prime of S is the radical of a principal ideal, in particular, if S is a UFD, then the extension $S \hookrightarrow T$ is height-one preserving. To see this, let P be a height-one prime of S and suppose that P is the radical of the principal ideal xS. Then $PT \neq T$ if and only if xT is a proper principal ideal of T. Every proper principal ideal of a Krull domain is contained in a height-one prime. Hence if $PT \neq T$, then PT is contained in a height-one prime of T.

With these results and remarks in hand, we return to the investigation of the structure of the intersection domain A mentioned in the introduction to this chapter: When does A equal the approximation domain B? We first consider the intersection domain A of Inclusion Construction 5.3 and the approximation ring B of Section 5.2. We show in Weak Flatness Theorem 8.7 that, if the base ring R of the construction is a normal Noetherian domain and the extension

$$R[\tau_1, \ldots, \tau_s] \hookrightarrow R^*[1/z]$$

is weakly flat, then the intersection domain A is equal to the approximation domain B; that is, $\tau_1, \ldots, \tau_s$ are limit-intersecting in the sense of Definition 5.10.

WEAK FLATNESS THEOREM 8.7. (Inclusion Version) *Let R be a normal Noetherian integral domain and let $z \in R$ be a nonzero nonunit. Let R^* denote the (z)-adic completion of R and let $\tau_1, \ldots, \tau_s \in R^*$ be algebraically independent over R. Assume that every nonzero element of the polynomial ring $R[\tau_1, \ldots, \tau_s]$ is regular on R^*. Let $A = \mathcal{Q}(R)(\tau_1, \ldots, \tau_s) \cap R^*$ and let B be the approximation domain defined in Section 5.2. Consider the following statements:*

(1) *$A = B$; that is, $\tau_1, \ldots, \tau_s$ are limit-intersecting in the sense of Definition 5.10.*
(2) *The extension $R[\tau_1, \ldots, \tau_s] \hookrightarrow R^*[1/z]$ is weakly flat.*
(3) *The extension $B \hookrightarrow R^*[1/z]$ is weakly flat.*
(4) *The extension $B \hookrightarrow R^*$ is weakly flat.*

Then

(a) *Items 2, 3, and 4 are equivalent.*
(b) *Item 2 $\implies$ item 1.*
(c) *If R^* is normal, then the four items are equivalent.*

PROOF. For (a), we show item 4 $\implies$ item 3 $\implies$ item 2 $\implies$ item 4. To see that item 4 $\implies$ item 3, we have

$$B \overset{\text{w.f.}}{\hookrightarrow} R^* \overset{\text{flat}}{\hookrightarrow} R^*[1/z].$$

Thus, for a height-one prime ideal P of B with $PR^*[1/z] \neq R^*[1/z]$, we have $PR^* \neq R^*$ and $z \notin P$, and so $PR^*[1/z] \cap B = PR^* \cap B = P$, where the last equality uses $B \overset{\text{w.f.}}{\hookrightarrow} R^*$. Thus item 3 holds.

Item 3 $\implies$ item 2: We have $B \overset{\text{w.f.}}{\hookrightarrow} R^*[1/z]$ implies $B[1/z] \overset{\text{w.f.}}{\hookrightarrow} R^*[1/z]$, by Remarks 8.5.b. By Construction Properties Theorem 5.14.2, $B[1/z]$ is a localization of $R[\tau_1, \ldots, \tau_s]$. Thus, by Remark 8.5.b, we have $R[\tau_1, \ldots, \tau_s] \overset{\text{w.f.}}{\hookrightarrow} R^*[1/z]$.

To see that item 2 $\implies$ item 4, let $P \in \operatorname{Spec} B$ have height one and suppose $PR^* \neq R^*$. If $z \in P$, then, by Construction Properties Theorem 5.14.3, we have $P/zB = PR^*/zR^*$, and so $PR^* \cap B = P$ in this case. Thus we assume $z \notin P$; then $PB[1/z] \cap B = P$.

By assumption, $R[\tau_1, \ldots, \tau_s] \hookrightarrow R^*[1/z]$ is weakly flat. Since $B[1/z]$ is a localization of $R[\tau_1, \ldots, \tau_s]$ and of B, Remark 8.5.b implies that $B[1/z] \hookrightarrow R^*[1/z]$ is weakly flat. Since $PR^*[1/z] \neq R^*[1/z]$, we have $PR^*[1/z] \cap B[1/z] = PB[1/z]$. Thus $PR^* \cap B = P$ and so $B \hookrightarrow R$ is weakly flat, as desired.

We show item 2 $\implies$ item 1: Since B is a Krull domain and the extension $B \hookrightarrow A$ is birational, by Proposition 8.3.2, it suffices to show that every height-one prime ideal $\mathbf{p}$ of B is contracted from A. As in the proof of item 2 $\implies$ item 4, Construction Properties Theorem 5.14.3 implies that each height-one prime of B containing zB is contracted from A.

Let $\mathbf{p}$ be a height-one prime of B that does not contain zB. Consider the prime ideal $\mathbf{q} = R[\tau_1, \ldots, \tau_s] \cap \mathbf{p}$. Since $B[1/z]$ is a localization of the ring $R[\tau_1, \ldots, \tau_s]$, we see that $B_{\mathbf{p}} = R[\tau_1, \ldots, \tau_s]_{\mathbf{q}}$ and so $\mathbf{q}$ has height one in $R[\tau_1, \ldots, \tau_s]$. The weakly flat hypothesis implies $\mathbf{q}R^* \cap R[\tau_1, \ldots, \tau_s] = \mathbf{q}$. Hence there exists a prime ideal $\mathbf{w}$ of R^* with $\mathbf{w} \cap R[\tau_1, \ldots, \tau_s] = \mathbf{q}$. This implies that $\mathbf{w} \cap B = \mathbf{p}$ and thus also $(\mathbf{w} \cap A) \cap B = \mathbf{p}$. Hence every height-one prime ideal of B is the contraction of a prime ideal of A. Thus $A = B$ as desired.

To prove (c), we assume R^* is a normal Noetherian domain. Thus R^* is a Krull domain; see Definition 2.8.1. We prove item 1 $\implies$ item 4: Since $B = A = \mathcal{Q}(B) \cap R^*$, Proposition 8.3 implies the extension $B \hookrightarrow R^*$ is weakly flat. □

8.2. Height-one primes in extensions of Krull domains

We observe in Proposition 8.8 that a weakly flat extension of Krull domains is height-one preserving.

PROPOSITION 8.8. *If $\phi : S \hookrightarrow T$ is a weakly flat extension of Krull domains, then ϕ is height-one preserving. Moreover, for every height-one prime ideal P of S with $PT \neq T$ there is a height-one prime ideal Q of T with $Q \cap S = P$.*

PROOF. Let $P \in \operatorname{Spec} S$ with $\operatorname{ht} P = 1$ be such that $PT \neq T$. Since T is weakly flat over S, we have $PT \cap S = P$. Then $S \setminus P$ is a multiplicatively closed subset of T and $PT \cap (S \setminus P) = \emptyset$. Let Q' be an ideal of T that contains PT and is maximal with respect to $Q' \cap (S \setminus P) = \emptyset$. Then Q' is a prime ideal of T and $Q' \cap S = P$. Let a be a nonzero element of P and let $Q \subseteq Q'$ be a minimal prime divisor of aT. Since T is a Krull domain, Q has height one. We have $a \in Q \cap S$. Hence $(0) \neq Q \cap S \subseteq P$. Since $\operatorname{ht} P = 1$, we have $Q \cap S = P$. □

The height-one preserving condition does *not* imply weak flatness as we demonstrate in Example 8.9.

EXAMPLE 8.9. Let x and y be variables over a field k, let $R = k[[x]][y]_{(x,y)}$ and let $C = k[[x, y]]$. There exists an element $\tau \in \mathbf{n} = (x, y)C$ that is algebraically independent over $\mathcal{Q}(R)$. For any such element τ, let $S = R[\tau]_{(\mathbf{m},\tau)}$. Since R is a UFD, the ring S is also a UFD and the local inclusion map $\varphi : S \hookrightarrow C$ is height-one preserving. There exists a height-one prime ideal P of S such that $P \cap R = 0$. Since the map $S \hookrightarrow C$ is a local map, we have $PC \neq C$. Because φ is height-one preserving, there exists a height-one prime ideal Q of C such that $PC \subseteq Q$. Since C is the $\mathbf{m}$-adic completion $\widehat{R}$ of R and the generic formal fiber of R is zero-dimensional, $\dim(C \otimes_R \mathcal{Q}(R)) = 0$. Hence $Q \cap R \neq 0$. We have $P \subseteq Q \cap S$ and $P \cap R = (0)$. It follows that P is strictly smaller than $Q \cap S$, so $Q \cap S$ has height greater than one. Therefore the extension $\varphi : S \hookrightarrow C$ is not weakly flat.

Proposition 8.10 describes weakly flat and PDE (pas d'éclatment[1]) extensions.

PROPOSITION 8.10. *Let $\varphi : S \hookrightarrow T$ be an extension of Krull domains.*

(1) *φ is weakly flat $\iff$ for every height-one prime ideal $P \in \operatorname{Spec} S$ such that $PT \neq T$ there is a height-one prime ideal $Q \in \operatorname{Spec} T$ with $P \subseteq Q \cap S$ such that the induced map on the localizations*

$$\varphi_Q : S_{Q\cap S} \longrightarrow T_Q$$

is faithfully flat.

(2) *φ satisfies PDE $\iff$ for every height-one prime ideal $Q \in \operatorname{Spec} T$, the induced map on the localizations*

$$\varphi_Q : S_{Q\cap S} \longrightarrow T_Q$$

is faithfully flat.

PROOF. We use in both (1) and (2) that for each height-one prime $P \in \operatorname{Spec} S$ the induced map $\varphi_P : S_P \longrightarrow (S \setminus P)^{-1}T$ is flat since a domain extension of a DVR is always flat by Remark 2.33.3; and φ_P is faithfully flat $\iff$ P does not extend to the whole ring in $(S \setminus P)^{-1}T$, a property that is equivalent to the existence of a prime in T lying over P in S.

For the proof of (1), to see ($\Longleftarrow$), we use that φ_Q a faithfully flat map implies φ_Q satisfies the going-down property; see Remark 2.31.10. Hence $Q \cap S$ is of height one, so $P = Q \cap S$, and thus $PT \cap S = P$. For ($\Longrightarrow$), suppose $P \in \operatorname{Spec} S$ has height one and φ is weakly flat. Then Proposition 8.8 implies the existence of $Q \in \operatorname{Spec} T$ of height one such that $Q \cap S = P$. Since T_Q is a localization of $(S \setminus P)^{-1}T$, we see that φ_Q is faithfully flat.

For the proof of (2), ($\Longrightarrow$) is clear by the remark in the first sentence of the proof, and ($\Longleftarrow$) follows from the fact that a faithfully flat map satisfies the going-down property. □

Proposition 8.11 is an immediate consequence of Proposition 8.10:

PROPOSITION 8.11. *Let $\varphi : S \hookrightarrow T$ be an extension of Krull domains. Then φ satisfies PDE if and only if φ satisfies LF_1.*

[1]See Definition 2.10

We show in Proposition 8.12 that an extension of Krull domains satisfying both the LF_1 condition and the height-one preserving condition is weakly flat. Example 8.13 shows that LF_1 alone does not imply weak flatness.

PROPOSITION 8.12. *Let $S \hookrightarrow T$ be an extension of Krull domains that is height-one preserving and satisfies PDE. Then T is weakly flat over S. That is, if $S \hookrightarrow T$ is height-one preserving and satisfies LF_1, then T is weakly flat over S.*

PROOF. Let $P \in \operatorname{Spec} S$ be such that $\operatorname{ht}(P) = 1$ and $PT \neq T$. Since $S \hookrightarrow T$ is height-one preserving, PT is contained in a prime ideal Q of T of height one. The PDE hypothesis on $S \hookrightarrow T$ implies that $Q \cap S$ has height one. It follows that $Q \cap S = P$, and so $PT \cap T = P$; that is, the extension is weakly flat. The last statement holds by Proposition 8.11. □

Without the assumption that the extension $S \hookrightarrow T$ is height-one preserving, it can happen that the extension satisfies PDE and yet is not weakly flat as we demonstrate in Example 8.13. Since PDE and height-one preserving imply weak flatness, this example also shows that PDE does not imply height-one preserving.

EXAMPLE 8.13. Let X, Y, Z, W be indeterminates over a field k and define

$$S \;:=\; k[x,y,z,w] \;=\; \frac{k[X,Y,Z,W]}{(XY - ZW)} \quad \text{and} \quad T \;:=\; S[\frac{x}{z}].$$

Since $w = \frac{yx}{z}$, the ring $T = k[y, z, \frac{x}{z}]$. Since $\mathcal{Q}(T)$ has transcendence degree 3 over k, the elements $y, z, \frac{x}{z}$ are algebraically independent over k and $T = k[y, z, \frac{x}{z}]$ is a polynomial ring in three variables over k. Let $A = k[X, Y, Z, W]$ and let $F = XY - ZW$. Then $S = A/FA$ and and the partials of F generate a maximal ideal of A. It follows that $S_{\mathbf{p}}$ is regular for each nonmaximal prime ideal $\mathbf{p}$ of S; see for example [**103**, Theorem 30.3]. Since S is Cohen-Macaulay, it follows from Serre's normality theorem [**103**, Theorem 23.8] that S is a normal Noetherian domain. Hence S is a Krull domain. The ideal $P := (y, z)S$ is a height-one prime ideal of S, because it corresponds to the height-one prime ideal $(Y, Z)A/FA$ of A/FA. Since $PT = (y, z)T$ and $(y, z)T \cap S = (y, z, x, w)S$, a maximal ideal of S, the extension $S \hookrightarrow T$ is not weakly flat.

Another way to realize this example is to let r, s, t be indeterminates over the field k, and let $S = k[r, s, rt, st] \hookrightarrow k[r, s, t] = T$. Here we set $r = y, s = z, rt = w$ and $st = x$. Then $P = (r, s)S$. We have

$$\begin{aligned} T \;&=\; \bigcap\{S_Q \mid Q \in \operatorname{Spec} S \;,\; \operatorname{ht} Q = 1 \;\text{ and }\; Q \neq P \;\} \\ &=\; \bigcup_{n=1}^{\infty} (S :_{\mathcal{Q}(S)} P^n) \;=\; S\,[\frac{1}{r}] \;\cap\; S\,[\frac{1}{s}]\;; \end{aligned} \tag{6.2.13.0}$$

for the last equality, see Exercise 3 at the end of this chapter and [**18**]. It is straightforward to see that $T \subseteq \bigcup_{n=1}^{\infty}(S :_{\mathcal{Q}(S)} P^n)$. The reverse inclusion follows because $\operatorname{ht} PT > 1$. To see the other equality in Equation 8.13.0, we use the uniqueness of the family of essential valuation rings of the Krull domain S and that an intersection of localizations of S is again a Krull domain for which the family of essential valuation rings is a subset of the family of essential valuation rings for S; see Definition 2.8.2. Therefore T is an intersection of localizations of S. Thus the extension $S \hookrightarrow T$ satisfies PDE.

REMARKS 8.14. (1) By Proposition 8.8, an injective map of Krull domains that is weakly flat is also height-one preserving. Thus the equivalent conditions of Theorem 8.7 imply that $B \hookrightarrow R^*$ is height-one preserving.

(2) If the ring B in Theorem 8.7 is Noetherian, then, by Noetherian Flatness Theorem 6.3, $A = B$. Since flatness implies weak flatness, the equivalent conclusions b, c and d of Theorem 8.7.

(3) Theorem 10.7 of Chapter 10 yields examples where the constructed rings A and B are equal, but are not Noetherian. The limit-intersecting property holds for these examples. These examples are described in Examples 10.9.

(4) As we note in Remark 20.37, Examples 20.34 and 20.36 give extensions of Krull domains that are weakly flat but do not satisfy PDE.

QUESTION 8.15. Let $(C, \mathbf{n})$ be a complete Noetherian local domain that dominates a quasilocal Krull domain $(D, \mathbf{m})$. Assume that the inclusion map $D \hookrightarrow C$ is height-one preserving, and that $\tau \in \mathbf{n}$ is algebraically independent over D. Does it follow that the local inclusion map $\varphi : \ S := D[\tau]_{(\mathbf{m},\tau)} \hookrightarrow C$ is height-one preserving?

DISCUSSION 8.16. If D has torsion divisor class group, then S also has torsion divisor class group and by item c of Remark 8.6, the extension $S \hookrightarrow C$ is height-one preserving, and so the answer to Question 8.15 is affirmative in this case. To consider the general case, let P be a height-one prime ideal of S that is not assumed to be the radical of a principal ideal. One may then consider the following cases:

Case (i): If $\mathrm{ht}(P \cap D) = 1$, then $P = (P \cap D)S$. Since $D \hookrightarrow C$ is height-one preserving, $(P \cap D)C \subseteq Q$, for some height-one prime ideal Q of C. Then $PC = (P \cap D)SC \subseteq Q$ as desired.

Case (ii): Suppose $P \cap D = (0)$. Let U denote the multiplicative set of nonzero elements of D. Let t be an indeterminate over D and let $S_1 = D[t]_{(\mathbf{m},t)}$. Consider the following commutative diagram where the map from S_1 to S is the D-algebra isomorphism taking t to τ and λ is the extension mapping $C[[t]]$ onto C.

$$\begin{array}{ccccccc}
 & & U^{-1}S_1 & \xrightarrow{\subseteq} & U^{-1}C[t]_{(\mathbf{n},t)} & & \\
 & & \cup\uparrow & & \cup\uparrow & & \\
D & \xrightarrow{\subseteq} & S_1 = D[t]_{(\mathbf{m},t)} & \xrightarrow{\subseteq} & C[t]_{(\mathbf{n},t)} & \xrightarrow{\subseteq} & C[[t]] \\
= \downarrow & & \cong \downarrow & & & & \lambda \downarrow \\
D & \xrightarrow{\subseteq} & S = D[\tau]_{(\mathbf{m},\tau)} & & & \xrightarrow{\varphi} & C.
\end{array}$$

Under the above isomorphism of S with S_1, the prime ideal P corresponds to a height-one prime ideal P_1 of S_1 such that $P_1 \cap D = (0)$. Since $U^{-1}S_1$ is a localization of a polynomial ring in one variable over a field, the extended ideal $P_1U^{-1}S_1$ is a principal prime ideal. Therefore P_1 is contained in a proper principal ideal of $U^{-1}C[t]_{(\mathbf{n},t)}$.

However, in the above diagram it can happen that the inclusion map

$$U^{-1}S_1 \ \hookrightarrow \ U^{-1}C[t]_{(\mathbf{n},t)}$$

may fail to be faithfully flat. As an example to illustrate this, let

$$D := k[x, y = e^x - 1]_{(x,y)} \hookrightarrow k[[x]] =: C$$

and let $P = (xt - y)D[t]$. Then P extends to the whole ring in $U^{-1}C[t]_{(\mathbf{n},t)}$ since $t - \frac{y}{x}$ is a unit of $U^{-1}C[t]_{(\mathbf{n},t)}$.

Exercises

(1) Let $T = k[x, y, z]$ be a polynomial ring in the 3 variables x, y, z over a field k, and consider the subring $S = k[xy, xz, yz]$ of T.
   (a) Prove that the field extension $\mathcal{Q}(T)/\mathcal{Q}(S)$ is algebraic with $[\mathcal{Q}(T) : \mathcal{Q}(S)] = 2$.
   (b) Deduce that $xy,\ xz,\ yz$ are algebraically independent over k, so S is a polynomial ring in 3 variables over k.
   (c) Prove that the extension $S \hookrightarrow T$ is height-one preserving, but is not weakly flat.
   (d) Prove that $T \cap \mathcal{Q}(S) = S[x^2, y^2, z^2]$ is a Krull domain that properly contains S.
   (e) Prove that the map $S \hookrightarrow T[\frac{1}{xyz}]$ is flat.
   (f) Prove that $S[\frac{1}{xyz}] = T[\frac{1}{xyz}]$. (Notice that $S[\frac{1}{xyz}]$ is not a localization of S since xyz is not in $\mathcal{Q}(S)$.)

(2) In the case where T is also a Krull domain, give a direct proof using primary decomposition of the assertion in Corollary 8.4 that $S = \mathcal{Q}(S) \cap T$ implies T is weakly flat over S.

   Suggestion. Let $\mathbf{p}$ be a height-one prime ideal of S and let $0 \neq a \in \mathbf{p}$. Since T is a Krull domain, the principal ideal aT has an irredundant primary decomposition

$$aT = Q_1 \cap \cdots \cap Q_s,$$

   where each Q_i is primary for a height-one prime ideal P_i of T.
   (b) Show that $aS = \mathcal{Q}(S) \cap aT$.
   (c) Show that after relabeling there exists an integer $t \in \{1, \ldots, s\}$ such that the ideal $Q_1 \cap \ldots \cap Q_t \cap S$ is the $\mathbf{p}$-primary component of aS. Conclude that $P_i \cap S = \mathbf{p}$, for some i.

(3) Let A be an integral domain and let I be an ideal generated by the nonzero elements $a_1, \ldots, a_r$ of A. Let $\mathcal{F} = \{P \in \operatorname{Spec} A \mid I \nsubseteq P\}$. For each $n \in \mathbb{N}$ define $I^{-n} := (A :_{\mathcal{Q}(A)} I^n)$. Show that

$$\bigcup_{n=1}^{\infty} I^{-n} = \bigcap_{i=1}^{r} A\,[\frac{1}{a_i}] = \bigcap_{P \in \mathcal{F}} A_P.$$

   Remark. Exercise 3 is a result proved by Jim Brewer [**18**, Prop. 1.4 and Theorem 1.5]

(4) Let $(R, \mathbf{m})$ be a 3-dimensional regular local domain with $\mathbf{m} = (x, y, z)R$, let $\mathbf{p} = xR$ and let $V = R_{\mathbf{p}}$. Thus V is an essential valuation ring for the Krull domain R, and $R/\mathbf{p}$ is a 2-dimensional regular local domain. Let $w = \frac{x - y^2}{z}$ and let $T = R[w]_{(y,z,w)R[w]}$.
   (a) Prove that T is a 3-dimensional regular local domain such that $T \subset V$.
   (b) Prove that V is also an essential valuation ring for the Krull domain T.

(c) Find a generator for the height-one prime ideal $\mathbf{q}$ of T such that $T_{\mathbf{q}} = V$.
(d) Prove that $T/\mathbf{q}$ is a 2-dimensional local domain that is not regular.

CHAPTER 9

Prototypes and excellence

In Section 9.1 we present *Prototypes*; they are intersection domains E obtained using Inclusion Construction 5.3 in the standard setting of Setting and Notation 9.1. The Noetherian Flatness Theorem 6.3 holds for Prototypes, and so the intersection domain equals the associated approximation domain. We show in Prototype Theorem 9.2 that Prototypes are localized polynomial rings over DVRs. As such, they are always excellent if the underlying DVR is excellent.

Prototype Theorem 9.2 is useful for many of our examples and it is vital to the Insider Construction in Section 6.2 and Chapter 10. We also demonstrate in Proposition 9.4 the importance of requiring characteristic zero in order to obtain excellence.

9.1. Localized polynomial rings over special DVRs

As stated above *Prototypes* are intersection domains A obtained using Inclusion Construction 5.3 with a standard setting. In this section we give the setting and we show that Prototypes are polynomial rings over special DVRs, and they are equal to their approximation domains B.

We use the following setting and notation for the Prototype and Theorem 9.2. For convenience we also include the definitions of the intersection and approximation domains corresponding to the construction from Section 5.2.

Setting and Notation 9.1. Let x be an indeterminate over a field k. Let r be a nonnegative integer and s a positive integer. Assume that $\tau_1, \ldots, \tau_s \in xk[[x]]$ are algebraically independent over $k(x)$ and let $y_1, \ldots, y_r$ be additional indeterminates. We define the following rings:

(9.1.a) $R := k[x, y_1, \ldots, y_r],\ R^* = k[y_1, \ldots, y_r][[x]],\ V = k(x, \tau_1, \ldots, \tau_s) \cap k[[x]].$

Notice that R^* is the (x)-adic completion of R and V is a DVR.

The "Prototype" is described using the Intersection Domain of Inclusion Construction 5.3 and the Approximation Domain of Section 5.2. Its development is similar to that of the Local Prototype of Definition 4.27:

(9.1.b) $$D := k(x, y_1, \ldots, y_r, \tau_1, \ldots \tau_s) \cap R^*, \quad E := (1 + xU)^{-1}U,$$

where $U := \bigcup U_{n\in\mathbb{N}} R[\tau_{1n}, \ldots \tau_{sn}]$, each τ_{in} is the n^{th} endpiece of τ_i and each $\tau_{in} \in R^*$, for $1 \leq i \leq s$. By Construction Properties Theorem 5.14.3, the ring R^* is the (x)-adic completion of each of the rings D, E and U.

Prototype Theorem 9.2. (Inclusion Version) *Assume Setting and Notation 9.1. Thus $R := k[x, y_1, \ldots, y_r]$ and $R^* = k[y_1, \ldots, y_r][[x]]$. Let V, D and E be as defined in Equations 9.1.a and 9.1.b. Then:*

(1) *The canonical map $\alpha : R[\tau_1, \ldots \tau_s] \hookrightarrow R^*[1/x]$ is flat.*

(2) *$D = E$ is Noetherian of dimension $r+1$ and is the localization $(1 + xV[y_1, \dots, y_r])^{-1}V[y_1, \dots, y_r]$ of the polynomial ring $V[y_1, \dots, y_r]$ over the DVR V. Thus D is a regular integral domain.*
(3) *E is a directed union of localizations of polynomial rings in $r + s + 1$ variables over k.*
(4) *If k has characteristic zero, then the ring $E = D$ is excellent.*

PROOF. The map $k[x, \tau_1, \dots \tau_s] \hookrightarrow k[[x]][1/x]$ is flat by Remark 2.31.4 since $k[[x]][1/x]$ is a field. By Fact 2.32

$$k[x, \tau_1, \dots \tau_s] \otimes_k k[y_1, \dots, y_r] \hookrightarrow k[[x]][1/x] \otimes_k k[y_1, \dots, y_r]$$

is flat. We also have $k[[x]][1/x] \otimes_k k[y_1, \dots, y_r] \cong k[[x]][y_1, \dots, y_r][1/x]$ and

$$k[x, \tau_1, \dots \tau_s] \otimes_k k[y_1, \dots, y_r] \cong k[x, y_1, \dots, y_r][\tau_1, \dots \tau_s].$$

Hence the natural inclusion map

$$k[x, y_1, \dots, y_r][\tau_1, \dots \tau_s] \overset{\beta}{\hookrightarrow} k[[x]][y_1, \dots, y_r][1/x]$$

is flat. Also $k[[x]][y_1, \dots, y_r] \hookrightarrow k[y_1, \dots, y_r][[x]]$ is flat since it is the map taking a Noetherian ring to an ideal-adic completion; see Remark 3.2.2. Therefore

$$k[[x]][y_1, \dots, y_r][1/x] \overset{\delta}{\hookrightarrow} k[y_1, \dots, y_r][[x]][1/x]$$

is flat. It follows that the map

$$k[x, y_1, \dots, y_r][\tau_1, \dots \tau_s] \overset{\delta\circ\beta}{\hookrightarrow} R^*[1/x] = k[y_1, \dots, y_r][[x]][1/x]$$

is flat. Thus the Noetherian Flatness Theorem 6.3 implies $E = D$ and E is Noetherian, and so we have proved items 1, 3 and part of item 2.

To see E is the localization described in item 2, we use that $V[y_1, \dots, y_r] \subseteq D$ and that x is in the Jacobson radical of D by Construction Properties Theorem 5.14. Thus every element of $1 + xV[y_1, \dots, y_r]$ is invertible in D. Hence

$$(1 + xV[y_1, \dots, y_r])^{-1}V[y_1, \dots, y_r] \subseteq D.$$

Since each U_n is contained in $V[y_1, \dots, y_r]$, we have $U \subseteq V[y_1, \dots, y_r]$. We also have $E = (1 + xU)^{-1}U$, and so $E \subseteq (1 + xV[y_1, \dots, y_r])^{-1}V[y_1, \dots, y_r]$. This completes item 2.

For item 4, if k has characteristic zero, then V is excellent by Remark 3.38; hence item 4 follows from item 2 since excellence is preserved under localization of a finitely generated algebra by Remark 3.38. For more details see [**101**, (34.B),(33.G) and (34.A)], [**51**, Chap. IV]. □

DEFINITION 9.3. For integers r and s, indeterminates $x, y_1, \dots, y_r$ over a field k, and elements $\tau_1, \dots \tau_s \in k[[x]]$ that are algebraically independent over $k(x)$, we refer to the ring

$$\begin{aligned} D &:= k(x, y_1, \dots, y_r, \tau_1, \dots \tau_s) \cap k[y_1, \dots, y_r][[x]] \\ &= (1 + xV[y_1, \dots, y_r)^{-1}V[y_1, \dots, y_r], \end{aligned}$$

where $k[x, y_1, \dots, y_r]$ and V is the DVR $k(x, \tau_1, \dots, \tau_s) \cap k[[x]]$, as a *Prototype*. The ring D depends upon the field k, the integers r and s, and the choice of $\tau_1, \dots \tau_s$, and D is also called an *Inclusion Construction Prototype*.

We observe in Proposition 9.4 that over a perfect field k of characteristic $p > 0$ (so that $k = k^{1/p}$) a one-dimensional form of the construction in Prototype Theorem 9.2 yields a DVR that is not a Nagata ring, defined in Definition 2.11, and thus is not excellent; see Remark 3.38, [**103**, p. 264], [**101**, Theorem 78, Definition 34.A].

PROPOSITION 9.4. *Let k be a perfect field of characteristic $p > 0$, let the element τ of $xk[[x]]$ be such that x and τ are algebraically independent over k and set $V := k(x, \tau) \cap k[[x]]$. Then V is a DVR for which the integral closure $\overline{V}$ of V in the purely inseparable field extension $k(x^{1/p}, \tau^{1/p})$ is not a finitely generated V-module. Hence V is not a Nagata ring and so is not excellent.*

PROOF. It is clear that V is a DVR with maximal ideal xV. Since x and τ are algebraically independent over k, $[k(x^{1/p}, \tau^{1/p}) : k(x, \tau)] = p^2$. Let W denote the integral closure of V in the field extension $k(x^{1/p}, \tau)$ of degree p over $k(x, \tau)$. Notice that

$$W \;=\; k(x^{1/p}, \tau) \;\cap\; k[[x^{1/p}]] \quad \text{and} \quad \overline{V} \;=\; k(x^{1/p}, \tau^{1/p}) \;\cap\; k[[x^{1/p}]]$$

are both DVRs having residue field k and maximal ideal generated by $x^{1/p}$. Thus $\overline{V} = W + x^{1/p}\overline{V}$. If $\overline{V}$ were a finitely generated W-module, then by Nakayama's Lemma it would follow that $W = \overline{V}$. This is impossible because $\overline{V}$ is not birational over W. It follows that $\overline{V}$ is not a finitely generated V-module, and hence V is not a Nagata ring. □

REMARK 9.5. Let $V = k(x, \tau) \cap k[[x]]$, let D be as in Setting and Notation 9.1 with $s = r = 1$, and suppose that k is a perfect field with characteristic $p > 0$. By Proposition 9.4, the ring V is not excellent. By Prototype Theorem 9.2.2, $D = (1 + xV[y])^{-1}V[y]$, and so the ring V is a homomorphic image of D. Since excellence is preserved by taking homomorphic images, the two-dimensional regular ring D is not excellent in this situation; see Remark 3.38. The same argument applies if we put more variables in place of y, that is, $y_1, ..., y_r$, as in Theorem 9.2. In general, over a perfect field of characteristic $p > 0$, the Noetherian regular ring $D = E$ obtained in Prototype Theorem 9.2 fails to be excellent.

We give below a localized form of Prototype Theorem 9.2, with the rings R, D, and E local. With Setting 9.1, the ring E is a localization of $U = \bigcup_{n=1}^{\infty} U_n$, where each $U_n = R[\tau_{in}]$, and E is also a localization of $U' = \bigcup_{n=1}^{\infty} U'_n$, where each $U'_n = k[x, y_j, \tau_{in}]$. This simpler second form U' of U is used in Chapters 15 and 16.

LOCAL PROTOTYPE THEOREM 9.6. *If we adjust Setting and Notation 9.1 so that the base ring is the regular local ring $R := k[x, y_1, \ldots, y_r]_{(x,y_1,\ldots,y_r)}$, then the conclusions of Prototype Theorem 9.2 are still valid. In particular:*

(1) *For Inclusion Construction 5.3, with the notation of Equation 9.1.b,*

$$D = E = V[y_1, \ldots, y_r]_{(x,y_1,\ldots,y_r)},$$

is a Noetherian regular local ring, and the extension $R[t_1, \ldots, t_s] \to R^*[1/x]$ *is flat. In addition,*

$$E = \bigcup_{n=1}^{\infty} (U_n)_{\mathbf{m}_n} = U_{\mathbf{m}_U} = \bigcup_{n=1}^{\infty} (U'_n)_{\mathbf{m}'_n} = U'_{\mathbf{m}_{U'}}, \text{ where } U = \bigcup_{n=1}^{\infty} U_n,\ U' = \bigcup_{n=1}^{\infty} U'_n,$$

$$U_n = k[x, y_1, \ldots, y_r]_{(x,y_1,\ldots,y_r)}[\tau_{1n}, \ldots, \tau_{sn}], \quad \mathbf{m}_n = (x, y_1, \ldots, y_r, \tau_{1n}, \ldots, \tau_{sn})U_n,$$

$$U'_n = k[x, y_1, \ldots, y_r, \tau_{1n}, \ldots, \tau_{sn}], \quad \mathbf{m}'_n = (x, y_1, \ldots, y_r, \tau_{1n}, \ldots, \tau_{sn})U'_n,$$

$$\mathbf{m}_U = (x, y_1, \ldots, y_r)U \quad \text{and} \quad \mathbf{m}'_U = (x, y_1, \ldots, y_r)U'.$$

(2) *If* k *has characteristic zero, then the Localized Prototype Domain* D *is excellent.*

PROOF. The proof of Theorems 9.2 applies to the localized polynomial rings. The statements about the rings U and U' follow from Remark 5.15. □

CHAPTER 10

Insider Construction details,

In this chapter we continue the development of the Insider (Inclusion) Construction begun in Section 6.2. Insider Construction 10.1 is a more general construction using Inclusion Construction 5.3 than the version given in Section 6.2. The base ring R is a Noetherian domain that is not necessarily a polynomial ring over a field.

We refer to an Interesection Domain $D = A$, constructed using Inclusion Construction 5.3, as a *Generalized Prototype* if the equivalent conditions of Noetherian Flatness Theorem 6.3 hold. A Generalized Prototype D is both an Intersection Domain and an Approximation Domain. The domain D is a directed union of localized polynomial rings over the base ring R; see Section 5.2.

We construct inside D two integral domains: an intersection domain A of a field with an ideal-adic completion of R as in Construction 5.3, and a domain B that is a nested union of localized polynomial rings over R that "approximates" A as in Section 5.2. We show that B is Noetherian and equal to A if a certain map of polynomial rings over R is flat.

In Section 10.1, we describe background and notation for Inclusion Construction 10.1. Theorem 10.3 of Section 10.2 gives necessary and sufficient conditions for the integral domains constructed with Insider Construction 10.1 to be Noetherian and equal. In Section 10.3, we use the analysis of flatness for polynomial extensions from Chapter 7 to obtain a general flatness criterion for the Insider Construction. This yields examples where the constructed domains A and B are equal and are not Noetherian.

In Section 10.4 we discuss the preservation of excellence for the insider Construction. That is, if the Intersection Domain A and the Approximation Domain B that result from the Insider Construction are equal and Noetherian and the base ring R is excellent, we consider conditions in order that A and B are excellent. We give in Theorem 10.10 necessary and sufficient conditions so that A is excellent.

Insider Construction 10.1 is a useful shortcut for constructing examples. In Chapter 11, we use Insider Construction to establish, for each integer $d \geq 3$ and each integer h with $2 \leq h \leq d-1$, the existence of a d-dimensional regular local domain $(A, \mathbf{n})$ that has a prime ideal P of height h such that the extension $P\widehat{A}$ is not integrally closed. In Chapter 15, we use the Insider Construction to obtain, for each positive integer n, an explicit example of a 3-dimensional non-Noetherian local unique factorization domain B such that the maximal ideal of B is 2-generated, B has precisely n prime ideals of height two, and each prime ideal of B of height two is not finitely generated.

10.1. Describing the construction

We use the following setting and details for the Insider Construction in this chapter. This setting includes Noetherian domains that are not necessarily local and thus generalizes Settings 6.12 and 6.15.

INSIDER CONSTRUCTION 10.1. Let R be a Noetherian integral domain. Let z be a nonzero nonunit of R and let R^* be the (z)-adic completion of R. The intersection domain of Construction 5.3 and the corresponding approximation domain of Section 5.2 are inside R^*. Assume that $\tau_1, \ldots, \tau_n \in zR^*$ are algebraically independent over R and are such that nonzero elements of $R[\tau_1, \ldots, \tau_n]$ are regular on R^*. Let $\underline{\tau}$ abbreviate the list $\tau_1, \ldots, \tau_n$. We define the *intersection domain corresponding to* $\underline{\tau}$ to be the ring $A_{\underline{\tau}} := K(\tau_1, \ldots, \tau_n) \cap R^*$. The Noetherian Flatness Theorem 6.3 implies that simultaneously $A_{\underline{\tau}}$ is Noetherian and $A_{\underline{\tau}}$ is a nested union $B_{\underline{\tau}}$ of certain associated localized polynomial rings over R using $\underline{\tau}$ if and only if the extension $T := R[\underline{\tau}] = R[\tau_1, \ldots, \tau_n] \overset{\psi}{\hookrightarrow} R^*[1/z]$ is flat. Moreover, if this flatness condition holds, then $A_{\underline{\tau}}$ is a localization of a subring of $T[1/z]$ and $A_{\underline{\tau}}[1/z]$ is a localization of T.

We assume that $\psi : T \hookrightarrow R^*[1/z]$ is flat, so that the intersection domain $A_{\underline{\tau}}$ is Noetherian and equals its associated approximation domain. We define the Prototype or Generalized Prototype domain $D := A_{\underline{\tau}} = B_{\underline{\tau}}$ and D is a directed union of localized polynomial rings. Then we construct new "insider" examples inside D as follows: We choose elements $f_1, \ldots, f_m$ of $T := R[\underline{\tau}]$, considered as polynomials in the τ_i with coefficients in R and abbreviated by $\underline{f}$. Assume that $f_1, \ldots, f_m$ are algebraically independent over K; thus $m \leq n$. As above we define $A := A_{\underline{f}} := K(f_1, \ldots, f_m) \cap R^*$ to be the *intersection domain corresponding to* $\underline{f}$. We let $B := B_{\underline{f}}$ be the *approximation domain corresponding to* $\underline{f}$ that approximates $A_{\underline{f}}$, obtained using the f_i as in Section 5.2.

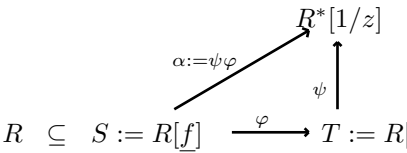
Recall that the nested union domains D and B are localizations of $R[\underline{\tau}]$ and $R[\underline{f}]$ respectively by Construction Properties Theorem 5.14.4. We have that $B \subseteq D$.

Set $S := R[\underline{f}] = R[f_1, \ldots, f_m]$, let φ be the embedding

$$\varphi : S := R[\underline{f}] \overset{\varphi}{\hookrightarrow} T := R[\underline{\tau}], \tag{10.1.1}$$

and let ψ be the inclusion map: $R[\underline{\tau}] \hookrightarrow R^*[1/z]$. Put $\alpha := \psi \circ \varphi : S \to R^*[1/z]$. Then we have

(10.1.2)



We show in Theorem 6.11 of Chapter 6 for the special case where R is a localized polynomial ring over a field in two or three variables that, if $\varphi_z : S \hookrightarrow T[1/z]$ is flat, then A is Noetherian and is equal to the corresponding approximation domain B. In Section 10.2 we make a more thorough analysis of conditions for A to be

Noetherian and equal to B. In Section 10.3, we present examples where $B = A$ is not Noetherian.

REMARK 10.2. If R is a Noetherian local domain, then R^* is local and hence the intersection domains D and A are also local with A possibly non-Noetherian. By Construction Properties Theorem 5.14.6, the approximation domain B is also local. Then $B \subseteq D$ and D dominates B.

10.2. The flat locus of the Insider Construction

We assume the notation of Insider Construction 10.1 for this discussion and refer the reader to Section 5.2 for details concerning the approximation domain B corresponding to the intersection domain A, respectively, of (10.1).

Noetherian Flatness Theorem 6.3 is the basis for our construction of examples.

In the notation of Diagram 10.1.2, let

$$F := \cap\{P \in \operatorname{Spec}(T) \mid \varphi_P : S \to T_P \text{ is not flat }\}. \tag{10.2.1}$$

Thus, as in (7.12.2), the ideal F defines the nonflat locus of the map $\varphi : S \to T$.

For $Q^* \in \operatorname{Spec}(R^*[1/z])$, we consider flatness of the localization $\varphi_{Q^* \cap T}$ of the map φ in Equation 10.1.1:

$$\varphi_{Q^* \cap T} : S \longrightarrow T_{Q^* \cap T} \tag{10.2.2}$$

Theorem 10.3 enables us to recover information about the flatness of α in Diagram 10.1.2 from the map $\varphi : S \to T$.

THEOREM 10.3. *Let R be a Noetherian domain, let z be a nonzero nonunit of R and let R^* be the z-adic completion of R. With the notation of Insider Construction 10.1, we have:*

1. *For $Q^* \in \operatorname{Spec}(R^*[1/z])$, the map $\alpha_{Q^*} : S \to (R^*[1/z])_{Q^*}$ is flat if and only if the map $\varphi_{Q^* \cap T}$ in Equation 10.2.2 is flat.*
2. *The following are equivalent:*
   - (i) *The ring A is Noetherian and $A = B$.*
   - (ii) *The ring B is Noetherian.*
   - (iii) *For every maximal $Q^* \in \operatorname{Spec}(R^*[1/z])$, the map $\varphi_{Q^* \cap T}$ in Equation 10.2.2 is flat.*
   - (iv) $FR^*[1/z] = R^*[1/z]$.
3. *The map $\varphi_z : S \hookrightarrow T[1/z]$ is flat if and only if $FT[1/z] = T[1/z]$. Moreover, either of these equivalent conditions implies B is Noetherian and $B = A$. It then follows that $A[1/z]$ is a localization of S.*
4. *If z is in the Jacobson radical of R and the conditions of item 2 or item 3 hold, then* $\dim R = \dim A = \dim R^*$.

PROOF. For item 1, we have $\alpha_{Q^*} = \psi_{Q^*} \circ \varphi_{Q^* \cap T} : S \to T_{Q^* \cap T} \to (R^*[1/z])_{Q^*}$. Since the map ψ_{Q^*} is faithfully flat, the composition α_{Q^*} is flat if and only if $\varphi_{Q^* \cap T}$ is flat [**103**, (1) and (3), p. 46].

For item 2, the equivalence of (i) and (ii) is part of Theorem 6.3. The equivalence of (ii) and (iii) follows from item 1 and Theorem 6.3. For the equivalence of (iii) and (iv), we use $FR^* \neq R^* \iff F \subseteq Q^* \cap T$, for some Q^* maximal in $\operatorname{Spec}(R^*[1/z]) \iff$ the map in Equation 10.2.2 fails to be flat.

The first statement of item 3 follows from the definition of F and the fact that the nonflat locus of $\varphi : S \to T$ is closed. Noetherian Flatness Theorem 6.3 implies the final statement of item 3.

Item 4 follows by Remark 3.2.4. □

COROLLARY 10.4. *Let R be a Noetherian domain, let z be a nonzero nonunit of R and let R^* be the (z)-adic completion of R. With notation as in Theorem 10.3, if $\varphi : S \to T$ is flat, then the ring B is Noetherian and $B = A$.*

10.3. The nonflat locus of the Insider Construction

To examine the map $\alpha : S \to R^*[1/z]$ in more detail, we consider the following:

PROPOSITION 10.5. *Let R be a normal Noetherian domain. With the notation of Insider Construction 10.1 and Equation 10.2.1, we have*

1. $\operatorname{ht}(FR^*[1/z]) > 1 \iff \alpha : S \to R^*[1/z]$ *satisfies* LF_1.
2. *Assume that R^* is a normal domain and that each height-one prime of R is the radical of a principal ideal. Then the equivalent conditions of item 1 imply that $B = A$.*

PROOF. Item 1 follows from the definition of LF_1; see Definition 8.1.3.

For item 2, assume R^* is a normal domain and each height-one prime of R is the radical of a principal ideal. Then the extension $R \hookrightarrow R^*$ is height-one preserving by Remark 8.6.c. By Proposition 8.12 the extension is weakly flat. Theorem 8.7 implies that $B = A$. □

QUESTION 10.6. Does item 2 of Proposition 10.5 hold without the condition that every height-one prime ideal is the radical of a principal ideal?

THEOREM 10.7. *Let R be a Noetherian integral domain, let z be a nonzero nonunit of R and let R^* be the z-adic completion of R. With the notation of Insider Construction 10.1, assume $m = 1$, that is, there is only one polynomial $f_1 = f$, $S := R[f]$ and $T := R[\tau_1, \ldots, \tau_n]$. Assume that $f \in T \setminus R$. Let B and A be the approximation domain and intersection domain associated to f over R, and let L be the ideal in R generated by the nonconstant coefficients of f as a polynomial in T. Then:*

1. *The ideal $LR^*[1/z]$ defines the nonflat locus of $\alpha : S \hookrightarrow R^*[1/z]$.*
2. *The ideal $LR^*[1/z]$ defines the nonflat locus of $\beta : B \hookrightarrow R^*[1/z]$.*
3. *The following are equivalent:*
   (a) *B is Noetherian.*
   (b) *B is Noetherian and $B = A$.*
   (c) *The extension $\alpha : S \hookrightarrow R^*[1/z]$ is flat.*
   (d) *For each $Q^* \in \operatorname{Spec} R^*[1/z]$, we have $LR^*[1/z]_{Q^*} = R^*[1/z]_{Q^*}$.*
   (e) *For each $Q^* \in \operatorname{Spec} R^*[1/z]$, we have $LR_q = R_q$, where $q = Q^* \cap R$.*
4. *If $\operatorname{ht} LR^*[1/z] = d$, then the map $\alpha : S \hookrightarrow R^*[1/z]$ satisfies LF_{d-1}, but not LF_d, as defined in Definition 8.1.3.*
5. *$\varphi_z : S \hookrightarrow T[1/z]$ is flat $\iff LT[1/z] = T[1/z] \iff LR[1/z] = R[1/z]$.*
6. *The equivalent conditions in item 5 imply the insider approximation domain B is Noetherian and is equal to the insider intersection domain A.*

PROOF. For item 1, let $Q^* \in \operatorname{Spec}(R^*[1/z])$. Theorem 10.3.1 implies the map

$$\alpha_{Q^*} : S \hookrightarrow (R^*[1/z])_{Q^*} \text{ is flat } \iff \varphi_{Q^*\cap T} : S \hookrightarrow T_{Q^*\cap T} \quad \text{is flat.}$$

By Corollary 7.24, the ideal LT defines the nonflat locus of $\varphi : S \hookrightarrow T$. Thus

$$\varphi_{Q^*\cap T} \quad \text{is flat} \iff LT \nsubseteq Q^* \cap T.$$

Since L is an ideal of R, we have

$$LT \nsubseteq Q^* \cap T \iff LR^*[1/z] \nsubseteq Q^*.$$

Thus $LR^*[1/z]$ defines the nonflat locus of α.

In view of item 1, to prove item 2, it suffices to show for each $Q^* \in \operatorname{Spec} R^*[1/z]$:

$$\alpha_{Q^*} : S \hookrightarrow R^*[1/z]_{Q^*} \text{ is flat } \iff \beta_{Q^*} : B \hookrightarrow R^*[1/z]_{Q^*} \text{ is flat.}$$

By Remarks 2.31.1, we have

$$\alpha_{Q^*} \text{ is flat } \iff S_{Q^*\cap S} \hookrightarrow R^*[1/z]_{Q^*} \text{ is flat.}$$

Similarly, we have

$$\beta_{Q^*} \text{ is flat } \iff B_{Q^*\cap B} \hookrightarrow R^*[1/z]_{Q^*} \text{ is flat.}$$

Since $z \notin Q^*$ and $B[1/z]$ is a localization of S, we have $S_{Q^*\cap S} = B_{Q^*\cap B}$ because $B_{Q^*\cap B}$ dominates and is a localizaton of $S_{Q^*\cap S}$. This completes the proof of item 2.

For item 3, (a), (b) and (c) are equivalent by Noetherian Flatness Theorem 6.3. By item 1, (c) and (d) are equivalent. Since L is an ideal of R, (d) is equivalent to (e); that is $L \nsubseteq Q^* \iff L \nsubseteq Q^* \cap R = q$.

For item 4, assume that $\operatorname{ht}(LR^*[1/z]) = d$. Let $Q^* \in \operatorname{Spec}(R^*[1/z])$. The map $\alpha_{Q^*} : S \hookrightarrow (R^*[1/z])_{Q^*}$ is flat $\iff L \nsubseteq Q^*$ by item 1. Thus α_{Q^*} is flat for every Q^* with $\operatorname{ht} Q^* < d$, and so α satisfies LF_{d-1}. On the other hand, there exists $Q^* \in \operatorname{Spec} R^*[1/z]$ such that $L \subseteq Q^*$ and $\operatorname{ht} Q^* = d$. By item 1, the map α_{Q^*} is not flat. Thus α does not satisfy LF_d.

For item 5, Corollary 7.24 states that LT is the nonflat locus of the map $\varphi : S \hookrightarrow T$. Thus $S \hookrightarrow T[1/z]$ is flat $\iff LT[1/z] = T[1/z]$. Since L is an ideal of R, and $T[1/z]$ is a polynomial ring over $R[1/z]$, we have $LT[1/z] = T[1/z] \iff LR[1/z] = R[1/z]$.

If $S \hookrightarrow T[1/z]$ is flat, then Theorem 10.3.3 implies that B is Noetherian and $B = A$. Thus item 6 holds. □

Example 10.8 illustrates that in Theorem 10.7 the map $\varphi_z : S \hookrightarrow T[1/z]$ may fail to be flat even though the map $\alpha : S \hookrightarrow R^*[1/z]$ is flat.

EXAMPLE 10.8. Let $R = k[z]_{(z)}$, where z is an indeterminate over the field k of characteristic zero Let $\tau \in zk[[z]]$ be such that z and τ are algebraically independent over k. Let $T = R[\tau]$, let $f = (1-z)\tau$, and let $S = R[f]$. The ideal L of R generated by the nonconstant coefficients of f is $L = (1-z)R$. The map $\varphi_z : S \hookrightarrow T[1/z]$ is not flat, but the map $\alpha : S \hookrightarrow R^*[1/z]$ is flat since $R^*[1/z]$ is a field.

We return to Example 6.18 and establish a more general result.

EXAMPLES 10.9. Let $d \in \mathbb{N}$ be greater than or equal to 2, and let $x, y_1, \ldots, y_d$ be indeterminates over a field k. Let R be either

(1) The polynomial ring $R := k[x, y_1, \ldots, y_d]$ with (x)-adic completion $R^* = k[y_1, \ldots, y_d][[x]]$, or

(2) The localized polynomial ring $R := k[x, y_1, \ldots, y_d]_{(x,y_1,\ldots,y_d)}$ with (x)-adic completion R^*.

Let $f := y_1\tau_1 + \cdots + y_d\tau_d$, where $\tau_1, \ldots, \tau_d \in xR^*$ are algebraically independent over R. Let $S := R[f]$ and let $T := R[\tau_1, \ldots, \tau_d]$. We regard f as a polynomial in $\tau_1, \ldots, \tau_d$ over R. By Theorem 10.7.4, the map $\varphi_x : S \hookrightarrow T[1/x]$ satisfies LF_{d-1}, but fails to satisfy LF_d because the ideal $L = (y_1, \ldots, y_d)R[1/x]$ of nonconstant coefficients of f has height d. Since $d \geq 2$, the map $\varphi_x : S \hookrightarrow T[1/x]$ satisfies LF_1. Moreover, S is a UFD and hence, by Proposition 10.5, we have $A = B$, that is, the element f is "limit-intersecting". However, since φ_x does not satisfy LF_d, the map φ_x is not flat and thus B is not Noetherian by Theorem 10.3.2.

10.4. Preserving excellence with the Insider Construction

In Theorem 10.10, we present conditions in order that Insider Construction 10.1 preserve excellence.

THEOREM 10.10. *Let $(R, \mathbf{m})$ be an excellent normal local domain with field of fractions K. Let z be a nonzero element of $\mathbf{m}$ and let R^* denote the (z)-adic completion of R. As in Insider Construction 10.1, assume the n elements $\tau_1, \ldots, \tau_n \in zR^*$ are algebraically independent over K, that $T := R[\tau_1, \ldots, \tau_n] \longrightarrow R^*[1/z]$ is flat, and $D := B_{\underline{\tau}} = A_{\underline{\tau}} := K(\tau_1, \ldots, \tau_n) \cap R^*$. Let $f_1, \ldots, f_m \in T := R[\underline{\tau}]$, considered as polynomials in the τ_i with coefficients in R. Assume $f_1, \ldots, f_m$ are algebraically independent over K; thus $m \leq n$. Let $S := R[f_1, \ldots, f_m]$ and $\varphi : S \hookrightarrow T$, and let J be the Jacobian ideal of φ as in Definition 7.12.1. Define $A := A_{\underline{f}} := K(f_1, \ldots, f_m) \cap R^*$, and define $B := B_{\underline{f}}$. If D is excellent, then the following are equivalent:*

(a) *The ring B is excellent.*
(b) *$JR^*[1/z] = R^*[1/z]$.*
(c) *$\alpha : S \to R^*[1/z]$ is a regular morphism.*

Moreover, if either of the following equivalent conditions holds, then B is excellent,

(b′) *$JT[1/z] = T[1/z]$.*
(c′) *$\varphi_z : S \to T[1/z]$ is a regular morphism.*

PROOF. That conditions (b′) and (c′) are equivalent follows from Theorem 7.14.1. Since T is a subring of R^* and J is an ideal of T, condition (b′) implies condition (b). For the other implications, consider the embeddings:

$$B \xrightarrow{\Phi} D \xrightarrow{\Psi} R^* \xrightarrow{\Gamma} \widehat{R}.$$

By Theorem 5.14.4, we have $B[1/z]$ is a localization of S, and $D[1/z]$ is a localization of T. Thus, for $Q \in \operatorname{Spec} R^*$ with $z \notin Q$, we have

(10.10.1)
$$\alpha_Q : S \longrightarrow S_{Q\cap S} = B_{Q\cap B} \xrightarrow{\Phi'} D_{Q\cap D} = T_{Q\cap T} \xrightarrow{\Psi'} R^*_Q.$$

(a)$\Longrightarrow$(b): Since B, D and R^* are all excellent with the same completion $\widehat{R}$, [**103**, Theorem 32.1] implies Φ is regular. Let $Q \in \operatorname{Spec}(R^*)$ with $z \notin Q$. The map $\Phi' : B_{Q\cap B} \hookrightarrow D_{Q\cap D}$ is also regular. It follows from Equation 10.10.1 that $\varphi_{Q\cap T} : S \hookrightarrow T_{Q\cap T}$ is regular. Thus $J \not\subseteq Q \cap T$. Since J is an ideal of T, we have $J \subseteq Q \cap T \iff J \subseteq Q$. We conclude that $JR^*[1/z] = R^*[1/z]$.

(b) $\iff$ (c): We show for every $Q \in Spec R^*$ with $z \notin Q$ that

$$(*) \qquad J \nsubseteq Q \cap T \iff \alpha_Q \quad \text{is regular.}$$

If $J \nsubseteq Q \cap T$, then α_Q is a composition of regular maps as shown in Equation 10.10.1. If α_Q is regular, then Ψ' faithfully flat implies $S \hookrightarrow T_{Q\cap T}$ is regular [**103**, Theorem 32.1 (ii)].

(b)$\Longrightarrow$(a) By Theorems 10.3.2 and 7.14.2, the ring B is Noetherian with z-adic completion R^*. Therefore the completion of B with respect to the powers of its maximal ideal is $\widehat{R}$. Therefore B is formally equidimensional. Hence by Ratliff's Equidimension Theorem 3.18, B is universally catenary.

To show B is excellent, it remains to show that B is a G-ring. Consider the morphisms

$$B \xrightarrow{\Phi} D \quad \text{and} \quad B \xrightarrow{\Gamma\circ\Psi\circ\Phi} \widehat{R}.$$

Since B and D are Noetherian, $\widehat{R}$ is faithfully flat over both B and D. Hence the map Φ is faithfully flat by Remark 2.31.14. A straightforward argument using Definition 3.35 of G-ring shows that B is a G-ring if the map Φ is regular in the sense of Definition 3.31; see [**103**, Theorem 32.2].

To see that Φ is regular, let $P \in \mathrm{Spec}(B)$. If $z \in P$, then we use that

$$B/zB = R^*/zR^* = R/zR = D/zD$$

from Construction Properties Theorem 5.14.2. Since R is excellent, the ring $\widehat{R} \otimes_B k(P)$ is geometrically regular over $k(P) = B_P/PB_P$.

If $z \notin P$, we show that the ring $D \otimes_B L$ is regular, for every finite field extension L of $k(P)$. Let $\overline{W}$ be a prime ideal in $D \otimes_B L$ and let W be the preimage in D of $\overline{W}$. Then $W \cap B = P$. By the faithful flatness of R^* over D, there exists $Q \in \mathrm{Spec}(R^*)$ such that $Q \cap D = W$. Then $P = W \cap B = Q \cap B$. Thus $z \notin Q$. Since $JR^*_Q = R^*_Q$, we have $J \nsubseteq Q$. Hence the morphism Φ' in Equation 10.10.1, is regular. We conclude that Φ is a regular morphism and B is excellent. □

Corollary 10.11. *Let k be a field of characteristic zero, and let D be the Local Protoype $D := k(x, y_1, \ldots, y_r, \tau_1, \ldots, \tau_n) \cap R^*$ of Theorem 9.6, where the base ring $R = k[x, y_1, \ldots, y_r]_{(x,y_1,\ldots,y_r)}$, R^* is the (x)-adic completion of R, and $\tau_1, \ldots, \tau_n$ are elements of $xk[[x]]$ that are algebraically independent over $k(x)$.*

Assume that $f_1, \ldots, f_m \in T := k[x, y_1, \ldots, y_r, \tau_1, \ldots, \tau_n]$ are algebraically independent over $k(x, y_1, \ldots, y_r)$, considered as polynomials in the τ_i with coefficients in R; thus $m \leq n$. Let $S := k[x, y_1, \ldots, y_r, f_1, \ldots, f_m]$, let $\varphi : S \hookrightarrow T$, and let J be the Jacobian ideal of φ as in Definition 7.12.1. Define $B = B_{\underline{f}}$ and $A = k(x, y_1, \ldots, y_r, f_1, \ldots, f_m) \cap R^*$. The following are equivalent:

(a) *The ring B is excellent.*
(b) *$JR^*[1/x] = R^*[1/x]$.*
(c) *$\varphi_x : S \to T[1/x]$ is a regular morphism.*

In relation to Corollary 10.11, we consider the historical examples of Nagata and Christel discussed in Chapters 4 and 6.

Remark 10.12. (1) For the example of Nagata described in Example 4.14 and in Proposition 6.13, we have $R = k[x, y]_{(x,y)}$, $R^* = k[y][[x]]_{(x,y)}$, $r = n = m = 1$, and k is a field of characteristic zero. Let $\tau = \tau_1$, let $T := R[\tau]$, and let D be the Local Prototype $D = k(x, y, \tau) \cap R^*$. Let $f = f_1 = (y+\tau)^2$ and let $A = k(x, yf) \cap R^*$. The Jacobian ideal J of the inclusion map $\varphi : S := R[f] \to T = R[\tau]$ is the ideal of

T generated by $\partial(y^2+2y\tau+\tau^2)/\partial\tau) = 2y+2\tau$. We see that $JR^*[1/x] \neq R^*[1/x]$. Thus, by Proposition 6.13 and Corollary 10.11, the approximation domain $B = A$ is not excellent, and the map φ_x is not regular.

(2) For Christel's example described in Examples 4.16 and 6.17, we have k a field of characteristic zero, $R := k[x,y,z]_{(x,y,z)}$, $r = n = 2$, and $m = 1$. The elements σ and $\tau \in xk[[x]]$ are algebraically independent over $k(x)$, and we have $f := (y+\sigma)(z+\tau)$. The Jacobian ideal J of the inclusion map $\varphi : S := R[f] \to T = R[\sigma,\tau]$ is the ideal of T generated by $z+\tau$ and $y+\sigma$. Again $JR^*[1/x] \neq R^*[1/x]$. Thus, by Example 6.17 and Corollary 10.11, the approximation domain $B = A$ is not excellent, and the map φ_x is not regular.

Examples 10.13 illustrates other applications of Corollary 10.11.

EXAMPLES 10.13. As in Corollary 10.11, let k be a field of characteristic zero, and let D be the Local Protoype $D := k(x,y,z,\sigma,\tau) \cap R^*$ of Local Prototype Theorem 9.6, where the base ring $R = k[x,y,z]_{(x,y,z)}$, R^* is the (x)-adic completion of R, and σ,τ are elements of $xk[[x]]$ that are algebraically independent over $k(x)$. The ring D is a three-dimensional regular local domain that is a directed union of five-dimensional regular local domains by Theorem 9.6.

With this setting we consider two intersection domains $A := k(x,y,z,f,g) \cap R^*$ formed from pairs of elements f and $g \in T := k[x,y,z,\sigma,\tau]$ that are algebraically independent over $k(x,y,z)$. By Construction Properties Theorem 5.14.4, the rings D and A have (x)-adic completion R^*. Let $S := k[x,y,z,f,g]$, let $\varphi : S \hookrightarrow T$, and let J be the Jacobian ideal of φ as in Definition 7.12.1.

(1) Let $f = (y-\sigma)^2$, $g = (z-\tau)^2$, and $A = k(x,y,z,f,g) \cap R^*$. Since $T = k[x,y,z,\sigma,\tau]$ is a free module over S with $\{1,\sigma,\tau,\sigma\tau\}$ as a free basis, the map $\varphi : S \to T$ is flat. By Corollary 10.4, A is Noetherian and is equal to its approximation domain. It follows that A is a 3-dimensional regular local domain.

Since the field k has characteristic zero, the Jacobian ideal of the map φ is $J = (\sigma-y)(\tau-z)T$, and $JR^*[1/x] \neq R^*[1/x]$. Hence by Corollary 10.11, the ring A is not excellent.

(2) Let $f = \sigma^2 + x\tau$, $g = \tau^2 + x\sigma$, and $A = k(x,y,z,f,g) \cap R^*$. The Jacobian ideal of the map $\varphi : S \hookrightarrow T$ is $J = (4\sigma\tau - x^2)T$, and $JR^*[1/x] = R^*[1/x]$. Hence by Corollary 10.11, the ring $B = A$ is excellent. However, $JT[1/x] \neq T[1/x]$. Thus it may happen that B is excellent, but conditions (b′) and (c′) of Theorem 10.10 do not hold

CHAPTER 11

Integral closure under extension to the completion

This chapter relates to the general question: "What properties of ideals of a Noetherian local ring $(A, \mathbf{n})$ are preserved under extension to the $\mathbf{n}$-adic completion $\widehat{A}$?" Our focus here is the integral closure property; see Definition 11.1.4.

Using Insider Construction 10.1 of Chapter 10, we present in Example 11.8 a height-two prime ideal P of a 3-dimensional regular local domain such that the extension $P\widehat{A}$ of P to the $\mathbf{n}$-adic completion $\widehat{A}$ of A is not integrally closed. The ring A in Example 11.8 is a nested union of 5-dimensional regular local domains.

More generally, we use this same technique to establish, for each integer $d \geq 3$ and each integer h with $2 \leq h \leq d-1$, the existence of a d-dimensional regular local domain $(A, \mathbf{n})$ having a prime ideal P of height h with the property that the extension $P\widehat{A}$ is not integrally closed, where $\widehat{A}$ is the $\mathbf{n}$-adic completion of A. A regular local domain having a prime ideal with this property is necessarily not a Nagata ring and is not excellent; see item 7 of Remark 11.7.

We discuss in Section 11.1 conditions in order that integrally closed ideals of a ring R extend to integrally closed ideals of R', where R' is an R-algebra. In particular, we consider conditions for integrally closed ideals of a Noetherian local ring A to extend to integrally closed ideals of the completion $\widehat{A}$ of A.

11.1. Integral closure under ring extension

The concept of "integrality over an ideal" is related to "integrality over a ring", defined in Section 2.1. For properties of integral closure of ideals, rings and modules we refer to the book of Swanson and Huneke [**149**].

Definitions and Remarks 11.1. Let I be an ideal of a ring R.

(1) An element $r \in R$ is *integral over* I if there exists a monic polynomial $f(x) = x^n + \sum_{i=1}^{n} a_i x^{n-i}$ such that $f(r) = 0$ and such that $a_i \in I^i$ for each i with $1 \leq i \leq n$.
(2) The *integral closure* $\overline{I}$ of I is the set of elements of R integral over I; the set $\overline{I}$ is an ideal.
(3) The integral closure of $\overline{I}$ is equal to $\overline{I}$ [**149**, Corollary 1.3.1].
(4) If $I = \overline{I}$, then I is said to be *integrally closed*.
(5) The ideal I is said to be *normal* if I^n is integrally closed for every $n \geq 1$.
(6) If J is an ideal contained in I and $JI^{n-1} = I^n$ for some integer $n \geq 1$, then J is said to be a *reduction* of I.

Remarks 11.2. We record the following facts about an ideal I of a ring R.

(1) An element $r \in R$ is integral over I if and only if I is a reduction of the ideal $L = (I, r)R$. To see this equivalence, observe that for a monic polynomial $f(x)$ as in Definition 11.1.1, we have

$$f(r) = 0 \iff r^n = -\sum_{i=1}^{n} a_i r^{n-i} \in IL^{n-1} \iff L^n = IL^{n-1}.$$

(2) An ideal is integrally closed if and only if it is not a reduction of a properly bigger ideal.
(3) A prime ideal is always integrally closed. More generally, a radical ideal is always integrally closed. This is Exercise 1.
(4) Let a, b be elements in a Noetherian ring R and let $I := (a^2, b^2)R$. The element ab is integral over I. If a, b form a regular sequence, then $ab \notin I$ and thus I is not integrally closed; see Exercise 2. More generally, if $h \geq 2$ and $a_1, \ldots, a_h$ form a regular sequence in R and $I := (a_1^h, \ldots, a_h^h)R$, then I is not integrally closed.

The Rees Algebra is relevant to the discussion of integral closure.

Definition and Remarks 11.3. Let I be an ideal of a ring R, and let t be a variable over R.

(1) The *Rees algebra of* I is the subring of $R[t]$ defined as

$$R[It] := \{ \sum_{i=0}^{n} a_i t^i \mid n \in \mathbb{N} \,;\, a_i \in I^i \} = \bigoplus_{n \geq 0} I^n t^n,$$

where $I^0 = R$.
(2) An element $a \in R$ is integral over I if and only if $at \in R[t]$ is integral over the subring $R[It]$.
(3) If R is a normal domain, then I is a normal ideal of R if and only if the Rees algebra $R[It]$ is a normal domain; see Swanson and Huneke [**149**, Prop. 5.2.1, p.95].

Our work in this chapter is motivated by the following questions:

Questions 11.4.

(1) Craig Huneke: "Does there exist an analytically unramified Noetherian local ring $(A, \mathbf{n})$ that has an integrally closed ideal I for which the extension $I\widehat{A}$ to the $\mathbf{n}$-adic completion $\widehat{A}$ is not integrally closed?"
(2) Sam Huckaba: "If there is such an example, can the ideal of the example be chosen to be a normal ideal?" See Definition 11.1.6.

Related to Question 11.4.1, we present in Theorem 11.10 a 3-dimensional regular local domain A having a height-two prime ideal $I = P = (f, g)A$ such that $I\widehat{A}$ is not integrally closed. Thus the answer to Question 11.4.1 is "yes". This example also shows that the answer to Question 11.4.2 is again "yes". Since f, g form a regular sequence and A is Cohen-Macaulay, the powers P^n of P have no embedded associated primes and therefore are P-primary [**101**, (16.F), p. 112], [**103**, Ex. 17.4, p. 139]. Since the powers of the maximal ideal of a regular local domain are integrally closed, the powers of P are integrally closed, that is, P is a normal ideal.

Thus, by Remarks 11.2.6, the Rees algebra $A[Pt] = A[ft, gt]$ is a normal domain while the Rees algebra $\widehat{A}[ft, gt]$ is not normal.

REMARKS 11.5. Without the assumption that A is analytically unramified, there exist examples even in dimension one where an integrally closed ideal of a Noetherian local domain A fails to extend to an integrally closed ideal in $\widehat{A}$. If A is reduced but analytically ramified, then the zero ideal of A is integrally closed, but its extension to $\widehat{A}$ is not integrally closed.

Examples of reduced analytically ramified Noetherian local rings have been known for a long time. By Remark 3.12.5, the examples of Akizuki and Schmidt mentioned in Classical Examples 1.4 of Chapter 1 are analytically ramified Noetherian local domains. Another example due to Nagata is given in [**117**, Example 3, pp. 205-207]. (See also [**117**, (32.2), p. 114], and Remarks 4.15.2.)

Let R be a commutative ring and let R' be an R-algebra. In Remark 11.7 we list cases where extensions to R' of integrally closed ideals of R are again integrally closed. In this connection we use the following definition as in Lipman [**92**, page 799].

DEFINITION 11.6. An R-algebra. R' is said to be *quasi-normal* over R if R' is flat over R and the condition $N_{R,R'}$ holds:

($N_{R,R'}$): If C is any R-algebra, and D is a C-algebra in which C is integrally closed[1], then also $C \otimes_R R'$ is integrally closed in $D \otimes_R R'$.

If condition $N_{R,R'}$ holds, we also say the map $R \to R'$ is *quasi-normal*.

REMARKS 11.7. Let R be a commutative ring and let R' be an R-algebra.

(1) By a result of Lipman [**92**, Lemma 2.4], if R' satisfies ($N_{R,R'}$) and I is an integrally closed ideal of R, then IR' is integrally closed in R'.

(2) A regular map of Noetherian rings is normal by Remark 3.32, and a normal map of Noetherian rings is quasi-normal [**51**, IV,(6.14.5)]. Hence a regular map of Noetherian rings is quasi-normal.

(3) Assume that R and R' are Noetherian rings and that R' is a flat R-algebra. Let I be an integrally closed ideal of R. The flatness of R' over R implies every $P' \in \operatorname{Ass}(R'/IR')$ contracts in R to some $P \in \operatorname{Ass}(R/I)$ [**103**, Theorem 23.2]. Thus by the previous item, if the map $R \to R'_{P'}$ is regular for each $P' \in \operatorname{Ass}(R'/IR')$, then IR' is integrally closed.

(4) Principal ideals of an integrally closed domain are integrally closed. This is Exercise 3.i.

(5) If I is an ideal of the Noetherian local domain A and $I\widehat{A}$ is integrally closed, then faithful flatness of the extension $A \hookrightarrow \widehat{A}$ implies that I is integrally closed.

(6) In general, integral closedness of ideals is a local condition. If R' is an R-algebra that is a normal ring in the sense that for every prime ideal P' of R', the local ring $R'_{P'}$ is an integrally closed domain, then the extension to R' of every principal ideal of R is integrally closed by item 4. In particular, if $(A, \mathbf{n})$ is an analytically normal Noetherian local domain, then every principal ideal of A extends to an integrally closed ideal of $\widehat{A}$.

[1]Let $h : C \to D$ be the structural map defining D as a C-algebra. Then "C is integrally closed in D" means the subring $h(C)$ of D is integrally closed in D.

(7) Let $(A, \mathbf{n})$ be a Noetherian local ring and let $\widehat{A}$ be the $\mathbf{n}$-adic completion of A. Since $A/\mathbf{q} \cong \widehat{A}/\mathbf{q}\widehat{A}$ for every $\mathbf{n}$-primary ideal $\mathbf{q}$ of A, the $\mathbf{n}$-primary ideals of A are in one-to-one inclusion preserving correspondence with the $\widehat{\mathbf{n}}$-primary ideals of $\widehat{A}$. It follows that an $\mathbf{n}$-primary ideal I of A is a reduction of a properly larger ideal of A if and only if $I\widehat{A}$ is a reduction of a properly larger ideal of $\widehat{A}$. Therefore an $\mathbf{n}$-primary ideal I of A is integrally closed if and only if $I\widehat{A}$ is integrally closed.

(8) If R is an integrally closed domain, then $\overline{xI} = x\overline{I}$, for every ideal I and element x of R; see Exercise 3.ii. If $(A, \mathbf{n})$ is analytically normal and also a UFD, then every height-one prime ideal of A extends to an integrally closed ideal of $\widehat{A}$ by item 4. In particular if A is a regular local domain, then A is a UFD by Remark 2.6.2, and so $P\widehat{A}$ is integrally closed for every height-one prime ideal P of A.

(9) If $(A, \mathbf{n})$ is a 2-dimensional local UFD, then every nonprincipal integrally closed ideal of A has the form xI, where I is an $\mathbf{n}$-primary integrally closed ideal and $x \in A$; see Exercise 4. In particular, this is the case if $(A, \mathbf{n})$ is a 2-dimensional regular local domain. It follows from items 7 and 8 that every integrally closed ideal of A extends to an integrally closed ideal of $\widehat{A}$ in the case where A is a 2-dimensional regular local domain.

(10) If $(A, \mathbf{n})$ is an excellent local ring, then the map $A \hookrightarrow \widehat{A}$ is quasi-normal by [**51**, (7.4.6) and (6.14.5)], and in this case every integrally closed ideal of A extends to an integrally closed ideal of $\widehat{A}$. If $(A, \mathbf{n})$ is a Nagata local ring, then for each prime ideal P of A, the ideal $P\widehat{A}$ is reduced, and hence integrally closed [**117**, Theorem 36.4].

(11) Let $(A, \mathbf{n})$ be a Noetherian local domain and let A^h denote the Henselization of A. Every integrally closed ideal of A extends to an integrally closed ideal of A^h. This follows because A^h is a filtered direct limit of étale A-algebras; see Lipman [**92**, (i), (iii), (vii) and (ix), pp. 800-801]. Since the map from A to its completion $\widehat{A}$ factors through A^h, every integrally closed ideal of A extends to an integrally closed ideal of $\widehat{A}$ if every integrally closed ideal of A^h extends to an integrally closed ideal of $\widehat{A}$.

11.2. Extending ideals to the completion

We present an example of a height-two prime ideal $I = (f, g)A$ of the 3-dimensional RLR $(A, \mathbf{n})$ such that the extension $I\widehat{A}$ to the $\mathbf{n}$-adic completion is not integrally closed. We use Example 10.13 and results from Chapters 5, 6, 9 and 10 to justify that $I\widehat{A}$ is not integrally closed.

In Example 11.8 we review the setting and basic description of the ring A of Example 10.13.1.

EXAMPLE 11.8. In the notation of Example 10.13.1, k is a field of characteristic zero, x, y and z are indeterminates over k, and the base ring $R := k[x, y, z]_{(x,y,z)}$. The (x)-adic completion of R is $R^* = k[y, z]_{(y,z)}[[x]]$, and σ, τ are elements of $xk[[x]]$ that are algebraically independent over $k(x)$. The Local Protoype $D := k(x, y, z, \sigma, \tau) \cap R^*$ of Local Prototype Theorem 9.6 is a three-dimensional regular local domain and a directed union of five-dimensional regular local domains.

With $f = (y-\sigma)^2$ and $g = (z-\tau)^2$, we define $A = k(x,y,z,f,g) \cap R^*$. By Example 10.13.1, the ring A is Noetherian, A is equal to its approximation domain, A is a 3-dimensional regular local domain, and A is not a Nagata ring.

The following commutative diagram, where all the labeled maps are the natural inclusions, displays this situation:

$$\begin{array}{ccccc} B = A = R^* \cap \mathcal{Q}(S) & \xrightarrow{\gamma_1} & D = R^* \cap \mathcal{Q}(T) & \xrightarrow{\gamma_2} & R^* = A^* \\ {\scriptstyle\delta_1}\uparrow & & {\scriptstyle\delta_2}\uparrow & & {\scriptstyle\psi}\uparrow \\ S = R[f,g] & \xrightarrow{\varphi} & T = R[\sigma,\tau] & = & T \end{array} \tag{11.1}$$

In order to better understand the structure of A, we recall some of the details of the approximation domain B associated to A.

Approximation Technique 11.9. With k, x, y, z, f, g, R and R^* as in Example 11.8, we have

$$f = y^2 + \sum_{j=1}^{\infty} b_j x^j, \qquad g = z^2 + \sum_{j=1}^{\infty} c_j x^j,$$

where $b_j \in k[y]$ and $c_j \in k[z]$. The r^{th} *endpiece*s for f and g are the sequences $\{f_r\}_{r=1}^{\infty}$, $\{g_r\}_{r=1}^{\infty}$ of elements in R^* defined for each $r \geq 1$ by:

$$f_r := \sum_{j=r+1}^{\infty} \frac{b_j x^j}{x^r} \qquad \text{and} \qquad g_r := \sum_{j=r+1}^{\infty} \frac{c_j x^j}{x^r}.$$

Then $f = y^2 + xb_1 + xf_1 = y^2 + xb_1 + x^2b_2 + x^2f_2 = \ldots$ and similar equations hold for g. Thus we have:

$$f = y^2 + xb_1 + x^2b_2 + \ldots x^tb_t + x^tf_t; \quad g = y^2 + xc_1 + x^2c_2 + \ldots x^tc_t + x^tg_t, \tag{11.9.0}$$

for each $t \geq 1$.

For each integer $r \geq 1$, we define the ring $B_r = (U_r)_{\mathbf{m}_r}$, where U_r is the polynomial ring $k[x,y,z,f_r,g_r]$ and $\mathbf{m}_r$ is the maximal ideal $(x,y,z,f_r,g_r)U_r$. We define the approximation domain $B := \bigcup_{r=1}^{\infty} B_r$.

Theorem 11.10. *With the notation of Example 11.8 and Approximation Technique 11.9, let $P = (f,g)A$. Then*

1. *The ring $A = B$ is a 3-dimensional regular local domain that has (x)-adic completion $A^* = R^* = k[y,z]_{(y,z)}[[x]]$. Moreover A is a nested union of five-dimensional regular local domains.*
2. *The ideal P is a height-two prime ideal of A.*
3. *The ideal PA^* is not integrally closed in A^*.*
4. *The completion $\widehat{A}$ of A is $\widehat{R} = k[[x,y,z]]$ and $P\widehat{A}$ is not integrally closed.*

Proof. Item 1 follows from Example 11.8 and Theorem 10.3, parts 3 and 4.

For item 2, it suffices to observe that P has height two and that, for each positive integer r, $P_r := (f,g)U_r$ is a prime ideal of U_r. We have $f = y^2 + xb_1 + xf_1$ and $g = z^2 + xc_1 + xg_1$. It is clear that $(f,g)k[x,y,z,f,g]$ is a height-two prime ideal.

Since $U_1 = k[x, y, z, f_1, g_1]$ is a polynomial ring over k in the variables x, y, z, f_1, g_1, we see that

$$P_1U_1[1/x] = (xb_1 + xf_1 + y^2, xc_1 + xg_1 + z^2)U_1[1/x]$$

is a height-two prime ideal of $U_1[1/x]$. Indeed, setting $f = g = 0$ is equivalent to setting $f_1 = -b_1 - y^2/x$ and $g_1 = -c_1 - z^2/x$. Therefore the residue class ring $(U_1/P_1)[1/x]$ is isomorphic to the integral domain $k[x, y, z][1/x]$. Since U_1 is Cohen-Macaulay and f, g form a regular sequence, and since $(x, f, g)U_1 = (x, y^2, z^2)U_1$ is an ideal of height three, we see that x is in no associated prime ideal of $(f, g)U_1$ (see, for example [**103**, Theorem 17.6]). Therefore $P_1 = (f, g)U_1$ is a height-two prime ideal, and so the same holds for P_1B_1.

For $r > 1$, by Equation 11.9.0, there exist elements $u_r \in k[x, y]$ and $v_r \in k[x, z]$ such that $f = x^r f_r + u_r x + y^2$ and $g = x^r g_r + v_r x + z^2$. An argument similar to that given above shows that $P_r = (f, g)U_r$ is a height-two prime ideal of U_r. Since U is the nested union of the U_r we have that $(f, g)U$ is a height-two prime ideal of U. Since B is a localization of U we see that $(f, g)B$ is a height-two prime ideal of $B = A$.

For items 3 and 4, $R^* = B^* = A^*$ by Example 11.8 and it follows that $\widehat{A} = k[[x, y, z]]$. To see that $PA^* = (f, g)A^*$ and $P\widehat{A} = (f, g)\widehat{A}$ are not integrally closed, observe that $\xi := (y - \alpha)(z - \beta)$ is integral over PA^* and $P\widehat{A}$ since $\xi^2 = fg \in P^2$. On the other hand, $y - \alpha = u$ and $z - \beta = v$ form a regular sequence in A^* and $\widehat{A}$. Since $P = (u^2, v^2)A$, an easy computation shows that $uv \notin P\widehat{A} = (u^2, v^2)\widehat{A}$; see Exercise 2. Since $PA^* \subseteq P\widehat{A}$, this completes the proof. □

In Example 11.11, we generalize the technique of Example 11.8 to obtain non-Nagata RLRs similar to Example 11.8 in higher dimensions.

Example 11.11. Let k be a field of characteristic zero and, for an integer $n \geq 2$, let $x, y_1, \ldots, y_n$ be indeterminates over k. Let h be an integer with $2 \leq h \leq n$, and let $\tau_1, \ldots, \tau_h \in xk[[x]]$ be algebraically independent over $k(x)$. Let $R := k[x, y_1, \ldots, y_n]_{(x,y_1,\ldots,y_n)}$, a $d := n + 1$-dimensional regular local ring. For each i with $1 \leq i \leq h$, define $f_i = (y_i - \tau_i)^h$, and set $u_i = y_i - \tau_i$. The rings

$$S := R[f_1, \ldots, f_h] \quad \text{and} \quad T := R[\tau_1, \ldots, \tau_h] = R[u_1, \ldots, u_h].$$

are polynomial rings in h variables over R, and T is a finite free integral extension of S. The set

$$\{u_1^{e_1} \cdot u_2^{e_2} \cdot \ldots \cdot u_h^{e_h} \mid 0 \leq e_i \leq h - 1\}$$

is a free module basis for T as an S-module. Therefore the map $S \hookrightarrow T[1/x]$ is flat. Let R^* denote the (x)-adic completion of R, and define D to be the Local Prototype domain $D := k(x, y_1, \ldots, y_n, \tau_1, \ldots, \tau_h) \cap R^*$ of Theorem 9.6. Let $A := k(x, y_1, \ldots, y_n, f_1, \ldots, f_h) \cap R^*$. By Construction Properties Theorem 5.14.4, the rings D and A have (x)-adic completion R^*. Since the map $S \hookrightarrow T$ is flat, Theorem 10.3 implies that the ring A is a d-dimensional regular local ring and is equal to its approximation domain B; thus A is a directed union of $(d + h)$-dimensional regular local domains.

The following commutative diagram where the labeled maps are the natural inclusions displays this situation:

$$\begin{array}{ccccc} B = A = R^* \cap \mathcal{Q}(S) & \xrightarrow{\gamma_1} & D = R^* \cap \mathcal{Q}(T) & \xrightarrow{\gamma_2} & R^* = A^* \\ {\scriptstyle\delta_1}\uparrow & & {\scriptstyle\delta_2}\uparrow & & {\scriptstyle\psi}\uparrow \\ S = R[f_1, \ldots, f_h] & \xrightarrow{\varphi} & T = R[\tau_1, \ldots, \tau_h] & = & T \end{array}$$

Since the field k has characteristic zero and $h \geq 2$, the Jacobian ideal of the map $\varphi : S \hookrightarrow T$ has radical $\sqrt{(J)} = \Pi_{i=1}^{h}(y_i - \tau_i)T$, and $JR^*[1/x] \neq R^*[1/x]$. Hence, by Corollary 10.11, the ring A is not excellent.

REMARK 11.12. Let k be a field of characteristic zero and for an integer $n \geq 2$ let $x, y_1, \ldots, y_n$ be indeterminates over k. Let R denote the $d := n+1$ dimensional regular local ring obtained by localizing the polynomial ring $k[x, y_1, \ldots, y_n]$ at the maximal ideal generated by $(x, y_1, \ldots, y_n)$. Let h be an integer with $2 \leq h \leq n$ and let $\tau_1, \ldots, \tau_h \in xk[[x]]$ be algebraically independent over $k(x)$. For each i with $1 \leq i \leq h$, define $f_i = (y_i - \tau_i)^h$, and set $u_i = y_i - \tau_i$. Consider the rings

$$S \ := \ R[f_1, \ldots, f_h] \quad \text{and} \quad T \ := \ R[\tau_1, \ldots, \tau_h] \ = \ R[u_1, \ldots, u_h].$$

Notice that S and T are polynomial rings in h variables over R and that T is a finite free integral extension of S. The set

$$\{u_1^{e_1} \cdot u_2^{e_2} \cdot \ldots \cdot u_h^{e_h} \mid 0 \leq e_i \leq h-1\}$$

is a free module basis for T as an S-module. Therefore the map $S \hookrightarrow T[1/x]$ is flat. Let R^* denote the (x)-adic completion of R, and define

$$A \ := \ R^* \ \cap \ \mathcal{Q}(S) \quad \text{and} \quad D \ := \ R^* \ \cap \ \mathcal{Q}(T).$$

The following commutative diagram where the labeled maps are the natural inclusions displays this situation:

$$\begin{array}{ccccc} B = A = R^* \cap \mathcal{Q}(S) & \xrightarrow{\gamma_1} & D = R^* \cap \mathcal{Q}(T) & \xrightarrow{\gamma_2} & R^* = A^* \\ {\scriptstyle\delta_1}\uparrow & & {\scriptstyle\delta_2}\uparrow & & {\scriptstyle\psi}\uparrow \\ S = R[f_1, \ldots, f_h] & \xrightarrow{\varphi} & T = R[\tau_1, \ldots, \tau_h] & = & T \end{array}$$

Since the map $S \hookrightarrow T$ is flat, Theorem 10.3 implies that the ring A is a d-dimensional regular local ring and is equal to its approximation domain B. Let $P := (f_1, \ldots, f_h)A$; we see that an argument similar to that given in Theorem 11.10 shows that P is a prime ideal of A of height h. We have $y_i - \tau_i = u_i \in A^*$. Let $\xi = \prod_{i=1}^{h} u_i$. Then $\xi^h = f_1 \cdots f_h \in P^h$ implies ξ is integral over PA^* and $P\widehat{A}$. Since $u_1, \ldots, u_h$ are a regular sequence in A^* and $\widehat{A}$, it follows that $\xi \notin P\widehat{A}$; see the thesis of Taylor [**152**, Theorem 1]. Therefore the extended ideals PA^* and $P\widehat{A}$ are not integrally closed.

11.3. Comments and Questions

In connection with Theorem 11.10 it is natural to ask the following question.

QUESTION 11.13. For P and A as in Theorem 11.10, is P the only prime ideal of A that does not extend to an integrally closed ideal of $\widehat{A}$?

COMMENTS 11.14. In relation to Example 11.8 and to Question 11.13, consider the following commutative diagram, where the labeled maps are the natural inclusions:

$$\begin{array}{ccccc} B = A = R^* \cap \mathcal{Q}(S) & \xrightarrow{\gamma_1} & D = R^* \cap \mathcal{Q}(T) & \xrightarrow{\gamma_2} & R^* = A^* \\ {\scriptstyle\delta_1}\uparrow & & {\scriptstyle\delta_2}\uparrow & & {\scriptstyle\psi}\uparrow \\ S = R[f,g] & \xrightarrow{\varphi} & T = R[\alpha,\beta] & = & T \end{array}$$

Referring to the diagram above, we observe the following:

(1) Theorem 10.3 implies that $A[1/x]$ is a localization of S and $D[1/x]$ is a localization of T. By Prototype Theorem 17.25 of Chapter 9, D is excellent. Notice, however, that A is not excellent since there exists a prime ideal P of A such that $P\widehat{A}$ is not integrally closed by Remark 11.7.10. The excellence of D implies the map $\gamma_2 : D \to A^*$ is regular [**51**, (7.8.3 v)]. Thus, for each $Q^* \in \operatorname{Spec} A^*$ with $x \notin Q^*$ the map $\psi_{Q^*} : T \to A^*_{Q^*}$ is regular. It follows that $\psi_x : T \to A^*[1/x]$ is regular.

(2) Let $Q^* \in \operatorname{Spec} A^*$ be such that $x \notin Q^*$ and let $\mathbf{q}' = Q^* \cap T$. Assume that $\varphi_{\mathbf{q}'} : S \to T_{\mathbf{q}'}$ is regular. By item 1 and [**103**, Theorem 32.1], the map $S \to A^*_{Q^*}$ is regular. Thus $(\gamma_2 \circ \gamma_1)_{Q^*} : A \to A^*_{Q^*}$ is regular.

(3) Let I be an ideal of A. Since D and A^* are excellent and both have completion $\widehat{A}$, Remark 11.7.10 shows that the ideals ID, IA^* and $I\widehat{A}$ are either all integrally closed or all fail to be integrally closed.

(4) In this setting, the Jacobian ideal of $\varphi : S \hookrightarrow T$ gives information about the smoothness and regularity of φ by Theorems 7.8 and 7.14.1. The Jacobian ideal of $\varphi : S := k[x,y,z,f,g] \hookrightarrow T := k[x,y,z,\alpha,\beta]$ is the ideal of T generated by the determinant of the matrix

$$\mathcal{J} := \begin{pmatrix} \frac{\partial f}{\partial \alpha} & \frac{\partial g}{\partial \alpha} \\ \frac{\partial f}{\partial \beta} & \frac{\partial g}{\partial \beta} \end{pmatrix}.$$

Since the characteristic of the field k is zero, this ideal is $(y-\alpha)(z-\beta)T$.

In Proposition 11.15, we relate the behavior of integrally closed ideals in the extension $\varphi : S \to T$ to the behavior of integrally closed ideals in the extension $\gamma_2 \circ \gamma_1 : A \to A^*$.

PROPOSITION 11.15. *With the setting of Theorem 11.10, let I be an integrally closed ideal of A such that $x \notin Q$ for each $Q \in \operatorname{Ass}(A/I)$. Let $J = I \cap S$. If JT is integrally closed, respectively a radical ideal, then IA^* is integrally closed, respectively a radical ideal.*

PROOF. Since the map $A \to A^*$ is flat, Remark 11.7.3 implies that x is not in any associated prime ideal of IA^*. Therefore IA^* is contracted from $A^*[1/x]$ and it suffices to show $IA^*[1/x]$ is integrally closed (resp. a radical ideal). Our hypothesis implies $I = IA[1/x] \cap A$. By Comment 11.14.1, the ring $A[1/x]$ is a localization of S. Thus every ideal of $A[1/x]$ is the extension of its contraction to S. It follows that $IA[1/x] = JA[1/x]$. Thus $IA^*[1/x] = JA^*[1/x]$.

By Comment 11.14.1, the map $T \to A^*[1/x]$ is regular. If JT is integrally closed, then Remark 11.7.3 implies that $JA^*[1/x]$ is integrally closed. If JT is a

radical ideal, then the zero ideal of $\frac{T}{JT}$ is integrally closed. The regularity of the map $\frac{T}{JT} \to \frac{A^*[1/x]}{JA^*[1/x]}$ implies that the zero ideal of $\frac{A^*[1/x]}{JA^*[1/x]}$ is integrally closed. Since the integral closure of the zero ideal is the nilradical, it follows that $JA^*[1/x]$ is a radical ideal. □

PROPOSITION 11.16. *With the setting of Theorem 11.10 and Comment 11.14, let $Q \in \operatorname{Spec} A$ be such that QA^* (or equivalently $Q\widehat{A}$) is not integrally closed. Then*

1. *Q has height two and $x \notin Q$.*
2. *There exists a minimal prime ideal Q^* of QA^* such that with $\mathbf{q}' = Q^* \cap T$, the map $\varphi_{\mathbf{q}'} : S \to T_{\mathbf{q}'}$ is not regular.*
3. *Q contains $f = (y-\alpha)^2$ or $g = (z-\beta)^2$.*
4. *Q is contained in $\mathbf{n}^2$, where $\mathbf{n}$ is the maximal ideal of A.*

PROOF. We have $\dim A = 3$, the maximal ideal of A extends to the maximal ideal of A^*, and principal ideals of A^* are integrally closed by Remark 11.7.8. Thus the height of Q is two. By Construction Properties Theorem 5.14, we have $A^*/xA^* = A/xA = R/xR$. Hence $x \notin Q$. This proves item 1.

By Remark 11.7.3, there exists a minimal prime ideal Q^* of QA^* such that $(\gamma_2 \circ \gamma_1)_{Q^*} : A \to A^*_{Q^*}$ is not regular. Thus item 2 follows from Comment 11.14.2.

For item 3, let Q^* and $\mathbf{q}'$ be as in item 2. Since $(\gamma_2 \circ \gamma_1)_{Q^*}$ is not regular it is not essentially smooth [**51**, 6.8.1]. By Theorem 7.14.1, $(y-\alpha)(z-\beta) \in \mathbf{q}'$. Hence $f = (y-\alpha)^2$ or $g = (z-\beta)^2$ is in $\mathbf{q}'$ and thus in Q. This proves item 3.

Suppose $w \in Q$ is a regular parameter for A; that is $w \in \mathbf{n} \backslash \mathbf{n}^2$. Then A/wA and A^*/wA^* are two-dimensional regular local domains. By Remark 11.7.8, QA^*/wA^* is integrally closed, but this implies that QA^* is integrally closed, which contradicts our hypothesis that QA^* is not integrally closed. This proves item 4. □

QUESTION 11.17. In the setting of Theorem 11.10 and Comment 11.14, let $Q \in \operatorname{Spec} A$ with $x \notin Q$ and let $\mathbf{q} = Q \cap S$. If QA^* is integrally closed, does it follow that $\mathbf{q}T$ is integrally closed?

QUESTION 11.18. In the setting of Theorem 11.10 and Comment 11.14, if a prime ideal Q of A contains f or g, but not both, and does not contain a regular parameter of A, does it follow that QA^* is integrally closed?

In Example 11.8, the three-dimensional regular local domain A contains height-one prime ideals P such that $\widehat{A}/P\widehat{A}$ is not reduced. This motivates us to ask:

QUESTION 11.19. Let $(A, \mathbf{n})$ be a three-dimensional regular local domain and let $\widehat{A}$ denote the $\mathbf{n}$-adic completion of A. If for each height-one prime ideal P of A, the extension $P\widehat{A}$ is a radical ideal, i.e., the ring $\widehat{A}/P\widehat{A}$ is reduced, does it follow that $Q\widehat{A}$ is integrally closed for each $Q \in \operatorname{Spec} A$?

REMARK 11.20. A problem analogous to that considered here in the sense that it also deals with the behavior of ideals under extension to completion is addressed by Loepp and Rotthaus in [**95**]. They construct nonexcellent Noetherian local domains to demonstrate that tight closure need not commute with completion.

Exercises

(1) If I is a radical ideal of a ring R, then I is an integrally closed ideal of R.

(2) Let u, v be a regular sequence in a commutative ring R. Prove that $uv \notin (u^2, v^2)R$.

Suggestion: Use that if a, b are in R and $au = bv$, then $b \in uR$.

(3) Let R be an integrally closed domain.
   (i) Prove that every principal ideal in R is integrally closed.
   (ii) Let $0 \neq x \in R$ and let I be an ideal of R. Prove that $x\overline{I} = \overline{xI}$.

Suggestion: Show that if $a \in \overline{xI}$, then a/x is in R.

(4) (i) Prove that if A is a UFD, then every nonzero ideal of A has the form xI, where I is an ideal of A that is not contained an any proper principal ideal of A.
   (ii) Prove every non principal integrally closed ideal of a two-dimensional local UFD $(A, \mathbf{n})$ has the form xI, where $x \in A$ and I is an $\mathbf{n}$-primary integrally closed ideal of A.

CHAPTER 12

The iterative examples

We present a family of examples contained in $k[[x,y]]$, where k is a field and x and y are indeterminates. We show that for certain values of the parameters that occur in the examples, one obtains an example of a 3-dimensional local Krull domain $(B,\mathbf{n})$ such that B is not Noetherian, $\mathbf{n} = (x,y)B$ is 2-generated and the $\mathbf{n}$-adic completion $\widehat{B}$ of B is a two-dimensional regular local domain; see Example 12.6. These examples are *iterative* in the sense that they arise from applying the inclusion construction twice, first using an (x)-adic completion and then using a (y)-adic completion.

Let R be the localized polynomial ring $R := k[x,y]_{(x,y)}$. If $\sigma, \tau \in \widehat{R} = k[[x,y]]$ are algebraically independent over R, then the polynomial ring $R[t_1,t_2]$ in two variables t_1, t_2 over R, can be identified with a subring of $\widehat{R}$ by means of an R-algebra isomorphism mapping $t_1 \to \sigma$ and $t_2 \to \tau$. The structure of the local domain $A = k(x,y,\sigma,\tau) \cap \widehat{R}$ depends on the residual behavior of σ and τ with respect to prime ideals of $\widehat{R}$. Theorem 12.3 illustrates this in the special case where $\sigma \in k[[x]]$ and $\tau \in k[[y]]$. A special case of this is given in Example 4.10.

Remark 12.1. In examining properties of subrings of the formal power series ring $k[[x,y]]$ over the field k, we use that the subfields $k((x))$ and $k((y))$ of the field $\mathcal{Q}(k[[x,y]])$ are linearly disjoint over k as defined for example in [**164**, page 109]. It follows that if $\alpha_1, \ldots, \alpha_n \in k[[x]]$ are algebraically independent over $k(x)$ and $\beta_1, \ldots, \beta_m \in k[[y]]$ are algebraically independent over $k(y)$, then the elements $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$ are algebraically independent over $k(x,y)$.

12.1. The iterative examples and their properties

We begin by fixing notation. We include several remarks concerning the integral domains that are used in the proof of Theorem 12.3.

Notation and Remarks 12.2. Let k be a field, let x and y be indeterminates over k, and let

$$\sigma := \sum_{i=1}^{\infty} a_i x^i \in xk[[x]] \qquad \text{and} \qquad \tau := \sum_{i=1}^{\infty} b_i y^i \in yk[[y]]$$

be formal power series that are algebraically independent over the fields $k(x)$ and $k(y)$, respectively. Let $R := k[x,y]_{(x,y)}$, and let σ_n, τ_n be the n^{th} endpieces of σ, τ defined as in Equation 5.41. We define the following rings; we explain the equalities below:

$$
\begin{aligned}
C_n &:= k[x,\sigma_n]_{(x,\sigma_n)}, \quad C := k(x,\sigma)\cap k[[x]] = \cup_{n=1}^{\infty} C_n;\\
D_n &:= k[y,\tau_n]_{(y,\tau_n)}, \quad D := k(y,\tau)\cap k[[y]] = \cup_{n=1}^{\infty} D_n;\\
U_n &:= k[x,y,\sigma_n,\tau_n], \qquad U := \cup_{n=1}^{\infty} U_n;\\
B_n &:= k[x,y,\sigma_n,\tau_n]_{(x,y,\sigma_n,\tau_n)} \quad B := \cup_{n=1}^{\infty} B_n;\\
A &:= k(x,y,\sigma,\tau)\cap k[[x,y]].
\end{aligned}
\tag{12.2.0}
$$

(i) Since $k[[x,y]]$ is the (x,y)-adic completion of the Noetherian ring R, Remark 3.2.4 implies that $k[[x,y]]$ is faithfully flat over R. By Remark 2.31.7

$$(x,y)^n k[[x,y]] \cap R = (x,y)^n R$$

for each $n \in \mathbb{N}$. Endpiece Recursion Relation 5.5 implies for each positive integer n the inclusions

$$C_n \subset C_{n+1}, \quad D_n \subset D_{n+1}, \quad \text{and} \quad B_n \subset B_{n+1}.$$

Moreover, for each of these inclusions we have birational domination of the larger local ring over the smaller, and the local rings C_n, D_n, B_n are all dominated by $k[[x,y]] = \widehat{R}$.

(ii) We observe that $(x,y)B \cap B_n = (x,y,\sigma_n,\tau_n)B_n$, for each $n \in \mathbb{N}$: To see this, let $h \in (x,y,\sigma_n,\tau_n)U_n$. Then Equation 5.5 implies that

$$\sigma_n = -xa_{n+1} + x\sigma_{n+1} \quad \text{and} \quad \tau_n = -yb_{n+1} + y\tau_{n+1}.$$

Hence $h \in (x,y)U_{n+1} \cap U_n \subseteq (x,y)U \cap U_n$. Since $(x,y,\sigma_n,\tau_n)U_n$ is a maximal ideal of U_n and is contained in $(x,y)U$, a proper ideal of U, it follows that

$$(x,y)U \cap U_n = (x,y,\sigma_n,\tau_n)U_n.$$

Thus also $(x,y)B \cap B_n = (x,y,\sigma_n,\tau_n)B$ for each $n \in \mathbb{N}$.

(iii) We also observe that $U_n[\frac{1}{xy}] = U[\frac{1}{xy}]$ for each $n \in \mathbb{N}$: By Equation 5.4.2 we have $\sigma_{n+1} \in U_n[\frac{1}{x}] \subseteq U_n[\frac{1}{xy}]$ and $\tau_{n+1} \in U_n[\frac{1}{y}] \subseteq U_n[\frac{1}{xy}]$. Hence $U_{n+1} \subseteq U_n[\frac{1}{xy}]$ for each $n \in \mathbb{N}$. Hence $U \subseteq U_n[\frac{1}{xy}]$, and $U_n[\frac{1}{xy}] = U[\frac{1}{xy}]$.

(iv) By Remark 2.1, the rings C and D are rank-one discrete valuation domains; as in Remark 4.19 they are the asserted directed unions. The rings B_n are four-dimensional regular local domains that are localized polynomial rings over the field k. Thus the approximation domain B is the directed union of a chain of four-dimensional regular local rings, with each ring birational over the previous ring.

In Theorem 12.3 we prove other properties of the rings A and B.

THEOREM 12.3. *Assume Notation 12.2. Then the ring A is a two-dimensional regular local domain that birationally dominates the ring B; A has maximal ideal $(x,y)A$ and completion $\widehat{A} = k[[x,y]]$. Moreover we have:*

1. *The rings U and B are UFDs, and $B = U_{(x,y)}$.*
2. *B is a local Krull domain with maximal ideal $\mathbf{n} = (x,y)B$.*
3. *B is Hausdorff in the topology defined by the powers of $\mathbf{n}$.*
4. *The $\mathbf{n}$-adic completion $\widehat{B}$ of B is canonically isomorphic to $k[[x,y]]$.*
5. *The dimension of B is either 2 or 3.*
6. *The following statements are equivalent:*
   (a) *$B = A$.*

(b) *B is a two-dimensional regular local domain.*
(c) $\dim B = 2$.
(d) *B is Noetherian.*
(e) *In the $\mathbf{n}$-adic topology every finitely generated ideal of B is closed.*
(f) *In the $\mathbf{n}$-adic topology every principal ideal of B is closed.*

To establish the asserted properties of the ring A of Theorem 5.3, we use the following consequence of the useful result of Valabrega stated as Theorem 4.8 above. Since the construction of A can be realized by a succession of two principal ideal-adic completions, first using power series in x, then using power series in y, we consider it an "iterative" example.

PROPOSITION 12.4. *With the notation of Theorem 5.3, let $C = k(x, \sigma) \cap k[[x]]$ and let L be the field of fractions of $C[y, \tau]$. Then the ring $A = L \cap C[[y]]$ is a two-dimensional regular local domain with maximal ideal $(x, y)A$ and completion $\widehat{A} = k[[x, y]]$.*

PROOF. The ring C is a rank-one discrete valuation domain with completion $k[[x]]$, and the field $k(x, y, \sigma, \tau) = L$ is an intermediate field between the fields of fractions of the rings $C[y]$ and $C[[y]]$. Hence, by Theorem 4.8, $A = L \cap C[[y]]$ is a regular local domain with completion $k[[x, y]]$. □

PROOF. The assertions about A follow from Proposition 12.4, a consequence of Valabrega's Theorem 5.3. Since $U_0 \subseteq B \subseteq A$ and the field of fractions of U_0 is $\mathcal{Q}(U_0) = k(x, y, \sigma, \tau) = \mathcal{Q}(A)$, the extension $B \hookrightarrow A$ is birational. By Remark 12.2.ii above, we have $(x, y)U \cap U_n = (x, y, \sigma_n, \tau_n)U_n$. It follows that $(x, y)U$ is a maximal ideal of U, and $B = U_{(x,y)}$ is local with maximal ideal $\mathbf{n} = (x, y)B$. Since B and A are both dominated by $k[[x, y]]$, we have that A dominates B.

We now prove that U and B are UFDs. By Equation 12.2.0 and Remark 12.2.iii, U_n is a polynomial ring over a field and $U_n[\frac{1}{xy}] = U[\frac{1}{xy}]$. Thus the ring $U[\frac{1}{xy}]$ is a UFD. For each $n \in \mathbb{N}$, the principal ideals xU_n and yU_n are prime ideals in the polynomial ring U_n. Therefore xU and yU are principal prime ideals of U. Moreover, $U_{xU} = B_{xB}$ and $U_{yU} = B_{yB}$ are DVRs since each is the contraction to the field $k(x, y, \sigma, \tau)$ of the (x)-adic or the (y)-adic valuations of $k[[x, y]]$; see Remark 2.5.

By applying Fact 2.22 with $S = U$ and $c = x$, and then with $S = U[\frac{1}{x}]$ and $c = y$, we obtain $U = U_{xU} \cap U_{yU} \cap U[\frac{1}{xy}]$. Since $U[\frac{1}{xy}] = U_n[\frac{1}{xy}]$, we have $U[\frac{1}{xy}]$ is a Krull domain, and so also U is a Krull domain; see Definition 2.7 and Remarks 2.8. Hence, by Nagata's Theorem 2.21, U is a UFD. Since B is a localization of U, the ring B is a UFD. This completes the proof of items 1 and 2.

Since B is dominated by $k[[x, y]]$, the intersection $\bigcap_{n=1}^{\infty} \mathbf{n}^n = (0)$. Thus B is Hausdorff in the topology defined by the powers of $\mathbf{n}$ [**121**, Proposition 4, page 381], as in Definitions 3.1. We have local injective maps $R \hookrightarrow B \hookrightarrow \widehat{R}$, and we have $\mathbf{m}^n B = \mathbf{n}^n$, $\mathbf{m}^n \widehat{R} = \widehat{\mathbf{m}}^n$ and $\widehat{\mathbf{m}}^n \cap R = \mathbf{m}^n$, for each positive integer n. Since the natural map $R/\mathbf{m}^n \to \widehat{R}/\mathbf{m}^n\widehat{R} = \widehat{R}/\widehat{\mathbf{m}}^n$ is an isomorphism, the map $R/\mathbf{m}^n \to B/\mathbf{m}^n B = B/\mathbf{n}^n$ is injective and the map $B/\mathbf{n}^n \to \widehat{R}/\mathbf{n}^n\widehat{R} = \widehat{R}/\widehat{\mathbf{m}}^n$ is surjective. Since $B/\mathbf{n}^n$ is a finite length R-module, it follows that for each $n \in \mathbb{N}$

$$R/\mathbf{m}^n \cong B/\mathbf{n}^n \cong \widehat{R}/\widehat{\mathbf{m}}^n.$$

Hence $\widehat{B} = \widehat{R} = k[[x,y]]$. Notice that B is a birational extension of the three-dimensional Noetherian domain $C[y,\tau]$. The dimension of B is at most 3 by Theorem 2.20, a theorem of Cohen; also see [**103**, Theorem 15.5]. The elements x and y are in the Jacobson radical $\mathcal{J}(B_n)$, of B_n for each $n \in \mathbb{N}$, and so $x, y \in \mathcal{J}(B)$. If $\dim B = 1$, then the local UFD B is a DVR, and so, by Remark 3.2.4,

$$1 = \dim B = \dim \widehat{B} = \dim(k[[x,y]]) = 2,$$

a contradiction. Hence $\dim B \geq 2$. This completes the proof of items 3, 4 and 5.

For item 6, by Proposition 12.4, we have A is a two-dimensional RLR. Thus $(a) \implies (b)$. Clearly $(b) \implies (c)$. By items 1 and 2, B is a local UFD with maximal ideal $\mathbf{n} = (x,y)B$. Hence every prime ideal of B is finitely generated. Thus by Cohen's Theorem 2.19 we have $(c) \implies (d)$. Since B is local and since the completion of a Noetherian local ring is a faithfully flat extension by Remark 3.2.4, we have $(d) \implies (e)$ by Remark 2.31.7. It is clear that $(e) \implies (f)$. To complete the proof of Theorem 12.3, it suffices to show that $(f) \implies (a)$. Since A birationally dominates B, we have $B = A$ if and only if $bA \cap B = bB$ for every element $b \in \mathbf{n}$; see Exercise 2.ii of Chapter 4. The principal ideal bB is closed in the $\mathbf{n}$-adic topology on B if and only if $bB = b\widehat{B} \cap B$. Also $\widehat{B} = \widehat{A}$ and $bA = b\widehat{A} \cap A$, for every $b \in B$. Thus (f) implies, for every $b \in B$,

$$bB = b\widehat{B} \cap B = b\widehat{A} \cap B = b\widehat{A} \cap A \cap B = bA \cap B,$$

and so $B = A$. This completes the proof of Theorem 12.3. □

Remark 12.5. With σ, τ and B as in Notation 12.2, items 5 and 6 of Theorem 12.3 establish that either the approximation domain B has dimension two and is Noetherian or B has dimension three and is not Noetherian. The theorem does not complete the proof of Theorem 5.3, however, because the statement of Theorem 5.3 asserts that both types of approximation domains exist and that B depends upon the choice of σ and τ. In the remainder of this chapter we establish the existence of both types for B, and illustrate the effect of the choice of σ and τ on the resulting approximation domain B; see Examples 12.6 and 12.19. This will complete the proof of Theorem 5.3.

Example 12.6 shows that in the setting of Theorem 12.3 the ring B can be non-Noetherian and strictly smaller than $A := k(x,y,\sigma,\tau) \cap k[[x,y]]$.

Example 12.6. With Notation 12.2, let $\tau \in k[[y]]$ be defined to be $\sigma(y)$, that is, set $b_i := a_i$, for every $i \in \mathbb{N}$. We then have that $\theta := \frac{\sigma - \tau}{x - y} \in A$. To see this, write

$$\sigma - \tau = a_1(x-y) + a_2(x^2 - y^2) + \cdots + a_n(x^n - y^n) + \cdots,$$

and so $\theta = \frac{\sigma - \tau}{x-y} \in k[[x,y]] \cap k(x,y,\sigma,\tau) = A$. As a specific example, one may take $k := \mathbb{Q}$ and set $\sigma := e^x - 1$ and $\tau := e^y - 1$.

Claim 12.7 below and Theorem 12.3 above together imply that, if $\tau = \sigma(y)$, then the approximation domain B is non-Noetherian and strictly contained in the corresponding intersection domain A.

Claim 12.7. *The element θ is not in B, and so $B \subsetneq A$.*

Proof. If θ is an element of B, then

$$\sigma - \tau \ \in \ (x-y)B \ \cap \ U \ = \ (x-y)U.$$

Let $S := k[x, y, \sigma, \tau]$ and let $U_n := k[x, y, \sigma_n, \tau_n]$ for each positive integer n. We have

$$U \;=\; \bigcup_{n\in\mathbb{N}} U_n \;\subseteq\; S[\frac{1}{xy}] \;\subset\; S_{(x-y)S},$$

where the last inclusion is because $xy \notin (x-y)S$. Thus $\theta \in B$ implies that

$$\sigma - \tau \in (x-y)S_{(x-y)S} \;\cap\; S \;=\; (x-y)S,$$

but this contradicts the fact that x, y, σ, τ are algebraically independent over k, and thus S is a polynomial ring over k in x, y, σ, τ. □

In contrast to Example 12.6, a Krull domain that birationally dominates a two-dimensional Noetherian local domain is Noetherian; see Exercise 13 in Chapter 2.

In Remarks 12.8 we justify using the words "Iterative Example" in the title of this chapter to describe the construction of the rings B and A of Notation 12.2.

REMARKS 12.8. Assume Notation 12.2. Thus $R = k[x, y]_{(x,y)}$; $C_n = k[x, \sigma_n]_{(x,\sigma_n)}$; $C = \; k(x, \sigma) \cap k[[x]] \;=\; \cup_{n=1}^{\infty} C_n$; $\quad B_n = \; k[x, y, \sigma_n, \tau_n]_{(x,y,\sigma_n,\tau_n)}$; $\quad B = \; \cup_{n=1}^{\infty} B_n$; and $A = k(x, y, \sigma, \tau) \cap k[[x, y]]$.

(1) We observe that the rings B and A may be obtained by "iterating" Inclusion Construction 5.3 and the approximation in Section 5.2. To see this, we define a ring T associated with A and B:

$$T_n : = k[x, y, \sigma_n]_{(x,y,\sigma_n)} = C_n[y]_{(x,y,\sigma_n)}, \quad T := \bigcup_{n=1}^{\infty} T_n.$$

The ring T is a Prototype Example, and so $T = k[y]_{(y)}[[x]] \cap k(x, y, \sigma) = C[y]_{(x,y)}$, a two-dimensional regular local domain, as in Localized Prototype Theorem 17.28.1. If $\operatorname{char} k = 0$, then T is excellent. The ring T is the result of one iteration of the construction, where we have taken an (x)-adic completion of R and used the power series σ.

For each positive integer n, $B_n \subset T[\tau_n]_{(x,y,\tau_n)} \subset B$. Hence by definition $B = \bigcup_{n=1}^{\infty} T[\tau_n]_{(x,y,\tau_n)}$. Thus, as in Construction Properties Theorem 5.14.6, B is the approximation domain obtained using the power series τ and applying the construction with T as the base ring.

(2) By applying Remark 12.8.1 we obtain alternate proofs of parts of Theorem 12.3. By Theorem 5.17 and its proof, T, U and B are UFDs and items 1 and 2 hold. By Construction Properties Theorem 5.14, item 4 holds. Moreover part d of item 6 implies part a, by Noetherian Flatness Theorem 6.3.

(3) In addition, item 1 justifies our use of the results of Chapters 5, 6 and 9 in the remainder of this chapter to show there exist σ and τ such that $A = B$.

As stated in Remark 12.5, the ring B may be Noetherian for certain choices of σ and τ. To obtain an example of a triple σ, τ and B fitting Notation 12.2 where B is Noetherian, we first establish in Example 12.19 below with $k = \mathbb{Q}$ that the elements $\sigma := e^x - 1$ and $\tau := e^{(e^y-1)} - 1$ give an example where $B = A$. As we show in Proposition 12.12, the critical property of τ used to prove B is Noetherian and $A = B$ is that, for $T = C[y]_{(x,y)}$, the image of τ in $\widehat{R}/Q$ is algebraically independent over $T/(Q \cap T)$, for each height-one prime ideal Q of $\widehat{R} = \mathbb{Q}[[x, y]]$ such that $Q \cap T \neq (0)$ and $xy \notin Q$. We use Noetherian Flatness Theorem 6.3 to prove Proposition 12.12. In order to show that the property of Proposition 12.12

holds for $\tau = e^{(e^y-1)} - 1$ in the proof of Theorem 12.16, we use results of Ax that yield generalizations of Schanuel's conjectures regarding algebraic relations satisfied by exponential functions [**14**, Corollary 1, p. 253].

REMARK 12.9. In Notation 12.2, It seems natural to consider the ring compositum $\widehat{C}[\widehat{D}]$ of $\widehat{C} = k[[x]]$ and $\widehat{D} = k[[y]]$. We outline in Exercise 3 of this chapter a proof due to Kunz that the subring $\widehat{C}[\widehat{D}]$ of $k[[x, y]]$ is not Noetherian.

12.2. Residual algebraic independence

Recall that an extension of Krull domains $R \hookrightarrow S$ satisfies the condition **PDE** ("pas d'éclatement", or in English "no blowing up") provided that $\mathrm{ht}(Q \cap R) \leq 1$ for each prime ideal of height one Q in S; see Definition 2.10. The iterative example leads us to consider in this section a related property as in the following definition.

DEFINITION 12.10. Let $R \hookrightarrow S$ denote an extension of Krull domains. An element $\nu \in S$ is *residually algebraically independent with respect to* S over R if ν is algebraically independent over R and for every height-one prime ideal Q of S such that $Q \cap R \neq 0$, the image of ν in S/Q is algebraically independent over $R/(Q \cap R)$.

REMARK 12.11. If $(R, \mathbf{m})$ is a regular local domain, or more generally an analytically normal Noetherian local domain, it is natural to consider the extension of Krull domains $R \hookrightarrow \widehat{R}$, where $\widehat{R}$ is the $\mathbf{m}$-adic completion of R, and to ask about the existence of an element $\nu \in \widehat{R}$ that is residually algebraically independent with respect to $\widehat{R}$ over R. If the dimension of R is at least two and R has countable cardinality, for example, if $R = \mathbb{Q}[x, y]_{(x,y)}$, then a cardinality argument implies the existence of an element $\nu \in \widehat{R}$ that is residually algebraically independent with respect to $\widehat{R}$ over R; see Theorems 20.20 and 20.27.

We show in Proposition 20.15 and Theorem 20.27 of Chapter 20 that, if $\nu \in \widehat{\mathbf{m}}$ is residually algebraically independent with respect to $\widehat{R}$ over R, then the intersection domain $A = \widehat{R} \cap \mathcal{Q}(R[\nu])$ is the localized polynomial ring $R[\nu]_{(\mathbf{m},\nu)}$. Therefore A is Noetherian and the completion $\widehat{A}$ of A is a formal power series ring in one variable over $\widehat{R}$. As in Exercise 6 of Chapter 3, the local inclusion maps $R \hookrightarrow A \hookrightarrow \widehat{R}$ determine a surjective map of $\widehat{A}$ onto $\widehat{R}$. Since $\dim \widehat{A} > \dim \widehat{R}$, this surjective map has a nonzero kernel. Hence A is not a a *subspace* of $\widehat{R}$, that is, the topology on A determined by the powers of the maximal ideal of A is not the same as the *subspace topology* on A defined by intersecting the powers of the maximal ideal of $\widehat{R}$ with A.

The existence of an element that is almost residually algebraically independent is important in completing the proof of the iterative examples of Section 12.1, as we demonstrate in Proposition 12.12 and Theorem 12.16. In the proof of these results we use Noetherian Flatness Theorem 6.3 of Chapter 6. In the proof of Proposition 12.12 we show that our setting here fits Inclusion Construction 5.3 and the approximation procedure of Section 5.2, and so Theorem 6.3 implies that the intersection domain equals the approximation domain and is Noetherian provided a certain extension is flat.

PROPOSITION 12.12. *With Notation 12.2, let* $T = C[y]_{(x,y)C[y]}$. *Thus* T *is a two-dimensional regular local domain with completion* $\widehat{T} = k[[x, y]] = \widehat{R}$. *If the*

image of τ in $C[[y]]/Q$ is algebraically independent over $T/(Q\cap T)$ for each height-one prime ideal Q of $C[[y]]$ such that $Q\cap T\neq(0)$ and $xy\notin Q$, then B is Noetherian and $B=A$.

PROOF. As observed in Remark 12.8, B is obtained from T by Inclusion Construction 5.3, and so Noetherian Flatness Theorem 6.3 applies. Thus, in order to show that B is Noetherian and $B=A$, it suffices to show that the map

$$\phi_y : T[\tau] \longrightarrow C[[y]][1/y]$$

is flat; see Definition 2.30. By Remark 2.31.1, flatness is a local property. Hence it suffices to show for each prime ideal Q of $C[[y]]$ with $y\notin Q$ that the induced map $\phi_Q : T[\tau]_{Q\cap T[\tau]} \longrightarrow C[[y]]_Q$ is flat. If $\mathrm{ht}(Q\cap T[\tau])\leq 1$, then $T[\tau]_{Q\cap T[\tau]}$ is either a field or a DVR. We have that $C[[y]]_Q$ is torsionfree over $T[\tau]_{Q\cap T[\tau]}$. Thus, by Remark 2.33.3, ϕ_Q is flat. Therefore it suffices to show that $\mathrm{ht}(Q\cap T[\tau])\leq 1$. This is clear for $Q=xC[[y]]$. On the other hand, if $xy\notin Q$, then by hypothesis, the image $\bar{\tau}$ of τ in $C[[y]]/Q$ is algebraically independent over $T/(Q\cap T)$, and we have the following maps:

$$\frac{T}{Q\cap T}[\bar{\tau}] \;=\; \frac{T[\tau]}{(Q\cap T)T[\tau]} \;\overset{\alpha}{\to}\; \frac{T[\tau]}{Q\cap T[\tau]} \;\overset{\beta}{\hookrightarrow}\; \frac{C[[y]]}{Q}.$$

The map α is surjective and the composition $\beta\circ\alpha$ is injective. Since $C[[y]]$ is faithfully flat over T, we have $\mathrm{ht}(Q\cap T)\leq 1$. If $\mathrm{ht}(Q\cap T[\tau])=2$, then the image of τ in $T[\tau]/(Q\cap T[\tau])$ is algebraic over $T/(Q\cap T)$, a contradiction. Therefore we have $\mathrm{ht}(Q\cap T[\tau])\leq 1$. We conclude that B is Noetherian and that $B=A$. □

REMARK 12.13. To establish the existence of examples to which Proposition 12.12 applies, we take k to be the field $\mathbb{Q}$ of rational numbers. Thus $R:=\mathbb{Q}[x,y]_{(x,y)}$ is the localized polynomial ring, and the completion of R with respect to its maximal ideal $\mathbf{m}:=(x,y)R$ is $\widehat{R}:=\mathbb{Q}[[x,y]]$, the formal power series ring in the variables x and y. Let $\sigma:=e^x-1\in\mathbb{Q}[[x]]$, and $C:=\mathbb{Q}[[x]]\cap\mathbb{Q}(x,\sigma)$. Thus C is an excellent DVR [1] with maximal ideal xC, and $T:=C[y]_{(x,y)C[y]}$ is an excellent countable two-dimensional regular local ring with maximal ideal $(x,y)T$ and with $(y)-$adic completion $C[[y]]$. The UFD $C[[y]]$ has maximal ideal $\mathbf{n}=(x,y)$. Using that T is countable, we give an elementary proof in Theorem 12.14 below that there exists $\tau\in C[[y]]$ such that, for each height-one prime Q of $C[[y]]$ with $Q\cap T\neq(0)$ and $y\notin Q$, the image of τ in $C[[y]]/Q$ is transcendental over $T/(Q\cap T)$. If the element τ can be found in $\mathbb{Q}[[y]]$, then by Proposition 12.12 we have B is Noetherian and $B=A$, for this choice of $\sigma\in\mathbb{Q}[[x]]$ and for $\tau\in\mathbb{Q}[[y]]$ as in Theorem 5.3.

We contrast this situation with that of Example 12.6: With $\sigma=e^x-1$, $\tau=e^y-1$ and $Q=(x-y)C[[y]]$, the element $\bar{\tau}$ is not transcendental over $T/(Q\cap T)$.

THEOREM 12.14. *Let C be an excellent countable rank-one DVR with maximal ideal xC and let y be an indeterminate over C. Let $T=C[y]_{(x,y)C[y]}$. Then there exists an element $\tau\in C[[y]]$ for which the image of τ in $C[[y]]/Q$ is transcendental over $T/(Q\cap T)$, for every height-one prime ideal Q of $C[[y]]$ such that $Q\cap T\neq(0)$ and $y\notin Q$. Moreover τ is transcendental over T.*

[1] Every Dedekind domain of characteristic zero is excellent [**101**, (34.B)]. See also Remark 3.38.

PROOF. Since C is a DVR, C is a UFD, and so are $T = C[y]_{(x,y)C[y]}$ and $C[[y]]$. Hence every height-one prime ideal Q_i of T is principal and is generated by an irreducible polynomial of $C[y]$, say $f_i(y)$. There are countably many of these prime ideals.

Let $\mathcal{U}$ be the countable set of all height-one prime ideals of $C[[y]]$ that are generated by some irreducible factor in $C[[y]]$ of some irreducible polynomial $f(y)$ of $C[y]$ *other* than y; that is, $yC[[y]]$ is *not* included in $\mathcal{U}$. Let $\{P_i\}_{i=1}^{\infty}$ be an enumeration of the prime ideals of $\mathcal{U}$. Let $\mathbf{n} := (x, y)C[[y]]$ denote the maximal ideal of $C[[y]]$.

CLAIM 12.15. *For each $i \in \mathbb{N}$, there are uncountably many distinct cosets in $((P_1 \cap \cdots \cap P_{i-1} \cap y^{i+1}C[[y]]) + P_i)/P_i$.*

PROOF. Since $y \notin P_i$, the image of y in the one-dimensional local domain $C[[y]]/P_i$ generates an ideal primary for the maximal ideal. Also $C[[y]]/P_i$ is a finite $C[[y]]$-module. Since $C[[y]]$ is (y)-adically complete it follows that $C[[y]]/P_i$ is a (y)-adically complete local domain [**103**, Theorem 8.7]. Hence, if we let $\mathcal{H}$ denote a subset of $C[[y]]$ that is a complete set of distinct coset representatives of $C[[y]]/P_i$, then $\mathcal{H}$ is uncountable.

Let a_i be an element of $P_1 \cap \cdots \cap P_{i-1} \cap y^{i+1}C[[y]]$ that is not in P_i. Then the set $a_i\mathcal{H} := \{a_i\beta \mid \beta \in \mathcal{H}\}$ represents an uncountable set of distinct coset representatives of $C[[y]]/P_i$, since, if $a_i\beta$ and $a_i\gamma$ are in the same coset of P_i and $\beta, \gamma \in \mathcal{H}$, then

$$a_i\beta - a_i\gamma \in P_i \implies \beta - \gamma \in P_i \implies \beta = \gamma,$$

Thus there are uncountably many distinct cosets of $C[[y]]/P_i$ of the form $a_i\beta + P_i$, where β ranges over $\mathcal{H}$, as desired for Claim 12.15. □

To return to the proof of Theorem 12.14, we use that

$$((P_1 \cap \cdots \cap P_{i-1} \cap y^{i+1}C[[y]]) + P_i)/P_i$$

is uncountable for each i as follows: Choose $f_1 \in y^2C[[y]]$ so that the image of $y - f_1$ in $C[[y]]/P_1$ is not algebraic over $T/(P_1 \cap T)$; this is possible since the set of cosets is uncountable and so some cosets are transcendental over the countable set $T/(P_1 \cap T)$. Then the element $y - f_1 \notin P_1$, and $f_1 \notin P_1$, since the image of y is not transcendental over $T/(T \cap P_1)$. Choose $f_2 \in P_1 \cap y^3C[[y]]$ so that the image of $y - f_1 - f_2$ in $C[[y]]/P_2$ is not algebraic over $T/(P_2 \cap T)$. Note that $f_2 \in P_1$ implies the image of $y - f_1 - f_2$ is the same as the image of $y - f_1$ in $C[[y]]/P_1$ and so it is not algebraic over P_1. Successively by induction, for each positive integer n, we choose f_n as in the display

$$f_n \in P_1 \cap P_2 \cap \cdots \cap P_{n-1} \cap y^{n+1}C[[y]]$$

so that the image of $y - f_1 - \ldots - f_n$ in $C[[y]]/P_n$ is transcendental over $T/(T \cap P_i)$ for each i with $1 \leq i \leq n$. Then we have a Cauchy sequence $\{f_1 + \cdots + f_n\}_{n=1}^{\infty}$ in $C[[y]]$ with respect to the $(yC[[y]])$-adic topology, and so it converges to an element $a \in y^2C[[y]]$. Now

$$y - a = (y - f_1 - \ \ldots\ - f_n) + (f_{n+1} + \ldots),$$

where the image of $(y - f_1 - \ldots - f_n)$ in $C[[y]]/P_n$ is transcendental over $T/(P_n \cap T)$ and $f_i \in yC[[y]]$ for all $1 \leq i \leq n$ and $(f_{n+1} + \ldots) \in \cap_{i=1}^{n} P_i \cap yC[[y]]$. Therefore the image of $y - a$ in $C[[y]]/P_n$ is transcendental over $T/(P_n \cap T)$, for every $n \in \mathbb{N}$, and we have $y - a \in yC[[y]]$, as desired.

For the "Moreover" statement, suppose that τ is a root of a polynomial $f(z)$ with coefficients in T. For each prime ideal Q such that the image of τ is transcendental over $T/(T \cap Q)$, the coefficients of $f(z)$ must all be in $T \cap Q$. Since this is true for infinitely many height-one primes $T \cap Q$, and the intersection of infinitely many height-one primes in a Noetherian domain is zero, $f(z)$ is the 0 polynomial, and so τ is transcendental over T. □

Theorem 12.16 yields explicit examples for which B is Noetherian and $B = A$ in Theorem 5.3.

THEOREM 12.16. *Let x and y be indeterminates over $\mathbb{Q}$, the field of rational numbers. Then:*

(1) *There exist elements $\sigma \in x\mathbb{Q}[[x]]$ and $\tau \in y\mathbb{Q}[[y]]$ such that the following two conditions are satisfied:*
   (i) *σ is algebraically independent over $\mathbb{Q}(x)$ and τ is algebraically independent over $\mathbb{Q}(y)$.*
   (ii) *$\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(y, \tau, \{\frac{\partial^n \tau}{\partial y^n}\}_{n\in\mathbb{N}}) > r := \operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(x, \sigma, \{\frac{\partial^n \sigma}{\partial x^n}\}_{n\in\mathbb{N}})$, where $\{\frac{\partial^n \tau}{\partial y^n}\}_{n\in\mathbb{N}}$ is the set of partial derivatives of τ with respect to y and $\{\frac{\partial^n \sigma}{\partial x^n}\}_{n\in\mathbb{N}}$ is the set of partial derivatives of σ with respect to x.*

(2) *If $\sigma \in x\mathbb{Q}[[x]]$ and $\tau \in y\mathbb{Q}[[y]]$ satisfy conditions i and ii and $T = C[y]_{(x,y)}$, where $C = \mathbb{Q}(x, \sigma) \cap \mathbb{Q}[[x]]$, as in Notation 12.2 and Remark 12.8, then the image of τ in $C[[y]]/Q$ is algebraically independent over $T/(Q \cap T)$, for every height-one prime ideal Q of $C[[y]]$ such that $Q \cap T \neq (0)$ and $xy \notin Q$.*

(3) *If $\sigma \in x\mathbb{Q}[[x]]$ and $\tau \in y\mathbb{Q}[[y]]$ satisfy conditions i and ii, then the ring B of Theorem 12.3 defined for this choice of σ and τ is Noetherian and $B = A$.*

PROOF. For item 1, to establish the existence of elements σ and τ satisfying properties (i) and (ii) of Theorem 12.16, let $\sigma = e^x - 1 \in \mathbb{Q}[[x]]$ and choose for τ a hypertranscendental element in $\mathbb{Q}[[y]]$. A power series $\tau = \sum_{i=0}^{\infty} b_i y^i \in \mathbb{Q}[[y]]$ is called *hypertranscendental* over $\mathbb{Q}[y]$ if the set of partial derivatives $\{\frac{\partial^n \tau}{\partial y^n}\}_{n\in\mathbb{N}}$ is infinite and algebraically independent over $\mathbb{Q}(y)$. Two examples of hypertranscendental elements are the Gamma function and the Riemann Zeta function. (The exponential function is, of course, far from being hypertranscendental.) Thus there exist elements σ, τ that satisfy conditions (i) and (ii) of Theorem 12.16.

Another way to obtain such elements is to set $\sigma = e^x - 1$ and $\tau = e^{(e^y - 1)} - 1$. In this case, conditions i and ii of Theorem 12.16 follow from [**14**, Conjecture Σ, p. 252], a generalization of Schanuel's conjectures, which is established in Ax's paper [**14**, Corollary 1, p. 253]. To see that conditions i and ii hold, it is convenient to restate Conjecture Σ of [**14**] with different letters for the power series; let y be a variable, and use only one or two power series $s, t \in \mathbb{C}[[y]]$. Thus Conjecture Σ states that, if s and t are elements of $\mathbb{C}[[y]]$ that are $\mathbb{Q}$-linearly independent, then

$$\begin{aligned} \operatorname{trdeg}_{\mathbb{Q}}(\mathbb{Q}(s, e^s)) &\geq 1 + \operatorname{rank} \begin{bmatrix} \frac{\partial s}{\partial y} \end{bmatrix}. \\ \operatorname{trdeg}_{\mathbb{Q}}(\mathbb{Q}(s, t, e^s, e^t)) &\geq 2 + \operatorname{rank} \begin{bmatrix} \frac{\partial s}{\partial y} & \frac{\partial t}{\partial y} \end{bmatrix}. \end{aligned} \tag{12.16.0}$$

Since the rank of the matrix $\left[\frac{\partial e^y}{\partial y}\right]$ is 1, $\operatorname{trdeg}_{\mathbb{Q}}(\mathbb{Q}(y, e^y)) \geq 2$, by Equation 12.16.0. By switching the variable to x, $\operatorname{trdeg}_{\mathbb{Q}}(\mathbb{Q}(x, e^x)) \geq 2$. Thus $\sigma = e^x - 1$ satisfies condition i.

Since just two transcendental elements generate the field $\mathbb{Q}(x, e^x)$ over $\mathbb{Q}$, we have $\operatorname{trdeg}_{\mathbb{Q}}(\mathbb{Q}(x, e^x)) = 2$. Furthermore $\partial^n\sigma/\partial x^n = e^x$ for every $n \in \mathbb{N}$, and so

$$\operatorname{trdeg}_{\mathbb{Q}}(\mathbb{Q}(x, \sigma, \{\frac{\partial^n \sigma)}{\partial x^n}\}_{n\in\mathbb{N}})) = 2;$$

that is, for r as in condition ii with this σ, we have $r = 2$.

Since the rank of the matrix $\left[\begin{matrix}\frac{\partial y}{\partial y} & \frac{\partial(e^y-1)}{\partial y}\end{matrix}\right]$ is 1, we have

$$\operatorname{trdeg}_{\mathbb{Q}}(\mathbb{Q}(y, e^y, e^{(e^y-1)})) = \operatorname{trdeg}_{\mathbb{Q}}(\mathbb{Q}(y, e^y - 1, e^y, e^{(e^y-1)})) \geq 3,$$

by Equation 12.16.0 with $s = y$ and $t = e^y - 1$.

For τ we have, $\partial\tau/\partial y = \partial(e^{(e^y-1)} - 1)/\partial y = e^{(e^y-1)} \cdot e^y$. Thus

$$\operatorname{trdeg}_{\mathbb{Q}}(\mathbb{Q}(y, \tau, \{\frac{\partial^n \tau}{\partial y^n}\}_{n\in\mathbb{N}})) \geq \operatorname{trdeg}_{\mathbb{Q}}(\mathbb{Q}(y, e^y, e^{(e^y-1)})) > 2,$$

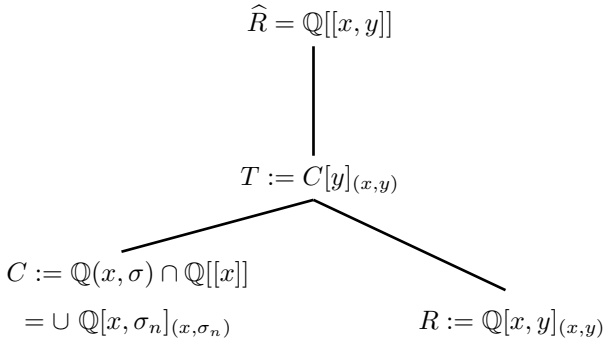
by the computation above, and so conditions i and ii both hold for τ. Thus item 1 is proved.

Item 3 follows from item 2 by Proposition 12.12.

For item 2, we observe that the ring $T = C[y]_{(x,y)}$ is an overring of $R = \mathbb{Q}[x, y]_{(x,y)}$ and a subring of $\widehat{R}$ and T has completion $\widehat{T} = \widehat{R}$:

$$R = \mathbb{Q}[x, y]_{(x,y)} \longrightarrow T = C[y]_{(x,y)} \longrightarrow \widehat{R} = \widehat{T} = \mathbb{Q}[[x, y]].$$

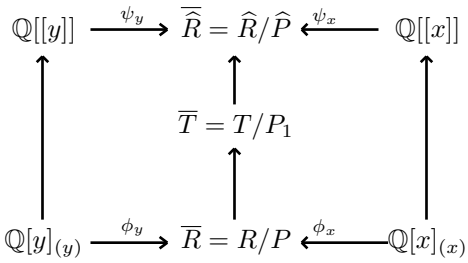
We display the relationships among these rings.



The rings of the example

Let $\widehat{P}$ be a height-one prime ideal of $\widehat{R}$, let bars (for example, $\bar{x}$), denote images in $\overline{\widehat{R}} = \widehat{R}/\widehat{P}$ and set $P := \widehat{P} \cap R$ and $P_1 := \widehat{P} \cap T$. Assume that $P_1 \neq 0$ and that $xy \notin \widehat{P}$.

In the following commutative diagram, we identify $\mathbb{Q}[[x]]$ with $\mathbb{Q}[[\bar{x}]]$ and $\mathbb{Q}[[y]]$ with $\mathbb{Q}[[\bar{y}]]$, etc.



All maps in the diagram are injective and $\overline{\widehat{R}}$ is finite over both of the rings $\mathbb{Q}[[x]]$ and $\mathbb{Q}[[y]]$. We divide into two cases: (i) $P \neq (0)$, and (ii) $P = (0)$; in each case we show that $\overline{T} \subseteq \mathbb{Q}(\bar{x}, \bar{\sigma})^a$, the algebraic closure of $\mathbb{Q}(\bar{x}, \bar{\sigma})$.

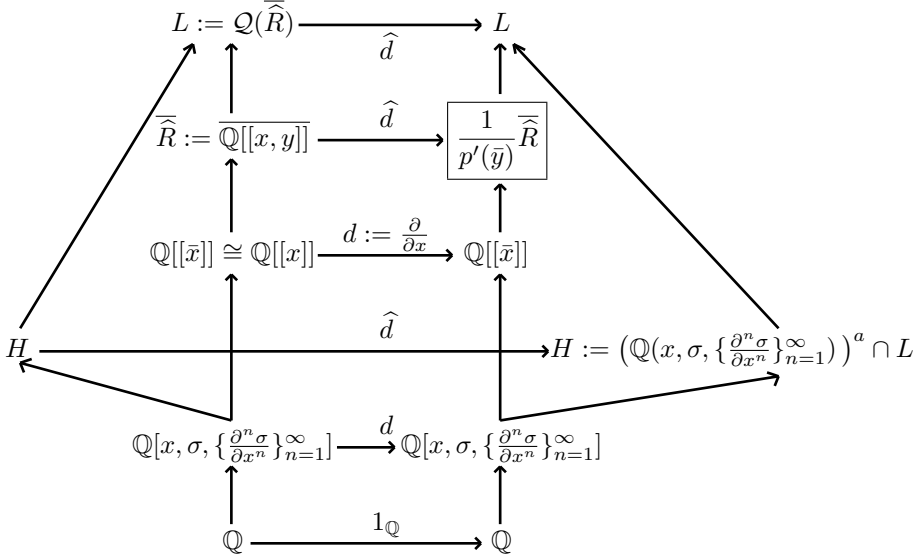
Case i: $P = R \cap \widehat{P} \neq (0)$. Since $\operatorname{trdeg}_{\mathbb{Q}} \mathcal{Q}(\overline{R}) = 1$, we have $\overline{R}$ is algebraic over $\mathbb{Q}[\bar{x}]_{(x)}$, Thus $\bar{y}$ is algebraic over $\mathbb{Q}[x]_{(x)}$. Thus $\overline{T} \subseteq \mathbb{Q}(\bar{x}, \bar{\sigma})^a$.

Case ii: $P = R \cap \widehat{P} = P_1 \cap R = (0)$. Then $P_1 \cap C = (0)$; otherwise $P_1 \cap C = xC$, since xC is the unique maximal ideal of the DVR C, and this would contradict $P_1 \cap R = (0)$. The integral domain T is a UFD since C is. Therefore the height-one prime ideal P_1 of T is generated by an element $f(y)$, which may be chosen in $C[y]$. Since $P_1 \cap C = (0)$, we have $\deg f(y) \geq 1$, where deg refers to the degree in y. Therefore we have $\bar{f}(\bar{y}) = 0$ in $\overline{T}$. Since the field of fractions of C is $\mathbb{Q}(x, \sigma)$, $\bar{y}$ is algebraic over the field $\mathbb{Q}(\bar{x}, \bar{\sigma})$. Hence $\overline{T}$ is contained in $\mathbb{Q}(\bar{x}, \bar{\sigma})^a$.

Let L denote the field of fractions of $\overline{\widehat{R}}$. We may consider $\mathbb{Q}(y, \tau, \{\frac{\partial^n \tau}{\partial y^n}\}_{n\in\mathbb{N}})$ and $\mathbb{Q}(x, \sigma, \{\frac{\partial^n \sigma}{\partial x^n}\}_{n\in\mathbb{N}})$ as subfields of L, where

$$\operatorname{trdeg}_{\mathbb{Q}} \left(\mathbb{Q}(y, \tau, \left\{ \frac{\partial^n \tau}{\partial y^n} \right\}_{n\in\mathbb{N}} \right) > \operatorname{trdeg}_{\mathbb{Q}} \left(\mathbb{Q}(x, \sigma, \left\{ \frac{\partial^n \sigma}{\partial x^n} \right\}_{n\in\mathbb{N}} \right).$$

Let d denote the partial derivative map $\frac{\partial}{\partial x}$ on $\mathbb{Q}((x))$. Since the extension L of $\mathbb{Q}((x))$ is finite and separable, d extends uniquely to a derivation $\widehat{d} : L \to L$, [**164**, Corollary 2, p. 124]. Let H denote the algebraic closure (shown in Picture 12.17.1 by a small upper a) in L of the field $\mathbb{Q}(x, \sigma, \{\frac{\partial^n \sigma}{\partial x^n}\}_{n\in\mathbb{N}})$. Let $\widehat{p}(x, y) \in \mathbb{Q}[[x, y]]$ be a prime element generating $\widehat{P}$. Claim 12.17 asserts that the images of H and $\overline{\widehat{R}}$ under $\widehat{d}$ are inside H and $(1/p')\overline{\widehat{R}}$, respectively, as shown in Picture 12.17.1.



Picture 12.17.1 The image of subrings of L via the extension $\widehat{d}$ of $d := \frac{\partial}{\partial x}$.

Claim 12.17. *With the notation above:*
(i) $\widehat{d}(H) \subseteq H$.
(ii) *There exists a polynomial* $p(x,y) \in \mathbb{Q}[[x]][y]$ *with* $p\mathbb{Q}[[x,y]] = \widehat{P}$ *and* $p(\bar{y}) = 0$.
(iii) $\widehat{d}(\bar{y}) \neq 0$ *and* $p'(\bar{y})\widehat{d}(\bar{y}) \in \overline{\widehat{R}}$, *where* $p'(y) := \frac{\partial p(x,y)}{\partial y}$.
(iv) *For every element* $\lambda \in \overline{\widehat{R}}$, *we have* $p'(\bar{y})\widehat{d}(\lambda) \in \overline{\widehat{R}}$, *and so* $\widehat{d}(\overline{\widehat{R}}) \subseteq (1/p'(\bar{y}))\overline{\widehat{R}}$.

Proof. For item i, since d maps $\mathbb{Q}(x, \sigma, \{\frac{\partial^n \sigma}{\partial x^n}\}_{n\in\mathbb{N}})$ into itself, $\widehat{d}(H) \subseteq H$.

For item ii, we have that x and y are not contained in $\widehat{P}$, and that the element $\widehat{p}(x,y) \in \mathbb{Q}[[x,y]]$ generates $\widehat{P}$ and is *regular in* y as a power series in $\mathbb{Q}[[x,y]]$ (in the sense of Zariski-Samuel [**165**, p.145]); that is, $\widehat{p}(0,y) \neq 0$. Thus by [**165**, Corollary 1, p.145] the element $\widehat{p}(x,y)$ can be written as:

$$\widehat{p}(x,y) = \epsilon(x,y)(y^n + c_{n-1}(x)y^{n-1} + \ldots + c_0(x)),$$

where $\epsilon(x,y)$ is a unit of $\mathbb{Q}[[x,y]]$ and each $c_i(x) \in \mathbb{Q}[[x]]$. Hence $\widehat{P}$ is also generated by

$$p(x,y) = p(y) := \epsilon^{-1}\widehat{p} \;=\; y^n + c_{n-1}y^{n-1} + \cdots + c_0,$$

and the $c_i \in \mathbb{Q}[[x]]$. Since $p(y)$ is the minimal polynomial of $\bar{y}$ over the field $\mathbb{Q}((x))$, we have $0 = p(\bar{y}) := \bar{y}^n + c_{n-1}\bar{y}^{n-1} + \cdots + c_1\bar{y} + c_0$.

For item iii, observe that

$$p'(y) = ny^{n-1} + c_{n-1}(n-1)y^{n-2} + \cdots + c_1,$$

and $p'(\bar{y}) \neq 0$ by minimality. Now

$$\begin{aligned}
0 = \widehat{d}(p(\bar{y})) &= \widehat{d}(\bar{y}^n + c_{n-1}\bar{y}^{n-1} + \cdots + c_1\bar{y} + c_0) \\
&= n\bar{y}^{n-1}\widehat{d}(\bar{y}) + c_{n-1}(n-1)\bar{y}^{n-2}\widehat{d}(\bar{y}) + d(c_{n-1})\bar{y}^{n-1} + \cdots \\
&\qquad + c_1\widehat{d}(\bar{y}) + d(c_1)\bar{y} + d(c_0) \\
&= \widehat{d}(\bar{y})(n\bar{y}^{n-1} + c_{n-1}(n-1)\bar{y}^{n-2} + \cdots + c_1) + \\
&\qquad + d(c_{n-1})\bar{y}^{n-1} + \cdots + d(c_1)\bar{y} + d(c_0) \\
&= \widehat{d}(\bar{y})(p'(\bar{y})) + \sum_{i=0}^{n-1} d(c_i)\bar{y}^i \\
\implies \widehat{d}(\bar{y})(p'(\bar{y})) &= -\big(\sum_{i=0}^{n-1} d(c_i)\bar{y}^i\big) \quad \text{and} \quad \widehat{d}(\bar{y}) = \frac{-1}{p'(\bar{y})}\sum_{i=0}^{n-1} d(c_i)\bar{y}^i.
\end{aligned}$$

In particular, $p'(\bar{y})\widehat{d}(\bar{y}) \in \overline{\widehat{R}}$. If $d(c_i) = 0$ for every i, then $c_i \in \mathbb{Q}$ for every i; this would imply that $p(x, y) \in \mathbb{Q}[[y]]$ and either $c_0 = 0$ or c_0 is a unit of $\mathbb{Q}$. If $c_0 = 0$, $p(x, y)$ could not be a minimal polynomial for $\bar{y}$, a contradiction. If c_0 is a unit, then $p(y)$ is a unit of $\mathbb{Q}[[y]]$, and so $\widehat{P}$ contains a unit, another contradiction. Thus $\widehat{d}(\bar{y}) \neq 0$, as desired for item iii.

For item iv, observe that every element $\lambda \in \overline{\widehat{R}}$ has the form:

$$\lambda = e_{n-1}(x)\bar{y}^{n-1} + \cdots + e_1(x)\bar{y} + e_0(x), \text{ where } e_i(x) \in \mathbb{Q}[[x]].$$

Therefore:

$$\widehat{d}(\lambda) = \widehat{d}(\bar{y})\big[(n-1)e_{n-1}(x)\bar{y}^{n-2} + \cdots + e_1(x)\big] + \sum_{i=0}^{n-1} d(e_i(x))\bar{y}^i.$$

The sum expression on the right is in $\overline{\widehat{R}}$ and, as established above, $p'(\bar{y})\widehat{d}(\bar{y}) \in \overline{\widehat{R}}$, and so $p'(\bar{y})\widehat{d}(\lambda) \in \overline{\widehat{R}}$. □

The next claim asserts an expression for $\widehat{d}(\bar{\tau})$ in terms of the partial derivative $\frac{\partial\bar{\tau}}{\partial\bar{y}}$ of $\bar{\tau}$ with respect to $\bar{y}$.

Claim 12.18. $\widehat{d}(\tau) = \widehat{d}(\bar{y})\frac{\partial\bar{\tau}}{\partial\bar{y}}$.

Proof. For every $m \in \mathbb{N}$, we have $\tau = \sum_{i=0}^{m} b_i y^i + y^{m+1}\tau_m$, where each $b_i \in \mathbb{Q}$ and each $\tau_m \in \mathbb{Q}[[y]]$ is defined as in Equation 5.4.1. Therefore

$$\widehat{d}(\bar{\tau}) = \widehat{d}(\bar{y}) \cdot \big(\sum_{i=1}^{m} ib_i\bar{y}^{i-1}\big) + \widehat{d}(\bar{y})(m+1)\bar{y}^m\bar{\tau}_m + \bar{y}^{m+1}\widehat{d}(\bar{\tau}_m).$$

Thus

$$p'(\bar{y})\widehat{d}(\bar{\tau}) = p'(\bar{y})\widehat{d}(\bar{y}) \cdot \sum_{i=1}^{m} ib_i\bar{y}^{i-1} + \bar{y}^m(p'(\bar{y})\widehat{d}(\bar{y})(m+1)\bar{\tau}_m + \bar{y}^{m+1}p'(\bar{y})\widehat{d}(\bar{\tau}_m)).$$

Since $\bar{\tau} = \sum_{i=0}^{\infty} b_i\bar{y}^i$ with $b_i \in \mathbb{Q}$, we have

$$\frac{\partial\bar{\tau}}{\partial\bar{y}} = \sum_{i=1}^{m} ib_i\bar{y}^{i-1} + \bar{y}^m \sum_{i=m+1}^{\infty} ib_i\bar{y}^{i-m-1}.$$

Thus, if we multiply the last equation by $p'(\bar{y})\widehat{d}(\bar{y})$, we obtain

$$p'(\bar{y})\widehat{d}(\bar{y})\frac{\partial\bar{\tau}}{\partial\bar{y}} = p'(\bar{y})\widehat{d}(\bar{y})\sum_{i=1}^{m} ib_i\bar{y}^{i-1} + p'(\bar{y})\widehat{d}(\bar{y})\bar{y}^m \sum_{i=m+1}^{\infty} ib_i\bar{y}^{i-m-1}.$$

Hence, by subtracting this last equation from the earlier expression for $p'(\bar{y})\widehat{d}(\bar{\tau})$, we obtain

$$p'(\bar{y})\widehat{d}(\bar{\tau}) - p'(\bar{y})\widehat{d}(\bar{y})\frac{\partial\bar{\tau}}{\partial\bar{y}} \in \bar{y}^m(\overline{\widehat{R}}),$$

for every $m \in \mathbb{N}$. Therefore $p'(\bar{y})\widehat{d}(\bar{\tau}) - p'(\bar{y})\widehat{d}(\bar{y})\frac{\partial\bar{\tau}}{\partial\bar{y}} \in \cap y^m(\overline{\widehat{R}}) = 0$, by Krull's Intersection Theorem 2.16. Thus $\widehat{d}(\tau) = \widehat{d}(\bar{y})\frac{\partial\bar{\tau}}{\partial\bar{y}}$, since $p'(\bar{y}) \neq 0$ and $\overline{\widehat{R}}$ is an integral domain. That is, Claim 12.18 holds. □

Completion of proof of Theorem 12.16. From above, in either case i or case ii, $\overline{T} \subseteq H$, where H is the algebraic closure of the field $\mathcal{Q}(\bar{x}T, \{\frac{\partial^n\bar{\sigma}}{\partial\bar{x}^n}\}_{n\in\mathbb{N}})$ in L. We have $\bar{\tau} \notin H$ if and only if $\bar{\tau}$ is transcendental over H. By hypothesis, the transcendence degree of $H/\mathbb{Q}$ is r. Since $\widehat{d}(H) \subseteq H$, if $\bar{\tau}$ were in H, then $\frac{\partial^n\bar{\tau}}{\partial\bar{y}^n} \in H$ for all $n \in \mathbb{N}$. This implies that the field $\mathbb{Q}(y, \tau, \{\frac{\partial^n\tau}{\partial y^n}\}_{n\in\mathbb{N}})$ is contained in H. This contradicts our hypothesis that $\operatorname{trdeg}_{\mathbb{Q}}\mathbb{Q}(y, \tau, \{\frac{\partial^n\tau}{\partial y^n}\}_{n\in\mathbb{N}}) > r$. Therefore the image of τ in $\widehat{R}/Q$ is algebraically independent over $T/(Q\cap T)$ for each height-one prime ideal Q of $\widehat{R}$ such that $Q\cap T \neq (0)$ and $xy \notin Q$. This completes the proof of Theorem 12.16. □

As explained in the proof of Theorem 12.16, Ax's results in [**14**] together with the arguments of the proof imply that the elements $\sigma = e^x - 1 \in \mathbb{Q}[[x]]$ and $\tau = e^{(e^y-1)} - 1 \in \mathbb{Q}[[y]]$ satisfy the conditions of Theorem 12.16. Thus we have the following example:

EXAMPLE 12.19. For $\sigma = e^x - 1 \in \mathbb{Q}[[x]]$ and $\tau = e^{(e^y-1)} - 1 \in \mathbb{Q}[[y]]$ in Theorem 5.3, the ring B is Noetherian and $B = A$.

This completes the proof of Theorem 5.3.

Exercises

(1) Let x and y be indeterminates over a field k and let R be the two-dimensional RLR obtained by localizing the mixed power series-polynomial ring $k[[x]][y]$ at the maximal ideal $(x, y)k[[x]][y]$.
   (i) For each height-one prime ideal P of R different from xR, prove that R/P is a one-dimensional complete local domain.
   (ii) For each nonzero prime ideal Q of $\widehat{R} = k[[x, y]]$ prove that $Q \cap R \neq (0)$. Conclude that the generic formal fiber of R is zero-dimensional.

   Suggestion. For part (ii), use Theorem 3.9. For more information about the dimension of the formal fibers, see the articles of Matsumura and Rotthaus [**102**] and [**134**].

(2) Let x and y be indeterminates over a field k and let $R = k[x, y]_{(x,y)}$. As in Remark 12.11, assume that $\nu \in \widehat{\mathbf{m}}$ is residually algebraically independent with respect to $\widehat{R} = k[[x, y]]$ over R. Thus $A = \widehat{R} \cap \mathcal{Q}(R[\nu])$ is the localized polynomial ring $R[\nu]_{(\mathbf{m},\nu)}$. Let $\mathbf{n} = (\mathbf{m}, \nu)A$ denote the maximal ideal of A. Give a direct proof that A is not a subspace of $\widehat{R}$.

Suggestion. Since $\nu \in \widehat{\mathbf{m}}$ is a power series in $\widehat{R} = k[[x, y]]$, for each positive integer n, there exists a polynomial $f_n \in k[x, y]$ such that $\nu - f_n \in \widehat{\mathbf{m}}^n$. Since A is a 3-dimensional regular local ring with $\mathbf{n} = (x, y, \nu)A$, the element $\nu - f_n \notin \mathbf{n}^2$. Hence for each positive integer n, the ideal $\widehat{\mathbf{m}}^n \cap A$ is not contained in $\mathbf{n}^2$.

(3) (Kunz) Let L/k be a field extension with L having infinite transcendence degree over k. Prove that the ring $L \otimes_k L$ is not Noetherian. Deduce that the ring $k[[x]] \otimes_k k[[x]]$, which has $k((x)) \otimes_k k((x))$ as a localization, is not Noetherian.

Suggestion. Let $\{x_\lambda\}_{\lambda \in \Lambda}$ be a transcendence basis for L/k and consider the subfield $F = k(\{x_\lambda\})$ of L. The ring $L \otimes_k L$ is faithfully flat over its subring $F \otimes_k F$, and if $F \otimes_k F$ is not Noetherian, then $L \otimes_k L$ is not Noetherian. Hence it suffices to show that $F \otimes_k F$ is not Noetherian. The module of differentials $\Omega^1_{F/k}$ is known to be infinite dimensional as a vector space over F [**89**, 5.4], and $\Omega^1_{F/k} \cong I/I^2$, where I is the kernel of the map $F \otimes_k F \to F$, defined by sending $a \otimes b \mapsto ab$. Thus the ideal I of $F \otimes_k F$ is not finitely generated.

CHAPTER 13

Excellent rings and related concepts

In the first two sections of this chapter we motivate and explain the concept of excellence.[1] We describe the desirable attributes of an excellent ring and discuss why they are useful. In considering this, we are led to a discussion of the singular locus and the Jacobian criterion. We discuss Nagata rings in Section 13.2 and Henselian rings and the Henselization of a Noetherian local ring in Section 13.3. For a Noetherian local ring $(R, \mathbf{m})$ with $\mathbf{m}$-adic completion $\widehat{R}$, the fibers of the map $R \hookrightarrow \widehat{R}$ play an important role in determining whether R is excellent or a Nagata ring.

13.1. Basic properties and background for excellent rings

In the 1950s, Nagata constructed an example in characteristic $p > 0$ of a normal Noetherian local domain $(R, \mathbf{m})$ such that the $\mathbf{m}$-adic completion $\widehat{R}$ is not reduced [**117**, Example 6, p.208], [**113**]. He constructed another example of a normal Noetherian local domain $(R, \mathbf{m})$ that contains a field of characteristic zero and has the property that $\widehat{R}$ is not an integral domain [**117**, Example 7, p.209]; see Example 4.14, Remarks 4.15 and Section 6.3 for information about this example. The existence of these examples motivated the search for conditions on a Noetherian local ring R that imply good behavior of the completion $\widehat{R}$.

We consider the following questions:

QUESTIONS 13.1.

(1) What properties should a "nice" Noetherian ring have?
(2) What properties of a Noetherian local ring ensure good behavior with respect to completion?
(3) What properties of a Noetherian ring ensure "nice" properties of finitely generated algebras over the given ring?

In the 1960s, Grothendieck systematically investigated Noetherian rings that are exceptionally well behaved. He called these rings "excellent". The intent of his definition of excellent rings is that these rings should have the same nice properties as the rings in classical algebraic geometry. Among the rings studied in classical algebraic geometry are the *affine rings*

$$A = k[x_1, \ldots, x_n]/I,$$

where k is a field and I is a ideal of the polynomial ring $S := k[x_1, \ldots, x_n]$.

There are four fundamental properties of affine rings that are relevant for the definition of excellent rings. The third property involves the concept of the singular locus as in Definition 1.

[1] Much of the material in this chapter comes from the article [**136**].

DEFINITION 13.2. Let A be a Noetherian ring. The *singular locus* of A is:

$$\operatorname{Sing} A = \{P \in \operatorname{Spec} A \mid A_P \text{ is not a regular local ring}\}.$$

Let $\mathcal{A}$ be a class of Noetherian rings that satisfy the following:

Property $\mathcal{A}$.1: If $A \in \mathcal{A}$ and B is an algebra of finite type over A, then $B \in \mathcal{A}$.

Property $\mathcal{A}$.2: If $A \in \mathcal{A}$, then A is universally catenary.

Property $\mathcal{A}$.3: If $A \in \mathcal{A}$, then the singular locus $\operatorname{Sing} A$ is *closed* in the Zariski topology of $\operatorname{Spec} A$, that is, there is an ideal $J \subseteq A$ such that $\operatorname{Sing} A = \mathcal{V}(J)$.[2]

Property $\mathcal{A}$.4: If $A \in \mathcal{A}$, then, for every maximal ideal $\mathbf{m} \in \operatorname{Spec} A$ and for every prime ideal $Q \in \operatorname{Spec}(\widehat{A_{\mathbf{m}}})$, we have:

$$(\widehat{A_{\mathbf{m}}})_Q \text{ is regular} \iff A_{Q\cap A} \text{ is regular.} \tag{13.1}$$

We discuss these properties in the remainder of this section. Properties $\mathcal{A}$.1-$\mathcal{A}$.4 hold for the class of affine rings; see Remark 13.17. It is straightforward that affine rings satisfy the first two properties, since an algebra of finite type over affine ring is again an affine ring, and every affine ring is universally catenary. The third and fourth properties are not as obvious for affine rings. They are, however, important properties for excellence.

David Mumford and John Tate discuss how Grothendieck's work revolutionized classical algebraic geometry in [**106**]. In particular, they write: Algebraic geometry "is the field where one studies the locus of solutions of sets of polynomial equations ...". One combines "the algebraic properties of the rings of polynomials with the geometric properties of this locus, known as a *variety*."

To apply this to the discussion of Property $\mathcal{A}$.3, let k be an algebraically closed field. For n a positive integer, let k^n denote affine n-space. An *affine algebraic variety* is a subset $\mathcal{Z}(I)$ of k^n, where $\mathcal{Z}(I)$ is the zero set of an ideal I of the polynomial ring $S = k[x_1, \ldots, x_n]$:

$$\mathcal{Z}(I) \;=\; \{a \in k^n \mid f(a) \;=\; 0, \text{ for all } f \in I\ \}.$$

It is clear that $\mathcal{Z}(I) = \mathcal{Z}(\sqrt{I})$. Let $A = S/\sqrt{I}$. We define the *singular locus* of $\mathcal{Z}(I)$ to be $\operatorname{Sing} A$.

The singular locus of a reduced affine ring A over an algebraically closed field is a proper closed subset of $\operatorname{Spec} A$; see for example Hartshorne's book [**53**, Theorem 5.3, page 33]. Thus Property $\mathcal{A}$.3 is satisfied for such a ring A.

Again quoting Mumford and Tate in [**106**]: Grothendieck "invented a class of geometric structures generalizing varieties that he called *schemes*". This applies to any commutative ring, and thus includes fields that are not algebraically closed and ideals that are not reduced.

Property $\mathcal{A}$.3 is related to the Jacobian criterion for smoothness over an arbitrary affine ring:

JACOBIAN CRITERION 13.3. Let $A = S/I$ be an affine ring, where I is an ideal of the polynomial ring $S = k[x_1, \ldots, x_n]$ over the field k. Let P be a prime ideal of S with $I \subseteq P$, let $\mathbf{p} = P/I$, and let r be the height of I_P in S_P. Assume that $I = (f_1, \ldots, f_s)S$. The *Jacobian criterion for smoothness* asserts the equivalence of the following statements:

(1) The map $\psi : k \hookrightarrow A_{\mathbf{p}}$ is smooth, or equivalently a regular homomorphism.

[2]Notation from Section 2.1.

(2) rank $(\partial f_i/\partial x_j) = r \pmod P$.

(3) The ideal generated by the $r \times r$ minors of $(\partial f_i/\partial x_j)$ is not contained in P.

These equivalent conditions imply that $A_{\mathbf{p}}$ is an RLR.

The rank of $(\partial f_i/\partial x_j) \pmod P$ is at most r; see Eisenbud's book [**38**, 16.19.a]. The Jacobian criterion for smoothness is proved in [**103**, Theorem 30.3].

REMARKS 13.4. Let $A = S/I$, where $S = k[x_1, \ldots, x_n]$ is a polynomial ring over a field k, and I is an ideal of S. Let the notation be as in Criterion 13.3.

(1) By Theorem 7.8, the morphism ψ is a regular morphism if and only if ψ is smooth, or equivalently $A_{\mathbf{p}}$ is a smooth k-algebra.[3] Since A is an affine k-algebra, $A_{\mathbf{p}}$ is essentially of finite type over k. Regularity of ψ is equivalent to ψ being flat with geometrically regular fibers. Equivalently, ψ is flat and, for each prime ideal Q of A and each finite algebraic field extension L of k, the ring $A_Q \otimes_k L$ is a regular local ring. Since k is a field, A_Q is a free k-module and so the extension ψ is flat by Remark 2.31.2.

(2) If k is a perfect field, then $A_{\mathbf{p}}$ is a regular local ring if and only if the equivalent conditions of Criterion 13.3 hold. To see this: Every algebraic extension is separable algebraic; this implies that, for every $Q \in \operatorname{Spec} A$ and every finite algebraic field extension L of k, $A_Q \otimes_k L$ is a regular local ring if A_Q is a regular local ring. Thus the map $k \hookrightarrow A_{\mathbf{p}}$ is regular if and only if $A_{\mathbf{p}}$ is a regular local ring.

(3) If k is a perfect field, the Jacobian criterion defines the singular locus of A. In this case the singular locus is $\mathcal{V}(J)$ where J is the ideal of S generated by I and the $r \times r$ minors of the Jacobian matrix $(\partial f_i/\partial x_j)$.

(4) If k is not a perfect field, then the equivalent conditions of Criterion 13.3 are stronger than the statement that $A_{\mathbf{p}}$ is a regular local ring [**103**, Theorem 30.3].

Example 13.5 is an example of a Noetherian local ring over a non-perfect field k that is a regular local ring, but is not smooth over k.

EXAMPLE 13.5. Let k be a field of characteristic $p > 0$ such that k is not perfect, that is, k^p is properly contained in k. Let $a \in k \setminus k^p$ and let $f = x^p - a$. Then $L = k[x]/(f)$ is a proper purely inseparable extension field of k. Since $\partial f/\partial x = 0$, the Jacobian criterion for smoothness implies L is not smooth over k. However, L is a field and thus a regular local ring.

REMARK 13.6. Zariski's Jacobian criterion for regularity in polynomial rings applies in the case where the ground field is not perfect; see [**103**, Theorem 30.5]. Assume the notation of Criterion 13.3. The singular locus of A is closed in $\operatorname{Spec} A$ and is defined by an ideal J of A; that is, $\operatorname{Sing}(A) = \mathcal{V}(J)$. In Criterion 13.3, the ideal J is generated by the $r \times r$ minors of the Jacobian matrix, whereas in Zariski's Jacobian criterion for regularity in polynomial rings if k has characteristic p and is not perfect, then the Jacobian matrix is extended by certain k^p-derivations of S, and J is generated by appropriate minors of the extended matrix. For Example 13.5, there exists a k^p-derivation $D : k[x] \to k[x]$ with $D(f) \neq 0$; see for example [**103**, page 202].

We return to properties for excellence. A first approach towards obtaining a class $\mathcal{A}$ of Noetherian rings that satisfy Properties $\mathcal{A}.1$, $\mathcal{A}.2$, $\mathcal{A}.3$ and $\mathcal{A}.4$ might

[3]Regularity is defined in Definition 3.31. For smoothness see Definition 7.7.

be to consider the rings satisfying "Jacobian criteria", similar to the conditions of Criterion 13.3. Unfortunately this class is rather small. Example 13.7 from [**136**, p. 319] is an excellent Noetherian local domain that fails to satisfy Jacobian criteria. This example is related to Theorem 12.16.

EXAMPLE 13.7. Let $\sigma = e^{(e^x-1)} \in \mathbb{Q}[[x]]$. By a result of Ax [**14**, Corollary 1, p. 253], σ and $\partial\sigma/\partial x$ are algebraically independent over $\mathbb{Q}(x)$; see the proof of item 1 of Theorem 12.16. As in Example 4.7, consider the intersection domain

$$A \;:=\; \mathbb{Q}(x,\sigma) \cap \mathbb{Q}[[x]].$$

By Remark 2.1, A is a DVR with maximal ideal xA and field of fractions $\mathbb{Q}(x,\sigma)$. We have $\mathbb{Q}[x]_{(x)} \subset A \subset \mathbb{Q}[[x]]$. If $d : A \hookrightarrow \mathbb{Q}[[x]]$ is a derivation, then $d(\sigma) = d(x)\partial\sigma/\partial x$. it follows that $d(\sigma) \notin A$ whenever $d(x) \neq 0$. Hence there is only the trivial derivation $d = 0$ from A into itself. Since every DVR containing a field of characteristic 0 is excellent, the ring A is excellent; see Remarks 3.38.

There is an important class of Noetherian local rings that admit Jacobian and regularity criteria, namely, the class of complete Noetherian local rings. These criteria were established by Nagata and Grothendieck and are similar to the above mentioned criterion. A principal objective of the theory of excellent rings is to exploit the Jacobian criteria for the completion $\widehat{A}$ of an excellent local ring A in order to describe certain properties of A, even if the ring A itself may fail to satisfy Jacobian criteria. This theory requires considerable theoretical background. Grothendieck's theory of formal smoothness and regularity was developed to work out the connection between a local ring A and its completion $\widehat{A}$; see [**51**, No 24, (6.8), pp. 150-153].

REMARK 13.8. Let $(A, \mathbf{m})$ be a Noetherian local ring. By Cohen's structure theorems, the $\mathbf{m}$-adic completion $\widehat{A}$ of A is the homomorphic image of a formal power series ring over a ring K, where K is either a field or a complete discrete valuation ring, that is, $\widehat{A} \cong K[[x_1, \ldots, x_n]]/I$; see Remarks 3.12.3. The singular locus $\operatorname{Sing}\widehat{A}$ of $\widehat{A}$ is closed by the Jacobian criterion on complete Noetherian local rings [**103**, Corollary to Theorem 30.10].

The following discussion relates to Properties $\mathcal{A}.3$ and $\mathcal{A}.4$ and the definition of excellence.

DISCUSSION 13.9. Let $\varphi : A \hookrightarrow C$ be a faithfully flat homomorphism of Noetherian rings. For example, let $(A, \mathbf{m})$ be a Noetherian local ring and let C be the $\mathbf{m}$-adic completion of A. We observe connections between the singular loci $\operatorname{Sing} A$ and $\operatorname{Sing} C$. Consider the following two conditions regarding $\operatorname{Sing} A$ and $\operatorname{Sing} C$ and regularity of localizations of A and C

(13.9.a) $\quad \operatorname{Sing} A \;=\; \mathcal{V}(J) \quad \text{and} \quad \operatorname{Sing} C \;=\; \mathcal{V}(JC), \quad$ for some ideal J of A.

(13.9.b) $\quad$ For every $Q \in \operatorname{Spec} C, \quad A_{Q\cap A}$ is regular $\iff C_Q$ is regular.

Condition 13.9.a implies that $\operatorname{Sing} A$ and $\operatorname{Sing} C$ are closed. We show in Theorem 13.10 below if $\operatorname{Sing} C$ is closed, then Condition 13.9.a is equivalent to Condition 13.9.b. We first make some observations about Condition 13.9.b.

(13.9.1) " $\Leftarrow$ " of Condition 13.9.b is always satisfied.

Proof. The induced morphism $A_{Q\cap A} \longrightarrow C_Q$ is faithfully flat. Since flatness descends regularity by Theorem 3.23.1, if C_Q is regular, then $A_{Q\cap A}$ is regular.

(13.9.2) If $\varphi : A \to C$ has regular fibers as in Definition 3.28, then Condition 13.9.b holds.

Proof. Let $P = Q \cap A$. Since the fiber over P is regular, the ring C_Q/PC_Q is regular. By Theorem 3.23.2, if A_P and C_Q/PC_Q are both regular, then the ring C_Q is regular. Thus " $\Rightarrow$ " of Condition 13.9.b holds. By part 1, we have $\Longleftrightarrow$ holds for the quantities in Condition 13.9.b, and so Condition 13.9.b holds.

THEOREM 13.10. *Let $\varphi : A \hookrightarrow C$ be a faithfully flat homomorphism of Noetherian rings. Assume* $\operatorname{Sing} C$ *is closed. Then:*

(1) *Condition 13.9.a is equivalent to Condition 13.9.b.*
(2) *If in addition the fibers of φ are regular, then* $\operatorname{Sing} A$ *is closed.*

PROOF. For item 1, it is clear that Condition 13.9.a implies Condition 13.9.b. Assume Condition 13.9.b and let I be the radical ideal of C such that $\operatorname{Sing} C = \mathcal{V}(I)$. Then $I = \bigcap_{i=1}^{n} Q_i$, where the Q_i are prime ideals of C. Let $P_i = Q_i \cap A$ for each i and let $I \cap A = J$. Then $J = \bigcap_{i=1}^{n} P_i$. We observe that $\operatorname{Sing} A = \mathcal{V}(J)$. Since C_{Q_i} is not regular, Condition 13.9.b implies that A_{P_i} is not regular. Let $P \in \operatorname{Spec} A$. If $J \subseteq P$, then $P_i \subseteq P$ for some i, and $P_i \subseteq P$ implies that A_{P_i} is a localization of A_P. Therefore A_P is not regular.

Assume that $J \not\subseteq P$. There exists $Q \in \operatorname{Spec} C$ such that $Q \cap A = P$, and it is clear that $I \not\subseteq Q$. Hence C_Q is regular, and thus by Condition 13.9.b, the ring A_P is regular. Therefore $\operatorname{Sing} A = \mathcal{V}(J)$.

It remains to observe that $\sqrt{JC} = I$. Clearly $\sqrt{JC} \subseteq I$. Let $Q \in \operatorname{Spec} C$ with $JC \subseteq Q$. Then $J \subseteq Q \cap A := P$ and A_P is not regular. By Condition 13.9.b, the ring C_Q is not regular, so $I \subseteq Q$.

For item 2, Condition 13.9.b holds for A, by (13.9.2). By item 1, Condition 13.9.b implies Condition 13.9.a. Hence the singular locus of A is closed in $\operatorname{Spec} A$. □

COROLLARY 13.11. *Let $\varphi : A \hookrightarrow C$ be a faithfully flat homomorphism of Noetherian rings. Let B be an essentially finite A-algebra such that* $\operatorname{Sing}(B \otimes_A C)$ *is closed. If the fibers of φ are geometrically regular, then* $\operatorname{Sing} B$ *is closed.*

PROOF. By Fact 2.32, the map $1_B \otimes_A \varphi : B \hookrightarrow B \otimes_A C$ is faithfully flat. Let $Q \in \operatorname{Spec}(B \otimes_A C)$, and let P' and P denote the contractions of Q to B and A, respectively. Since B is essentially finite over A, the field $k(P') = (B \setminus P')^{-1}B_{P'}$ is a finite algebraic extension of the field $k(P) = (A \setminus P)^{-1}A_P$. The fiber over P of the map φ is $\operatorname{Spec}(k(P) \otimes_A C)$; see Discussion 3.22. Since φ has geometrically regular fibers, $\operatorname{Spec}(k(P') \otimes_A C)$ is regular, that is, $(k(P') \otimes_A C)_{Q'}$ is a regular local ring for every prime ideal Q' of $k(P') \otimes_A C$.

Also the fiber over P' of the map $1_B \otimes_A \varphi$ is $\operatorname{Spec}(k(P') \otimes_B (B \otimes_A C))$. Since $k(P') \otimes_B (B \otimes_A C) = k(P') \otimes_A C$, the map $1_B \otimes_A \varphi$ has regular fibers.

By Theorem 13.10.2, $\operatorname{Sing} B$ is closed. □

COROLLARY 13.12. *Let $(A, \mathbf{m})$ be a Noetherian local ring and let $\varphi : A \hookrightarrow \widehat{A}$ be the canonical map from A to its $\mathbf{m}$-adic completion $\widehat{A}$.*

(1) *Condition 13.9.a is equivalent to Condition 13.9.b.*
(2) *If the formal fibers of A are regular, then* $\operatorname{Sing} A$ *is closed.*

PROOF. By Remark 13.8, Theorem 13.10 applies. □

REMARK 13.13. Let A be a Noetherian local ring with regular formal fibers. By Corollary 13.12, $\operatorname{Sing} A$ is closed. In order to obtain that every algebra essentially of finite type over A also has the property that its singular locus is closed, the stronger condition that the formal fibers of A are geometrically regular as in Definition 3.29 is needed. This is demonstrated by an example of Rotthaus of a regular local ring A that is a Nagata ring and has the property that its formal fibers are not geometrically regular; the example is described in Remark 19.7. In the example of Rotthaus, the ring A contains a prime element ω such that the singular locus of the quotient ring $A/(\omega)$ is not closed.

The following two theorems are due to Nagata.

THEOREM 13.14. [**101**, Theorem 73]. *Let A be a Noetherian ring. Then the following two statements are equivalent:*

(1) *For every A algebra B that is essentially finite over A, the singular locus* $\operatorname{Sing} B$ *is closed in* $\operatorname{Spec} B$.
(2) *For every A algebra B that is essentially of finite type over A, the singular locus* $\operatorname{Sing} B$ *is closed in* $\operatorname{Spec} B$.

THEOREM 13.15. [**101**, Theorem 74]. *If A is a complete Noetherian local ring, then A satisfies the equivalent conditions of Theorem 13.14*

From Theorems 13.14 and 13.15, we have:

COROLLARY 13.16. *Let A be a Noetherian local ring. If the formal fibers of A are geometrically regular, then for every A-algebra B essentially of finite type over A, the singular locus* $\operatorname{Sing} B$ *is closed in* $\operatorname{Spec} B$.

PROOF. Let B be an A-algebra that is essentially finite over A. Then $B \otimes_A \widehat{A}$ is an essentially finite $\widehat{A}$-algebra. By Theorem 13.15, $\operatorname{Sing}(B \otimes_A \widehat{A})$ is closed. By Corollary 13.11 with C replaced by $\widehat{A}$, we have $\operatorname{Sing} B$ is closed. This holds for every A-algebra B that is essentially finite over A. Thus by Theorem 13.14, $\operatorname{Sing} B$ is closed for every A-algebra B that is essentially of finite type over A. □

REMARK 13.17. Let A be an affine k-algebra, $\mathbf{m}$ a maximal ideal of A, and $\widehat{A_{\mathbf{m}}}$ the $\mathbf{m}$-adic completion of A. By Jacobian Criterion 13.3 and Remarks 13.4.3 and 13.6, the singular locus of A is closed and $\operatorname{Sing} A = \mathcal{V}(J)$, for an ideal J defined by partial derivatives and derivations. For every maximal ideal $\mathbf{m}$ of A, $\operatorname{Sing} \widehat{A_{\mathbf{m}}} = \mathcal{V}(J\widehat{A_{\mathbf{m}}})$, since the partial derivates $\partial f_i/\partial x_j$ and the k^p derivations on A and $A_{\mathbf{m}}$ extend to derivations of $\widehat{A_{\mathbf{m}}}$. That is, every $A_{\mathbf{m}}$ satisfies Condition 13.9.a, and, by Theorem 13.10.1, Condition 13.9.b holds. Therefore every affine algebra A satisfies Property $\mathcal{A}.3$ and Property $\mathcal{A}.4$.

13.2. Nagata rings and excellence

Developments leading to the concept of excellent rings were made by Zariski, Cohen, Chevalley, Abhyankar, Nagata, Rees, Tate, Hironaka, and Grothendieck among others over the two decades from the early 1940's to the 1960's. These authors were investigating ideal-theoretic properties of rings, the behavior of these

properties under certain kinds of extension, and the relations among these properties.

For the class of Nagata rings, defined in Definition 2.11, algebras essentially of finite type over Nagata rings are again Nagata, by Nagata's Polynomial Theorem 2.12. Rees Finite Integral Closure Theorem 3.14 gives a connection between the integral closure of a Noetherian local ring $(R, \mathbf{m})$ and its completion.

Nagata also proved the following:

THEOREM 13.18. [**101**, Theorem 70] [**117**, 36.4, p. 132, p. 219] *Let R be a Noetherian local Nagata domain. Then R is analytically unramified.*

Theorem 13.18 implies every Noetherian local Nagata ring R satisfies:

$(*)$ For every $P \in \operatorname{Spec} R$, the ring R/P is analytically unramified.

There exist Noetherian local domains $(R, \mathbf{m})$ with $\mathbf{m}$-adic completion $\widehat{R}$ that satisfy condition $(*)$, but are not Nagata; that is, R is not a Nagata ring, but for every $P \in \operatorname{Spec}(R)$, the ring $\widehat{R}/P\widehat{R} = \widehat{R} \otimes_R k(P)$ is reduced. Proposition 9.4 and Remark 9.5 describe examples of DVRs and other Noetherian regular rings that are not Nagata rings.

Condition $(*)$ requires only that the formal fibers of R are reduced. It does not require for a finite field extension L of $k(P)$ that the fibers of the map $R \otimes_R L \hookrightarrow \widehat{R} \otimes_R L$ are reduced.

A necessary and sufficient condition for a Noetherian local ring $(R, \mathbf{m})$ to be Nagata, is that the formal fibers of R are *geometrically* reduced.

THEOREM 13.19. [**51**, No 24, (7.6.4)] *A Noetherian local ring R is a Nagata ring if and only if the formal fibers of R are geometrically reduced.*

If R is a Nagata ring, the normal locus:

$$\operatorname{Nor} R = \{P \in \operatorname{Spec}(R) \mid R_P \text{ is a normal ring}\}$$

is open in $\operatorname{Spec}(R)$. Theorem 13.19 implies that every Noetherian local ring with geometrically reduced formal fibers has an open normal locus. For Noetherian non-local rings this is no longer true. Nishimura has constructed an example of a Noetherian ring R with geometrically regular formal fibers so that $\operatorname{Nor}(R)$ is not open in $\operatorname{Spec} R$. Theorem 13.20 characterizes Nagata rings in general:

THEOREM 13.20. [**51**, No 24, (7.6.4), (7.7.2)] *A Noetherian ring R is a Nagata ring if and only if the following two conditions are satisfied:*

(a) The formal fibers of R are geometrically reduced.

(b) For every finite R-algebra S that is a domain $Nor(S)$ is open in $Spec S$.

Theorem 13.21 stated below is another way to deduce that the examples described in Proposition 9.4 and Remark 9.5 are non-Nagata rings.

THEOREM 13.21. [**101**, Theorem 71] *Let $(R, \mathbf{m})$ be a Nagata local domain, let $\widehat{R}$ be the $\mathbf{m}$-adic completion of R, and let $\widehat{P}$ be a minimal prime ideal of $\widehat{R}$. Then $k(\widehat{P}) = \mathcal{Q}(\widehat{R}/\widehat{P})$ is separable over the field of fractions $\mathcal{Q}(R)$ of R.*

Grothendieck defined excellence for a Noetherian local ring as follows:

DEFINITION 13.22. Let A be a Noetherian local ring. Then A is *excellent* if

(a) The formal fibers of A are geometrically regular, that is, for every prime ideal P of A and, for every finite purely inseparable field extension L of the field of fractions $k(P)$ of A/P, the ring $\widehat{A} \otimes_A L$ is regular.
(b) A is universally catenary.

For a non-local Noetherian ring A an additional condition is needed in the definition of excellence: the singular locus of every finitely generated algebra over A is closed. This condition is not included in Definition 13.22; by Corollary 13.12.2, the singular locus is closed for a Noetherian local ring that has geometrically regular formal fibers.

If A is an excellent local ring, then its completion $\widehat{A}$ inherits many properties from A. In particular, we have:

THEOREM 13.23. [**51**, No 24,(7.8.3.1), p. 215] *Let $(A, \mathbf{m})$ be an excellent local ring with $\mathbf{m}$-adic completion $\widehat{A}$. Let $Q \in \operatorname{Spec} \widehat{A}$, and let $P = Q \cap A$. Then the ring A_P is regular (normal, reduced, Cohen-Macaulay, respectively) if and only if the ring $\widehat{A}_Q$ is regular (normal, reduced, Cohen-Macaulay, respectively).*

If A is not a local ring, the formal fibers of A are the formal fibers of the local rings $A_{\mathbf{m}}$, where $\mathbf{m}$ is a maximal ideal of A. We say that A has geometrically regular formal fibers if the local rings $A_{\mathbf{m}}$ for all maximal ideals $\mathbf{m}$ of A have geometrically regular formal fibers. If A is a semilocal ring with geometrically regular formal fibers, then $\operatorname{Sing} A$ is again closed in $\operatorname{Spec} A$. If A is a non-semilocal ring with geometrically regular formal fibers then it is possible that $\operatorname{Sing} A$ is no longer closed in $\operatorname{Spec} A$; see the example of Nishimura, [**119**]. Therefore an additional condition is needed for the singular locus of A and of all algebras of finite type over A to be closed. See Definition 3.37.

13.3. Henselian rings

Let $(R, \mathbf{m})$ be a local ring. Recall from Definition 2.13 that R is *Henselian* if Hensel's Lemma holds for R.

The Henselian property was first observed in algebraic number theory around 1910 for the ring of p-adic integers. Many popular Noetherian local rings fail to be Henselian; see for example Exercise 13.2.

In this section we describe an approach to the construction of the Henselization of the local ring R developed by Raynaud in [**129**] and discussed in [**136**]. This approach is different from that used in Nagata's book [**117**] and discussed in Remarks 2.15. Raynaud defines a local ring R to be Henselian if every finite R-algebra B is a finite product of local rings [**129**, Definition 1, p.1]. Raynaud's approach uses the concept of an étale morphism as in Definitions 13.24.[4]

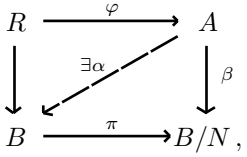
DEFINITIONS 13.24. Let $(R, \mathbf{m})$ be a local ring.

(1) Let $\varphi : (R, \mathbf{m}) \to (A, \mathbf{n})$ be a local homomorphism with A essentially finite over R; that is A is a localization of an R-algebra that is a finitely generated R-module. Then A is *étale* over R if the following condition holds: for every R-algebra B and ideal N of B with $N^2 = 0$, every

[4]David Mumford mentions that the word *étale* "refers to the appearance of the sea at high tide under a full moon in certain types of weather" [**105**, p. 344]. Another meaning for *étale*, given on dictionary.revers.net, is "slack" and *étaler* is translated as "spread or display". A sentence given there "Il s'est étale de tout son long"—is translated as "He fell flat on his face."

R-algebra homomorphism $\beta : A \to B/N$ has a unique lifting to an R-algebra homomorphims $\alpha : A \to B$. Thus A is *étale* over R if for every commutative diagram of the form below, where the maps from $R \to A$ and $R \to B$ are the canonical ring homomorphisms that define A and B as R-algebras and the map $\pi : B \to B/N$ is the canonical quotient ring map

(13.24.1)



there exists a unique R-algebra homomorphism $\alpha : A \to B$ that preserves commutativity of the diagram.

(2) A local ring $(A, \mathbf{n})$ is an *étale neighborhood* of R if A is étale over R and $R/\mathbf{m} \cong A/\mathbf{n}$; that is, there is no residue field extension.

Raynaud proves that Henselian local rings are closed under étale neighborhoods.

THEOREM 13.25. [**129**, Corollary 2, p. 84] *Let R be a local Henselian ring. Then R is closed under étale neighborhoods, that is, for every étale neighborhood $\phi : R \longrightarrow A$, we have that $R \cong A$ considered as R-algebras.*

Structure Theorem 13.26 is essential for Raynaud's approach to the construction of the Henselization.

THEOREM 13.26. *(Structure Theorem for étale neighborhoods)* [**129**, Theorem 1, p. 51] *Let $\varphi : (R, \mathbf{m}) \longrightarrow (A, \mathbf{n})$ be a local morphism with A essentially finite over R. Then A is étale over R if and only if*

$$A \cong (R[x]/(f))_Q$$

where $R[x]$ is the polynomial ring over R in one variable and

(a) $f \in R[x]$ is a monic polynomial.

(b) $Q \in R[x]$ is a prime ideal with $Q \cap R = \mathbf{m}$.

(c) $f' \notin Q$, that is, the derivative of f is not in Q.

The proof of the structure theorem involves a form of Zariski's Main Theorem [**125**], [**40**].

Using the structure theorem Raynaud defines a representative set of étale neighborhoods of R:

$$\Lambda = \{(f, Q) \mid f \in R[x] \text{ monic},\ Q \in \operatorname{Spec}(R[x]),\ f \in Q,\\ f' \notin Q,\ Q \cap R = \mathbf{m},\ (R[x]/Q)_Q = R/\mathbf{m}\}.$$

The set Λ is a subset of the product set $R[x] \times \operatorname{Spec}(R[x])$.

Raynaud defines the Henselization of R via a direct limit over the set Λ. Let $\lambda_1 = (f_1, Q_1)$ and $\lambda_2 = (f_2, Q_2)$ be elements of Λ, and let $S_1 = (R[x]/(f_1))_{Q_1}$, respectively, $S_2 = (R[x]/(f_2))_{Q_2}$, denote the corresponding étale neighborhoods. We define a partial order on Λ by $\lambda_1 \leq \lambda_2$ if and only if there is a local R-algebra morphism $\tau : S_1 \longrightarrow S_2$. In order to define a direct limit over the set Λ two conditions must be satisfied. First, the set of R-algebra morphisms between S_1 and

S_2 has to be rather small in order to restrict each choice of R-algebra morphisms to one for which "it all fits together". Second, the partially ordered set Λ must be directed, that is, for every λ_1 and $\lambda_2 \in \Lambda$, there must be a third element $\lambda_3 \in \Lambda$ with $\lambda_1 \leq \lambda_3$ and $\lambda_2 \leq \lambda_3$. The following result is what is needed:

THEOREM 13.27. [**129**, Proposition 2, p. 84] *Let $\lambda_1, \lambda_2 \in \Lambda$ with corresponding étale neighborhoods $S_i = (R[x]/(f_i))_{Q_i}$. Then:*

(a) There is at most one R-algebra morphism $\tau : S_1 \longrightarrow S_2$.

(b) There is an element $\lambda_3 \in \Lambda$ with corresponding étale neighborhood S_3 that contains S_1 and S_2, i.e. $\lambda_1 \leq \lambda_3$ and $\lambda_2 \leq \lambda_3$.

Theorem 13.27 implies that the set

$$\{(R[x]/(f))_Q \,|\, (f, Q) \in \Lambda\}$$

is directed in a natural way. Raynaud defines the direct limit of this system to be the Henselization of R:

$$R^h = \varinjlim_{\lambda=(f,Q)\in\Lambda} (R[x]/(f))_Q.$$

We list several properties of the Henselization:

REMARKS 13.28.
(1) A local ring R is Noetherian if and only if its Henselization R^h is Noetherian [**129**, Chapitre VIII].
(2) If R is a Noetherian local ring, then the natural injection $R \hookrightarrow R^h$ is a regular morphism [**51**, No 32, (18.6.9), p. 139].
(3) The formal fibers of a Noetherian local ring R are geometrically regular, respectively, geometrically normal, geometrically reduced, if and only if the formal fibers of R^h are geometrically regular, respectively, geometrically normal, geometrically reduced. Moreover, R is a Nagata ring if and only if R^h is a Nagata ring. In addition, if R is excellent so is R^h. These results are in [**51**, No. 32, (18.7.4), (18.7.2), (18.7.3), and (18.7.6), pp. 143-144]; see also Remark 18.3.
(4) The Henselization R^h of a Noetherian local ring R is in general much smaller than its completion $\widehat{R}$. The Henselization R^h of R is an algebraic extension of R whereas the completion $\widehat{R}$ is usually of infinite (uncountable) transcendence degree over R, if R is a domain; see Fact 3.5
(5) If R is an excellent normal local domain, then its Henselization R^h is the algebraic closure of R in $\widehat{R}$, that is, every element of R^h is algebraic over R and every element of $\widehat{R} - R^h$ is transcendental over R. With the definition of the Henselization in Nagata's book, this is given in [**117**, Corollary 44.3].

Theorem 13.29 is an extension of Remark 3.16.2 to integral domains of dimension bigger than one that have geometrically normal formal fibers.

THEOREM 13.29. [**129**, Corollaire, p. 99] *Let R be a Noetherian local domain with geometrically normal formal fibers. Then there is a one-to-one correspondence between the maximal ideals of the integral closure of R in its field of fractions $\mathcal{Q}(R)$ and the minimal prime ideals of its completion $\widehat{R}$.*

Exercises

(1) Let $\varphi : A \hookrightarrow C$ be a faithfully flat homomorphism of Noetherian rings.
   (a) If the fibers of φ are regular and for each $Q \in \operatorname{Spec} C$ the formal fiber over Q is regular, prove that for each $P \in \operatorname{Spec} A$, the formal fiber over P is regular.
   (b) If the fibers of φ are geometrically regular and for each $Q \in \operatorname{Spec} C$ the formal fiber over Q is geometrically regular, prove that for each $P \in \operatorname{Spec} A$, the formal fiber over P is geometrically regular.

(2) Let x be an indeterminate over a field k and let R denote the localized polynomial ring $k[x]_{(xk[x])}$. Show that R is *not* Henselian.
   Suggestion. Consider the polynomial $f(y) = y^2 + y + x \in R[y]$.

(3) Let A be a Nagata ring and let $S \subset A$ be a multiplicatively closed subset of A. Show that $S^{-1}A$ is a Nagata ring.

(4) Let $A \hookrightarrow B$ be Noetherian rings with B a finite A-module. If B is a Nagata ring prove that A is also a Nagata ring.

CHAPTER 14

Approximating discrete valuation rings by regular local rings

Let k be a field of characteristic zero and let $(V, \mathbf{n})$ be a rank-one discrete valuation domain (DVR) containing k and having residue field $V/\mathbf{n} \cong k$. If the field of fractions L of V has finite transcendence degree s over k, we prove that for every positive integer $d \leq s$, the ring V can be realized as a directed union of regular local rings each of which is a k-subalgebra of V of dimension d. We use a technique inspired by Nagata [**115**] and examined in Chapters 4, 5, 6, and 12, for the construction of Noetherian domains.

14.1. Local quadratic transforms and local uniformization

The concepts of local quadratic transformations and local uniformization are relevant for our work in this chapter.

DEFINITIONS 14.1. Let $(R, \mathbf{m})$ be a Noetherian local domain and let $(V, \mathbf{n})$ be a valuation domain that birationally dominates R.

(1) The *first local quadratic transform* of $(R, \mathbf{m})$ along $(V, \mathbf{n})$ is the ring
$$R_1 = R[\mathbf{m}/a]_{\mathbf{m}_1},$$
where $a \in \mathbf{m}$ is such that $\mathbf{m}V = aV$ and $\mathbf{m}_1 := \mathbf{n} \cap R[\mathbf{m}/a]$. The ring R_1 is also called the *dilatation of* R *by the ideal* $\mathbf{m}$ *along* V [**117**, page 141].

(2) More generally, if $I \subseteq \mathbf{m}$ is a nonzero ideal of R, the *dilatation of* R *by* I *along* V is the ring $R[I/a]_{\mathbf{m}_1}$, where $a \in I$ is such that $IV = aV$ and $\mathbf{m}_1 = \mathbf{n} \cap R[I/a]$; moreover, R_1 is uniquely determined by R, V and the ideal I [**117**, page 141].

(3) For each positive integer i, the $(i+1)^{\text{st}}$ *local quadratic transform* R_{i+1} *of* R *along* V is defined inductively: R_{i+1} is the first local quadratic transform of R_i along V.

REMARKS 14.2. Let $(R, \mathbf{m})$ be a regular local ring and let $(V, \mathbf{n})$ be a valuation domain that birationally dominates R.

(1) It is well known that the local quadratic transform R_1 of R along V is again a regular local ring [**117**, 38.1]; moreover, R_1 is uniquely determined by R and V [**117**, page 141].

(2) With the notation of Definition 14.1.3, we have the following relationship among iterated local quadratic transforms:
$$R_{i+j} = (R_i)_j \quad \text{for all} \quad i, j \geq 0.$$
Associated with the set $\{R_i\}_{i \in \mathbb{N}}$ of local quadratic transformations of R along V, it is natural to consider the subring $R_\infty := \bigcup_{i=1}^{\infty} R_i$ of V.

(3) If $(R, \mathbf{m})$ is a regular local ring of dimension 2 and V is a valuation domain that birationally dominates R, a classical result of Zariski and Abhyankar is that $R_\infty = \bigcup_{n=1}^{\infty} R_n = V$ [**2**, Lemma 12].
(4) In the case where R is a regular local ring of dimension $d \geq 3$, for certain valuation rings V that birationally dominate R, the union $\bigcup_{n=1}^{\infty} R_n$ of the local quadratic transforms of R along V is strictly smaller than V [**144**, 4.13]. In many cases Shannon proves in [**144**, (4.5), page 308] that V is a directed union of iterated monoidal transforms of R, where a *monoidal transform of* R is a dilatation of R by a prime ideal P for which the residue class ring R/P is regular.
(5) Assume that $R \subseteq S \subseteq V$, where S is regular local ring birationally dominating R and V is a valuation domain birationally dominating S. Using monoidal transforms, Cutkosky has shown in [**31**] and [**32**] that there exists an iterated local monoidal transform T of S along V such that T is an iterated local monoidal transform of R .

In the case where V is a DVR that birationally dominates a regular local ring, the following useful result is proved by Zariski [**163**, pages 27-28] and Abhyankar [**2**, page 336]. In this connection, for a related result, see Remark4.19.

PROPOSITION 14.3. *Let $(V, \mathbf{n})$ be a DVR that birationally dominates a regular local ring $(R, \mathbf{m})$, and let R_n be the n^{th} local quadratic transform of R along V. Then $R_\infty = \bigcup_{n=1}^{\infty} R_n = V$. In the case where V is essentially finitely generated over R, we have $R_n = V$ for some positive integer n, and thus $R_{n+i} = R_n$ for all $i \geq 0$.*

PROOF. A nonzero element η of V has the form $\eta = b/c$, where $b, c \in R$. If $(b, c)V = V$, then $b/c \in V$ implies $cV = V$. Since V dominates R, it follows that $cR = R$, so $b/c \in R$ in this case. If $\eta = b/c$, with $b, c \in R$ and $(b, c)V = \mathbf{n}^n$, we prove by induction on n that $\eta \in R_n$. We have already done the case where $n = 0$. Assume for every regular local domain $(S, \mathbf{p})$ birationally dominated by V, and every nonzero element $\beta/\gamma \in V$ with $\beta, \gamma \in S$, $(\beta, \gamma)V = \mathbf{n}^j$ and $0 \leq j < n$ we have $\beta/\gamma \in S_j$, where S_j is the j-th iterated local quadratic transform of S along V. Suppose $\beta, \gamma \in S$, $\beta/\gamma \in V$ with $(\beta, \gamma)V = \mathbf{n}^n$. Let $S_1 = S[\mathbf{p}/a]_{\mathbf{p}_1}$, where $a \in \mathbf{p}$ is such that $\mathbf{p}V = aV$ and $\mathbf{p}_1 := \mathbf{n} \cap S[\mathbf{p}/a]$; that is, S_1 is the first local quadratic transform of S along V. Then $\beta_1 := \beta/a$ and $\gamma_1 := \gamma/a$ are in S_1. Thus $a \in \mathbf{n}$ implies $(\beta_1, \gamma_1)V = \mathbf{n}^j$ where $0 \leq j < n$, so by induction

$$\beta/\gamma = \beta_1/\gamma_1 \in (S_1)_j = S_{j+1} \subseteq S_n.$$

This completes the proof of Proposition 14.3. □

DEFINITION 14.4. Let $(R, \mathbf{m})$ be a Noetherian local domain that is essentially finitely generated over a field k and let $(V, \mathbf{n})$ be a valuation domain that birationally dominates R. In algebraic terms *local uniformization of R along V* asserts the existence of a regular local domain extension S of R such that S is essentially finitely generated over R and S is dominated by V.

If R is a regular local ring and P is a prime ideal of R, *embedded local uniformization of R along V* asserts the existence of a regular local domain extension S of R such that S is essentially finitely generated over R and is dominated by V, and has the property that there exists a prime ideal Q of S with $Q \cap R = P$ such that the residue class ring S/Q is a regular local ring.

DISCUSSION 14.5. The classical approach for obtaining embedded local uniformization, introduced by Zariski in the 1940's [**162**], uses local quadratic transforms of R along V. Let $(R, \mathbf{m})$ be a s-dimensional regular local ring. If $(V, \mathbf{n})$ is a DVR that birationally dominates R and has the property that $V/\mathbf{n}$ is algebraic over $R/\mathbf{m}$, then the classical method of taking local quadratic transforms of R along V gives by Proposition 14.3 a representation for V as a directed union $R_\infty = \bigcup_{n\in\mathbb{N}} R_n = V$, where each R_n is an iterated local quadratic transforms of R. The dimension formula [**103**, page 119] implies that $\dim R_n = \dim R = s$ for each n. Thus the DVR V has been represented as a directed union of s-dimensional RLRs. We prove in Theorem 14.6 that certain rank-one discrete valuation rings (DVRs) can be represented as a directed union of regular local domains of dimension d for every positive integer d less than or equal to $s = \dim R$.

14.2. Expressing a DVR as a directed union of regular local rings

We prove the following theorem:

THEOREM 14.6. *Let k be a field of characteristic zero and let $(V, \mathbf{n})$ be a DVR containing k with $V/\mathbf{n} = k$. Assume that the field of fractions L of V has finite transcendence degree s over k. Then for every integer d with $1 \leq d \leq s$, there exists a nested family $\{C_n^{(\alpha)} : n \in \mathbb{N}, \alpha \in \Gamma\}$ of d-dimensional regular local k-subalgebras of V such that V is the directed union of the $C_n^{(\alpha)}$ and V dominates each $C_n^{(\alpha)}$.*

Moreover, if the field L is finitely generated over k, then V is a countable union $\bigcup_{n=1}^{\infty} C_n$, where, for each $n \in \mathbb{N}$,

1. *C_n is a d-dimensional regular local k-subalgebra of V,*
2. *C_n has field of fractions L,*
3. *C_{n+1} dominates C_n and*
4. *V dominates C_n.*

We have the following corollary to Theorem 14.6.

COROLLARY 14.7. *Let k be a field of characteristic zero and let $(R, \mathbf{m})$ be a local domain essentially of finite type over k with coefficient field $k = R/\mathbf{m}$ and field of fractions L. Let $(V, \mathbf{n})$ be a DVR birationally dominating R with $V/\mathbf{n} = k$. For every integer d with $1 \leq d \leq s$, $s = trdeg_k(L)$, there exists a sequence of d-dimensional regular local k-subalgebras C_n of V such that $V = \cup_{n=1}^{\infty} C_n$, and for each $n \in \mathbb{N}$, C_{n+1} dominates C_n and V dominates C_n. Moreover C_n dominates R for all sufficiently large n.*

DISCUSSION 14.8. (1) If L/k is finitely generated of transcendence degree s, then the fact that V is a directed union of s-dimensional regular local rings follows from classical theorems of Zariski. The local uniformization theorem of Zariski [**162**] implies the existence of a regular local domain $(R, \mathbf{m})$ containing the field k such that V birationally dominates R. Since k is a coefficient field for V, we have

- $k \hookrightarrow V \twoheadrightarrow V/\mathbf{n} \cong k$; thus k is relatively algebraically closed in L.
- $R/\mathbf{m} = k$ (because V dominates R).
- Every iterated local quadratic transform of R along V has dimension s.

Now by Proposition 14.3, V is a directed union of s-dimensional RLRs.

(2) If $d = 1$, the main theorem is trivially true by taking each $C_n = V$. Thus if L/k is finitely generated of transcendence degree $s = 2$, then the theorem is saying nothing new.

(3) If $s > 2$, then the classical local uniformization theorem says nothing about expressing V as a directed union of d-dimensional RLRs, where $2 \leq d \leq s-1$. If $(S, \mathbf{p})$ is a Noetherian local domain containing k and birationally dominated by V with $\dim(S) = d < s$, then S does not satisfy the dimension formula. It follows that S is not essentially finitely generated over k [**103**, page 119].

We use the following remark and notation in the proof of Theorem 14.6.

REMARK 14.9. With the notation of Theorem 14.6, let $y \in \mathbf{n}$ be such that $yV = \mathbf{n}$. Then the $\mathbf{n}$-adic completion $\widehat{V}$ of V is $k[[y]]$, and we have

$$k[y]_{(y)} \subseteq V \subseteq k[[y]].$$

Then $V = L \cap k[[y]]$, for example since $V \hookrightarrow k[[y]]$ is flat. Since the transcendence degree of L over $k(y)$ is $s-1$, there are $s-1$ elements $\sigma_1, \ldots, \sigma_{s-d}, \tau_1, \ldots, \tau_{d-1} \in yV$ such that L is algebraic over $F := k(y, \sigma_1, \ldots, \sigma_{s-d}, \tau_1, \ldots, \tau_{d-1})$.

NOTATION 14.10. Continuing with the terminology of (14.9), we set

$$K := k(y, \sigma_1, \ldots, \sigma_{s-d}) \quad \text{and} \quad R := V \cap K.$$

Then R is a DVR and the (y)-adic completion of R is $R^* = k[[y]]$. We also have $B_0 := R[\tau_1, \ldots, \tau_{d-1}]_{(y,\tau_1,\ldots,\tau_{d-1})}$ is a d-dimensional regular local ring and $V_0 := V \cap F$ is a DVR that birationally dominates B_0 and has y-adic completion $\widehat{V_0} = k[[y]]$. The following diagram displays these domains:

$$\begin{array}{ccccccccc}
k & \xrightarrow{\subseteq} & K & \xrightarrow{\subseteq} & F & = & F & \xrightarrow{\subseteq} & L := \mathcal{Q}(V) \\
\| \uparrow & & \cup | \uparrow & & \cup | \uparrow & & \cup | \uparrow & & \cup | \uparrow \\
k & \xrightarrow{\subseteq} & R := V \cap K & \xrightarrow{\subseteq} & B_0 & \xrightarrow{\subseteq} & V_0 := V \cap F & \xrightarrow{\subseteq} & V
\end{array}$$

Let R^* denote the (y)-adic completion of R. Then $\tau_1, \ldots, \tau_s \in yR^*$ are regular elements of R^* that are algebraically independent over K. As in Notation 5.4 we represent each of the τ_i by a power series expansion in y; we use these representations to obtain for each positive integer n the n^{th}-endpieces τ_{in} and corresponding n^{th}-localized polynomial ring B_n. For $1 \leq i \leq s$, and $\tau_i := \sum_{j=1}^{\infty} r_{ij} y^j$, where the $r_{ij} \in R$, we set, for each $n \in \mathbb{N}$,

$$\begin{aligned}
&\tau_{in} := \sum_{j=n+1}^{\infty} r_{ij} y^{j-n}, \qquad B_n := R[\tau_{1n}, \ldots, \tau_{sn}]_{(\mathbf{m}, \tau_{1n}, \ldots, \tau_{sn})} \\
&B := \bigcup_{n=0}^{\infty} B_n = \varinjlim B_n \qquad \text{and} \qquad A := K(\tau_1, \ldots, \tau_s) \cap R^*.
\end{aligned} \tag{14.10.1}$$

Recall that A birationally dominates B. Also by Proposition 5.9, the definition of B_n is independent of the representations of the τ_i.

Since $V_0 = F \cap k[[y]]$, each $\tau_{in} \in V_0$ and $B_n = R[\tau_{1n}, \ldots, \tau_{d-1,n}]_{(y,\tau_{1n},\ldots,\tau_{d-1,n})}$ is, for each $n \in \mathbb{N}$, the first quadratic transform of B_{n-1} along V_0.

In the proof of Theorem 14.6, we make use of Theorem 17.13 of Chapter 6. We also use Theorem 14.11 in the proof of Theorem 14.6:

THEOREM 14.11. *With the notation of (14.10), for each positive integer n, let B_n^h denote the Henselization of B_n. Then $\bigcup_{n=1}^{\infty} B_n^h = V_0^h = V^h$.*

PROOF. Since R_y^* is a field, it is flat as an $R[\tau_1, \ldots, \tau_{d-1}]$-module. By Theorem 17.13, $V_0 = \bigcup_{n=1}^{\infty} B_n$. An alternate way to justify this description of V_0 is to use Proposition 14.3, where the ring R is B_0, and V is V_0, and each $R_n = B_n$. We have

$$V_0 \underset{\text{alg}}{\longrightarrow} V \longrightarrow k[[y]],$$

where V_0 and V are DVRs of characteristic zero having completion $k[[y]]$. Since V_0 and V are excellent, their Henselizations V_0^h and V^h are the set of elements of $k[[y]]$ algebraic over V_0 or V [**117**, (44.3)]. Thus $V_0^h = V^h$ and V is a directed union of étale extensions of V_0, see Definition 13.24.

The ring $C := \bigcup B_n^h$ is Henselian and contains V_0, so $V_0^h = V^h \subseteq C$. Moreover, the inclusion map $V \to C = \bigcup B_n^h$ extends to a map $V^h \xrightarrow{\sigma} C = \bigcup B_n^h$. On the other hand, the maps $B_n \to V$ extend to maps: $B_n^h \to V^h$ yielding a map $\rho : C \to V^h$ with $\sigma\rho = 1_C$, and $\rho\sigma = 1_{V^h}$. Thus $\bigcup_{n=1}^{\infty} B_n^h = V^h$. □

Proof of Theorem 14.6 if the field L is finitely generated over k.

PROOF. Since L is algebraic over F, it follows that L is finite algebraic over F. Since $\bigcup_{n=1}^{\infty} B_n^h = V^h$, we have $\bigcup_{n=1}^{\infty} \mathcal{Q}(B_n^h) = \mathcal{Q}(V^h)$ and $L \subseteq \mathcal{Q}(V^h)$. Since L/F is finite algebraic, $L \subseteq \mathcal{Q}(B_n^h)$ for all sufficiently large n. By relabeling, we may assume $L \subseteq \mathcal{Q}(B_n^h)$ for all n. Let $C_n := B_n^h \cap L$. Since B_n is a regular local ring, C_n is a regular local ring with $C_n^h = B_n^h$. [**135**, (1.3)].

We observe that for every n, C_{n+1} dominates C_n and V dominates C_n. Also $\bigcup_{n=1}^{\infty} C_n = V$. Indeed, since B_{n+1} dominates B_n, we have B_{n+1}^h dominates B_n^h and hence $C_{n+1} = B_{n+1}^h \cap L$ dominates $C_n = B_n^h \cap L$. Since $C_n = B_n^h \cap L \subseteq V^h \cap L = V$, it follows that V dominates C_n and $V_0 \subseteq \bigcup_{n=1}^{\infty} C_n \subseteq V$. Since V birationally dominates $\bigcup_{n=1}^{\infty} C_n$, it suffices to show that $\bigcup_{n=1}^{\infty} C_n$ is a DVR.

But by the same argument as before, $\bigcup_{n=1}^{\infty} C_n^h = (\bigcup_{n=1}^{\infty} C_n)^h = V^h$. This shows that $\bigcup_{n=1}^{\infty} C_n$ is a DVR, and therefore $\bigcup_{n=1}^{\infty} C_n = V$. Thus in the case where L/k is finitely generated we have completed the proof of Theorem 14.6 including the "moreover" statement. □

REMARK 14.12. An alternate approach to the definition of C_n is as follows. Since V is a directed union of étale extensions of V_0 and $\mathbb{Q}(V) = L$ is finite algebraic over $\mathbb{Q}(V_0) = F$, V is étale over V_0 and therefore $V = V_0[\theta] = V_0[X]/(f(X))$, where $f(X)$ is a monic polynomial such that $f(\theta) = 0$ and $f'(\theta)$ is a unit of V. Let B_n' denote the integral closure of B_n in L and let $C_n = (B_n')_{(\mathbf{n} \cap B_n')}$. Since $\cup_{n=1}^{\infty} B_n = V_0$, it follows that $\cup_{n=1}^{\infty} C_n = V$. Moreover, for all sufficiently large n, $f(X) \in B_n[X]$ and $f'(\theta)$ is a unit of C_n. Therefore C_n is a regular local ring for all sufficiently large n [**117**, (38.6)]. As we note in (14.13) below, this allows us to deduce a version of Theorem 14.6 also in the case where k has characteristic $p > 0$ provided the field F can be chosen so that L/F is separable.

Proof of Theorem 14.11 if the field L is not finitely generated over k.

PROOF. If L is not finitely generated over k, we choose a nested family of fields L_α, with $\alpha \in \Gamma$, such that

(1) $F \subseteq L_\alpha$, for all α.
(2) L_α is finite algebraic over F.

(3) $\cup_{\alpha\in\Gamma} L_\alpha = L$.

The rings $V_\alpha = L_\alpha \cap V$ are DVRs with $\cup_{\alpha\in\Gamma} V_\alpha = V$ and $V_\alpha^h = V^h$, since $V_0 \subseteq V_\alpha$, for each $\alpha \in \Gamma$.

As above, $\cup_{n=1}^\infty B_n^h = V^h$, $\cup_{n=1}^\infty \mathcal{Q}(B_n^h) = \mathcal{Q}(V^h)$ and $L \subseteq \mathcal{Q}(V^h)$. Thus we see that for each $\alpha \in \Gamma$, there is an $n_\alpha \in \mathbb{N}$ such that $L_\alpha \subseteq \mathcal{Q}(B_n^h)$ for all $n \geq n_\alpha$.

Put $C_n^{(\alpha)} = L_\alpha \cap B_n^h$ for each $n \geq n_\alpha$. Then $V_\alpha = \cup_{n=n_\alpha}^\infty C_n^{(\alpha)}$ and V_α birationally dominates $C_n^{(\alpha)}$. Hence

$$V = \underset{\alpha\in\Gamma, n\geq n_\alpha}{\rightarrow} \bigcup C_n^{(\alpha)}.$$

This completes the proof of Theorem 14.6. □

REMARK 14.13. If the characteristic of k is $p > 0$ then the Henselization V_0^h of $V_0 = F \cap k[[y]]$ may not equal the Henselization V^h of $V = L \cap k[[y]]$, because the algebraic field extension L/F may not be separable. But in the case where L/F is separable algebraic, the fact that the DVRs V and V_0 have the same completion implies that V is a directed union of étale extensions of V_0 (see, for example, [**5**, Theorem 2.7]). Therefore in the case where L/F is separable algebraic, V is a directed union of regular local rings of dimension d.

Thus for a local domain $(R, \mathbf{m})$ essentially of finite type over a field k of characteristic $p > 0$, a result analogous to Corollary 1.3 is true provided there exists a subfield F of L such that F is purely transcendental over k, L/F is separable algebraic, and F contains a generator for the maximal ideal of V.

In characteristic $p > 0$, with V excellent and the extension separable, the ring V_0 need not be excellent (see for example Proposition 9.4 or [**60**, (3.3) and (3.4)]).

14.3. More general valuation rings as directed unions of RLRs

A useful method for constructing rank-one valuation rings is to make use of generalized power series rings as in [**165**, page 101].

DEFINITION 14.14. Let k be a field and let $e_0 < e_1 < \cdots$ be real numbers such that $\lim_{n\to\infty} e_n = \infty$. For a variable t and elements $a_i \in k$, consider the *generalized power series expansion*

$$z(t) := a_0 t^{e_0} + a_1 t^{e_1} + \cdots + a_n t^{e_n} + \cdots$$

The *generalized power series ring* $k\{t\}$ is the set of all generalized power series expansions $z(t)$ with the usual addition and multiplication.

REMARKS 14.15. With the notation of Definition 14.14, we have:

(1) The generalized power series ring $k\{t\}$ is a field.
(2) The field $k\{t\}$ admits a valuation v of rank one defined by setting $v(z(t))$ to be the order of the generalized power series $z(t)$. Thus $v(z(t)) = e_0$ if a_0 is a nonzero element of k.
(3) The valuation ring V of v is the set of generalized power series of nonnegative order together with zero. The value group of v is the additive group of real numbers.
(4) If $x_1, \ldots, x_r$ are variables over k, then every k-algebra isomorphism of the polynomial ring $k[x_1, \ldots, x_r]$ into $k\{t\}$ determines a valuation ring of rank one on the field $k(x_1, \ldots, x_r)$. Moreover, every such valuation ring has residue field k.

(5) Thus if $z_1(t), \ldots, z_r(t) \in k\{t\}$ are algebraically independent over k, then the k-algebra isomorphism defined by mapping $x_i \mapsto z_i(t)$ determines a valuation on the field $k(x_1, \ldots, x_r)$ of rank one. MacLane and Schilling prove in [**97**] a result that implies for a field k of characteristic zero the existence of a vaulation on $k(x_1, \ldots, x_r)$ of rank one with any preassigned value group of rational rank less than r. In particular, if $r \geq 2$, then every additive subgroup of the group of rational numbers is the value group of a suitable valuation on the field of rational functions in r variables over k.

(6) As a specific example, let k be a field of characteristic zero and consider the k-algebra isomorphism of the polynomial ring $k[x, y]$ into $k\{t\}$ defined by mappping $x \mapsto t$ and $y \mapsto \sum_{n=1}^{\infty} t^{e_1+\cdots+e_n}$, where $e_i = 1/i$ for each positive integer i. The result of MacLane and Schilling [**97**] mentioned above implies that the value group of the valuation ring V defined by this embedding is the group of all rational numbers.

Exercises

(1) Let $(R, \mathbf{m})$ be a two-dimensional regular local ring with $\mathbf{m} = (x, y)R$ and let $a \in R \setminus \mathbf{m}$. Define:

$$S := R\big[\frac{y}{x}\big] = R\big[\frac{\mathbf{m}}{x}\big], \quad \mathbf{n} := (x, \frac{y}{x} - a)S \quad \text{and} \quad R_1 := S_{\mathbf{n}} = \big(R\big[\frac{y}{x}\big]\big)_{\mathbf{n}}.$$

Thus R_1 is a first local quadratic transform of R. Prove that there exists a maximal ideal $\mathbf{n}'$ of the ring $S' := R\big[\frac{x}{y}\big]$ such that $R_1 = S'_{\mathbf{n}'}$, and describe generators for $\mathbf{n}'$.

Suggestion: Notice that $\frac{y}{x}$ is a unit of R_1.

(2) Let $(R, \mathbf{m})$ be a two-dimensional regular local ring with $\mathbf{m} = (x, y)R$. Define:

$$S := R\big[\frac{x}{y}\big] = R\big[\frac{\mathbf{m}}{y}\big], \quad \mathbf{n} := (y, \frac{x}{y})S \quad \text{and} \quad R_1 := (S)_{\mathbf{n}} = \big(R\big[\frac{x}{y}\big]\big)_{\mathbf{n}},$$

and define $P = (x^2 - y^3)R$.

(a) Prove that R/P is a one-dimensional local domain that is not regular.

(b) Prove that there exists a prime ideal Q of R_1 such that $Q \cap R = P$ and R_1/Q is a DVR and hence is regular.

Comment: This is an example of embedded local uniformization.

CHAPTER 15

Non-Noetherian insider examples of dimension 3,

In this chapter we use Insider Construction 10.1 of Section 10.1 to construct examples where the insider approximation domain B is local and non-Noetherian, but is very close to being Noetherian. The localizations of B at all nonmaximal prime ideals are Noetherian, and most prime ideals of B are finitely generated. Sometimes just one prime ideal is not finitely generated.

In Section 15.1 we describe, for each positive integer m, a three-dimensional local unique factorization domain B such that the maximal ideal of B is two-generated, B has precisely m prime ideals of height two, each prime ideal of B of height two is not finitely generated and all the other prime ideals of B are finitely generated. We give more details about a specific case where there is precisely one nonfinitely generated prime ideal. Section 15.2 contains the verification of the properties of the three-dimensional examples. A similar example is given by John David in [**33**]. In Chapter 16 we present a generalization to dimension four.

15.1. A family of examples in dimension 3

In this section we construct examples as described in Examples 15.1. In Discussion 15.6 we give more details for a special case of the example with exactly one nonfinitely generated prime ideal. We display the prime spectrum for this special case in Diagram 15.4.2.

EXAMPLES 15.1. For each positive integer m, we construct an example of a non-Noetherian local integral domain $(B, \mathbf{n})$ such that:

(1) $\dim B = 3$.
(2) The ring B is a UFD that is not catenary, as defined in (3.17.3).
(3) The maximal ideal $\mathbf{n}$ of B is generated by two elements.
(4) The $\mathbf{n}$-adic completion of B is a two-dimensional regular local domain.
(5) For every non-maximal prime ideal P of B, the ring B_P is Noetherian.
(6) The ring B has precisely m prime ideals of height two.
(7) Every prime ideal of B of height two is not finitely generated; all other prime ideals of B are finitely generated.

To establish the existence of the examples in Examples 15.1, we use the following notation:

NOTATION 15.2. Let k be a field, let x and y be indeterminates over k, and set

$$R := k[x,y]_{(x,y)}, \quad K := k(x,y) \quad \text{and} \quad R^* := k[y]_{(y)}[[x]].$$

The power series ring R^* is the xR-adic completion of R. Let $\tau \in xk[[x]]$ be transcendental over $k(x)$. For each integer i with $1 \le i \le m$, let $p_i \in R \setminus xR$ be such that $p_1R^*, \ldots, p_mR^*$ are m prime ideals. For example, if each $p_i \in R \setminus (x,y)^2R$,

then each p_iR^* is prime in R^*. In particular one could take $p_i = y - x^i$. Let $p := p_1 \cdots p_m$. We set $f := p\tau$ and consider the injective R-algebra homomorphism $S := R[f] \hookrightarrow R[\tau] =: T$. In this construction the polynomial rings S and T have the same field of fractions $K(f) = K(\tau)$. Hence the intersection domain

$$A \;:=\; A_f \;:=\; R^* \,\cap\, K(f) \;=\; R^* \,\cap\, K(\tau) := \; A_\tau. \tag{15.1.0}$$

By Valabrega's Theorem 4.8, A is a two-dimensional regular local domain with maximal ideal $(x, y)A$ and the $(x, y)A$-adic completion of A is $k[[x, y]]$.

Let $\tau := c_1x + c_2x^2 + \cdots + c_ix^i + \cdots \in xk[[x]]$, where the $c_i \in k$ and define for each $n \in \mathbb{N}_0$ the "n^{th} endpiece" τ_n of τ by

$$\tau_n \;:=\; \sum_{i=n+1}^{\infty} c_i x^{i-n} \;=\; \frac{\tau - \sum_{i=1}^{n} c_i x^i}{x^n}. \tag{15.1.a}$$

As in Equation 5.4.2 we have the following relation between the n^{th} and $(n+1)^{\text{st}}$ endpieces τ_n and τ_{n+1}:

$$\tau_n \;=\; c_{n+1}x \;+\; x\tau_{n+1}. \tag{15.1.b}$$

Define $f_n := p\tau_n$, set $U_n = k[x, y][f_n] = k[x, y, f_n]$, a three-dimensional polynomial ring over R, and set $B_n = (U_n)_{(x,y,f_n)} = k[x, y, f_n]_{(x,y,f_n)}$, a three-dimensional localized polynomial ring. Similarly set $U_{\tau n} = k[x, y, \tau_n]$, a three-dimensional polynomial ring containing U_n, and $B_{\tau n} = k[x, y, \tau_n]_{(x,y,\tau_n)}$, a localized polynomial ring containing $U_{\tau n}$ and B_n.

Let U, B, U_τ and B_τ be the nested union approximation domains defined as follows:

$$U \;:=\; \bigcup_{n=0}^{\infty} U_n \;\subseteq\; U_\tau \;:=\; \bigcup_{n=0}^{\infty} U_{\tau n}; \qquad B \;:=\; \bigcup_{n=0}^{\infty} B_n \;\subseteq\; B_\tau \;:=\; \bigcup_{n=0}^{\infty} B_{\tau n} \;\subseteq\; A.$$

Remark 15.3. By Construction Properties Theorem 5.14.4, with adjustments using Remark 5.15, parts 2 and 3, we have

(15.3.0)
$$U[1/x] = U_0[1/x] = k[x, y, f][1/x]; \;\; U_\tau[1/x] = U_{\tau,0}[1/x] = k[x, y, \tau][1/x],$$

$B[1/x]$ is a localization of $S = R[f]$ and $B[1/x]$ is a localization of B_n. Similarly, $B_\tau[1/x]$ is a localization of $T = R[\tau]$.

We establish in Theorem 15.10 of Section 15.2 that the rings B of Examples 15.1 have properties 1 through 7 and also some additonal properties.

Assuming properties 1 through 7 of Examples 15.1, we describe the ring B of Examples 15.1 in the case where $m = 1$ and $p = p_1 = y$ as follows:

Example 15.4. Assume Notation 15.2. Thus

$$R = k[x, y]_{(x,y)}, \quad f = y\tau, \quad f_n = y\tau_n, \quad B_n = R[y\tau_n]_{(x,y,y\tau_n)}, \quad B = \bigcup_{n=0}^{\infty} B_n.$$

As we show in Section 15.2, the ideal $Q := (y, \{y\tau_n\}_{n=0}^{\infty})B$ is the unique prime ideal of B of height 2. Moreover, Q is not finitely generated and is the only prime ideal of B that is not finitely generated. We also have $Q = yA \cap B$, and $Q \cap B_n = (y, y\tau_n)B_n$ for each $n \geq 0$.

To identify the ring B up to isomorphism, we include the following details: By Equation 15.1.b, we have $\tau_n = c_{n+1}x + x\tau_{n+1}$. Thus we have

$$f_n \;=\; xf_{n+1} \;+\; yxc_{n+1}. \tag{15.4.1}$$

The family of equations (15.4.1) uniquely determines B as a nested union of the three-dimensional RLRs $B_n = k[x, y, f_n]_{(x,y,f_n)}$.

We recall the following terminology of [**165**, page 325].

DEFINITION 15.5. If a ring C is a subring of a ring D, a prime ideal P of C is *lost* in D if $PD \cap C \neq P$.

DISCUSSION 15.6. Assuming properties 1 through 7 of Examples 15.1, if q is a height-one prime of B, then B/q is Noetherian if and only if q is not contained in Q. This is clear since q is principal, Q is the unique prime of B that is not finitely generated, and, by Cohen's Theorem 2.19, a ring is Noetherian if each prime ideal of the ring is finitely generated.

The height-one primes q of B may be separated into several types as follows:
Type I. The primes $q \not\subseteq Q$ have the property that B/q is a one-dimensional Noetherian local domain. These primes are contracted from A, i.e., they are not lost in A. To see this, consider $q = gB$ where $g \notin Q$. Then gA is contained in a height one prime P of A. Hence $g \in (P \cap B) \setminus Q$, and so $P \cap B \neq Q$. Since $\mathbf{m}_B A = \mathbf{m}_A$, we have $P \cap B \neq \mathbf{m}_B$. Therefore $P \cap B$ is a height-one prime containing q, so $q = P \cap B$ and $B_q = A_P$.

There are infinitely many primes q of type I, because every element of $\mathbf{m}_B \setminus Q$ is contained in a prime q of type I. Thus $\mathbf{m}_B \subseteq Q \cup \bigcup\{q \text{ of Type I}\}$. Since $\mathbf{m}_B$ is not the union of finitely many strictly smaller prime ideals, there are infinitely many primes q of Type I.

Type I*. Among the primes of Type I, we label the prime ideal xB as Type I*. The prime ideal xB is special since it is the unique height-one prime q of B for which R^*/qR^* is not complete. If q is a height-one prime of B such that $x \notin qR^*$, then $x \notin q$ by Proposition 5.16.3. Thus R^*/qR^* is complete with respect to the powers of the nonzero principal ideal generated by the image of x mod qR^*. Notice that $R^*/xR^* \cong k[y]_{yk[y]}$.

If q is a height-one prime of B not of Type I, then $\overline{B} = B/q$ has precisely three prime ideals. These prime ideals form a chain: $(\overline{0}) \subset \overline{Q} \subset \overline{(x, y)B} = \overline{\mathbf{m}_B}$.

Type II. We define the primes of Type II to be the primes $q \subset Q$ such that q has height one and is contracted from a prime p of $A = k(x, y, f) \cap R^*$, i.e., q is not lost in A. For example, the prime $y(y+\tau)B$ is of Type II by Lemma 15.14. For q of Type II, the domain B/q is dominated by the one-dimensional Noetherian local domain A/p. Thus B/q is a non-Noetherian generalized local ring in the sense of Cohen; that is, the unique maximal ideal $\overline{\mathbf{n}}$ of B/q is finitely generated and $\cap_{i=1}^{\infty} \overline{\mathbf{n}}^{\,i} = (0)$, [**28**].

For q of Type II, the maximal ideal of B/q is not principal. This follows because a generalized local domain having a principal maximal ideal is a DVR [**117**, (31.5)].

There are infinitely many height-one primes of Type II, for example, $y(y+x^t\tau)B$ for each $t \in \mathbb{N}$ by Lemma 15.13. For q of Type II, the DVR B_q is birationally dominated by A_p. Hence $B_q = A_p$ and the ideal $\sqrt{qA} = p \cap yA$. [1]

That each element $y(y + x^t\tau)$ is irreducible and thus generates a height-one prime ideal, is done in greater generality in Lemma 15.13.

Type III. The primes of Type III are the primes $q \subset Q$ such that q has height one and is not contracted from A, i.e., q is lost in A. For example, the prime yB and the prime $(y + x^t y\tau)B$ for $t \in \mathbb{N}$ are of Type III by Lemma 15.14. Since the elements y and $y + x^t y\tau$ are in $\mathbf{m}_B$ and are not in $\mathbf{m}_B^2$ and since B is a UFD, these elements are necessarily prime. There are infinitely many such prime ideals by Lemma 15.13. For q of Type III, we have $\sqrt{qA} = yA$.

If $q = yB$ or $q = (y+x^t y\tau)B$, then the image $\overline{\mathbf{m}_B}$ of $\mathbf{m}_B$ in B/q is principal. It follows that the intersection of the powers of $\overline{\mathbf{m}_B}$ is Q/q and B/q is not a generalized local ring. To see that $\bigcap_{i=1}^{\infty} \overline{\mathbf{m}_B}^i \neq (0)$, we argue as follows: If P is a principal prime ideal of a ring and P' is a prime ideal properly contained in P, then P' is contained in the intersection of the powers of P; see [**85**, page 7, ex. 5] and Exercise 15.3

The picture of Spec(B) is shown below.

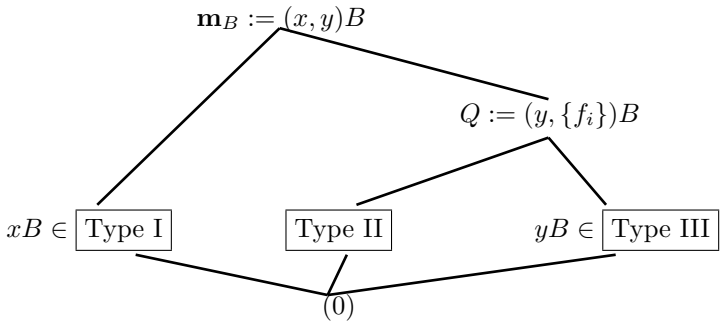


Diagram 15.4.2

In Remarks 15.7 we examine the height-one primes of B from a different perspective.

Remarks 15.7. (1) Assume the notation of Example 15.4. If w is a nonzero prime element of B such that $w \notin Q$, then wA is a prime ideal in A and is the unique prime ideal of A lying over wB. To see this, observe that $w \notin yA$ since $w \notin Q = yA \cap B$. It follows that $y \notin p$, for every prime ideal $p \in \operatorname{Spec} A$ that is a minimal prime of wA. Thus $p \cap B \neq Q$. Since we assume the properties of Examples 15.1 hold, $p \cap B$ has height one. Therefore $p \cap B = wB$. Hence the DVR B_{wB} is birationally dominated by A_p, and thus $B_{wB} = A_p$. This implies that p

[1] Bruce Olberding has pointed out that the existence of prime ideals q of Type II answers a question asked by Anderson-Matijevic-Nichols in [**12**, page 17]. Their question asks whether in an integral domain every nonzero finitely generated prime ideal P that satisfies $\bigcap_{n=1}^{\infty} P^n = (0)$ and that is minimal over a principal ideal has ht $P = 1$. For q of Type II, the ring $\overline{B} = B/q$ is a generalized local domain with precisely 3 prime ideals. An element in the maximal ideal $\overline{\mathbf{m}_B}$ not in the other nonzero prime ideal generates an ideal primary for $\overline{\mathbf{m}_B}$. Since ht $\overline{\mathbf{m}_B} = 2$, this yields a negative answer to the question.

is the unique prime of A lying over wB. We also have $wB_{wB} = pA_p$. Since A is a UFD and p is the unique minimal prime of wA, it follows that $wA = p$. In particular, q is not lost in A; see Definition 15.5.

If q is a height-one prime ideal of B that is contained in Q, then yA is a minimal prime of qA, and q is of Type II or III depending on whether or not qA has other minimal prime divisors.

To see this, observe that if yA is the only prime divisor of qA, then qA has radical yA and $yA \cap B = Q$ implies that Q is the radical of $qA \cap B$. Thus q is lost in A and q is of Type III.

On the other hand, if there is a minimal prime ideal $p \in \operatorname{Spec} A$ of qA that is different from yA, then y is not in $p \cap B$ and hence $p \cap B \neq Q$. Since Q is the only prime ideal of B of height two, it follows that $p \cap B$ is a height-one prime and thus $p \cap B = q$. Thus q is not lost in A and q is of Type II.

We observe that for every Type II prime q there are exactly two minimal primes of qA; one of these is yA and the other is a height-one prime p of A such that $p \cap B = q$. For every height-one prime ideal p of A such that $p \cap B = q$, we have B_q is a DVR that is birationally dominated by A_p and hence $B_q = A_p$. The uniqueness of $B_q = A_p$ as a DVR overring of A implies that there is precisely one such prime ideal p of A.

An example of a height-one prime ideal q of Type II is $q := (y^2 + y\tau)B$. The prime ideal $qA = (y^2 + y\tau)A$ has the two minimal primes yA and $(y + \tau)A$.

(2) The ring B/yB is a rank 2 valuation domain. This can be seen directly or else one may apply a result of Heinzer and Sally[**79**, Prop. 3.5(iv)]; see Exercise 15.3. For other prime elements g of B with $g \in Q$, it need not be true that B/gB is a valuation domain. If g is a prime element contained in $\mathbf{m}_B^2$, then the maximal ideal of B/gB is 2-generated but not principal and thus B/gB cannot be a valuation domain. For a specific example over the field $\mathbb{Q}$, let $g = x^2 + y^2\tau$.

15.2. Verification of the three-dimensional examples

In Theorem 15.10 we record and establish the properties asserted in Examples 15.1 and other properties of the ring B. We make some preliminary remarks:

REMARKS 15.8. (1) Assume that R is a Noetherian local domain with maximal ideal $\mathbf{m}$, let $z \in R$ be a nonzero nonunit, let R^* denote the z-adic completion of R, and let $\tau_1, \ldots, \tau_s \in zR^*$ be algebraically independent elements over R, as in the setting of Construction 5.3. Then, by Theorem 5.14.6, B as defined in Equation 5.4.5 is the same as the ring B defined as a directed union of the localized polynomial rings $B_r := U_{P_r}$, where $P_r := (\mathbf{m}, \tau_{1r}, \ldots, \tau_{sr})U_{\underline{\tau}r}$, with notation as in Section 5.2.

(2) Notation 15.2 used in Examples 15.1 fits that of Construction 5.3 and the modification given in Remarks 5.15.2 of the procedures in Section 5.2, where R is the localized polynomial ring $k[x, y]_{(x,y)}$ over a field k, $R^* = k[y]_{(y)}[[x]]$ is the (x)-adic completion of R and $f \in xR^*$ is transcendental over K. Thus, by Remark 5.15.2, the ring $B = \bigcup B_r$, where $B_r = (U_r)_{P_r}$, $U_r = k[x, y, f_r]$ and $P_r = (x, y, f_r)U_r$, is the same ring B as the ring B described in Equation 5.4.6. A similar remark applies to B_τ with appropriate modifications to $B_{\tau,r}$, $U_{\tau,r}$ and $P_{\tau,r}$. The corresponding rings called B, B_r, etc. in Example 16.1 of Chapter 16 also satisfy the relations from Section 5.2. Furthermore B_τ is the same ring B as in the setting in Localized Prototype Theorem 17.28 where $r = 1 = s$.

(3) Thus the results of Noetherian Flatness Theorem 6.3, Construction Properties Theorem 5.14, Proposition 5.16 and Theorem 5.17 hold for the rings B and B_τ of Examples 15.1 and for the rings of Example 16.1 in Chapter 16. Also Localized Prototype Theorem 17.28 holds for B_τ. With the rings U_r and U defined as non-localized polynomial rings as in Examples 15.1 and Example 16.1 in Chapter 16, we have the relations $U_0[1/x] = U_r[1/x] = U[1/x]$. We might not, however, have all of the same conclusions for $U = \bigcup U_r$ as for the domains called U or U_r in those results.

In order to examine more closely the prime ideal structure of the ring B of Examples 15.1, we establish in Proposition 15.9 some properties of its overring A and of the map $\operatorname{Spec} A \to \operatorname{Spec} B$.

PROPOSITION 15.9. *With Notation 15.2, we have*

(1) $A = B_\tau$ *and* $A[1/x]$ *is a localization of* $R[\tau]$.
(2) *For* $P \in \operatorname{Spec} A$ *with* $x \notin P$, *the following are equivalent:*

(**a**) $A_P = B_{P\cap B}$ (**b**) $\tau \in B_{P\cap B}$ (**c**) $p \notin P$.

PROOF. By Localized Prototype Theorem 17.28 with $r = 1, y = y_1, s = 1$, and $\tau = \tau_1$, as discussed in Remarks 15.8.3, we have $A = B_\tau$ and is Noetherian. By Remark 15.3, the ring $A[1/x]$ is a localization of $R[\tau]$. Thus item 1 holds.

For item 2, since $\tau \in A$, (a) $\implies$ (b) is clear. For (b) $\implies$ (c) we show that $p \in P \implies \tau \notin B_{P\cap B}$. By Remark 15.3, $B[1/x]$ is a localization of $R[f]$. Since $x \notin P$, the ring $B_{P\cap B}$ is a localization of $R[f]$, and thus $B_{P\cap B} = R[f]_{P\cap R[f]}$. The assumption that $p \in P$ implies that some $p_i \in P$, and so $R[f]_{P\cap R[f]}$ is contained in $V := R[f]_{p_i R[f]}$, a DVR. Since $R[f]$ is a polynomial ring over R, f is a unit in V. Hence $\tau = f/p \notin V$ and thus $\tau \notin R[f]_{P\cap R[f]}$. This shows that (b) $\implies$ (c).

For (c) $\implies$ (a), notice that $f = p\tau$ implies that $R[f][1/xp] = R[\tau][1/xp]$. By item 1, $A[1/x]$ is a localization of $R[\tau][1/x]$ and so $A[1/xp]$ is a localization of $R[\tau][1/xp] = R[f][1/xp]$. Thus $A[1/xp]$ is a localization of $R[f]$. By Remark 15.3, $B[1/x]$ is a localization of $R[f]$. Since $xp \notin P$ and $x \notin P \cap B$, we have that A_P and $B_{P\cap B}$ are both localizations of $R[f]$. Thus we have

$$A_P = R[f]_{PA_P\cap R[f]} = R[f]_{(P\cap B)B_{P\cap B}\cap R[f]} = B_{P\cap B}.$$

This completes the proof of Proposition 15.9. □

THEOREM 15.10. *As in Notation 15.2, let* $R := k[x,y]_{(x,y)}$, *where* k *is a field, and* x *and* y *are indeterminates. Set* $R^* = k[y]_{(y)}[[x]]$, *let* $\tau \in xk[[x]]$ *be transcendental over* $k(x)$, *and, for each integer* i *with* $1 \le i \le m$, *let* $p_i \in R \setminus xR$ *be such that* $p_1R^*, \ldots, p_mR^*$ *are* m *prime ideals. Let* $p := p_1 \cdots p_m$ *and set* $f := p\tau$. *With the approximation domain* B *and the intersection domain* A *defined as in Examples 15.1,* $A := A_f = A_\tau$. *Set* $Q_i := p_iR^* \cap B$, *for each* i *with* $1 \le i \le m$. *Then:*

(1) *The ring* B *is a three-dimensional non-Noetherian local UFD with maximal ideal* $\mathbf{n} = (x,y)B$, *and the* $\mathbf{n}$*-adic completion of* B *is the two-dimensional regular local ring* $k[[x,y]]$.
(2) *The rings* $B[1/x]$ *and* B_P, *for each nonmaximal prime ideal* P *of* B, *are regular Noetherian UFDs, and the ring* B/xB *is a DVR.*
(3) *The ring* A *is a two-dimensional regular local domain with maximal ideal* $\mathbf{m}_A := (x,y)A$, *and* $A = B_\tau$. *The ring* A *is excellent if the field* k *has*

characteristic zero. If k is a perfect field of characteristic p, then A is not excellent

(4) *The ideal $\mathbf{m}_A$ is the only prime ideal of A lying over $\mathbf{n}$.*
(5) *The ideals Q_i are the only height-two prime ideals of B.*
(6) *The ideals Q_i are not finitely generated and they are the only nonfinitely generated prime ideals of B.*
(7) *The ring B has saturated chains of prime ideals from (0) to $\mathbf{n}$ of length two and of length three, and hence is not catenary.*

PROOF. For item 1, since B is a directed union of three-dimensional regular local domains, $\dim B \leq 3$. By Proposition 5.16.5, B is local with maximal ideal $(x, y)B$, xB and p_iB are prime ideals, and, by Construction Properties Theorem 5.14.3, the (x)-adic completion of B is equal to R^*, the (x)-adic completion of R. Thus the $\mathbf{n}$-adic completion of B is $k[[x, y]]$. Since each $Q_i = \bigcup_{i=1}^{\infty} Q_{in}$, where $Q_{in} = p_iR^* \cap B_n$, we see that each Q_i is a prime ideal of B with $p_i, f \in Q_i$ and $x \notin Q_i$. Since $p_iB = \bigcup p_iB_n$, we have $f \notin p_iB$. Thus

$$(0) \subsetneq p_iB \subsetneq Q_i \subsetneq (x, y)B.$$

This chain of prime ideals of length at least three yields that $\dim B = 3$ and that the height of each Q_i is 2.

The prime ideal $p_iR^*[1/x]$ has height one, whereas $p_iR^*[1/x] \cap S = (p_i, f)S$ has height two. Since flat extensions satisfy the going-down property, by Remark 2.31.10, the map $S = R[f] \to R^*[1/x]$ is not flat. Therefore Noetherian Flatness Theorem 6.3 implies that the ring B is not Noetherian. By Theorem 5.17, B is a UFD, and so item 1 holds.

For item 2, by Construction Properties Theorem 5.14.2, $B/xB = R/xR$, and so B/xB is a DVR. By Theorem 5.17, $B[1/x]$ is a regular Noetherian UFD. If $x \in P$ and P is nonmaximal, then, again by Theorem 5.14.2, $P = xB$ and so B_P is a DVR and a regular Noetherian UFD. If $x \notin P$, the ring B_P is a localization of $B[1/x]$ and so is a regular Noetherian UFD. Thus item 2 holds.

The statements in item 3 that A is a two-dimensional regular local domain with maximal ideal $\mathbf{m}_A = (x, y)A$ and $A = B_\tau$ follow from Localized Prototype Theorem 17.28. If the field k has characteristic zero, then A is also excellent by Theorem 9.2 (if the non-localized ring is excellent, so is the localization).

If the field k is perfect with characteristic $p > 0$, then the ring A is not excellent by Remark‘9.5. This completes the proof of item 3.

By Theorem 5.14.2, $A/xA = R/xR$, and so $\mathbf{m}_A = (x, y)A$ is the unique prime ideal of A lying over $\mathbf{n} = (x, y)B$. Thus item 4 holds and for item 5 we see that x is not in any height-two prime ideal of B.

To complete the proof of item 5, it remains to consider $P \in \operatorname{Spec} B$ with $x \notin P$ and $\operatorname{ht} P > 1$. By Proposition 5.16.3, we have $x^n \notin PR^*$ for each $n \in \mathbb{N}$. Thus $\operatorname{ht}(PR^*) \leq 1$. Since $A \hookrightarrow R^*$ is faithfully flat, $\operatorname{ht}(PA) \leq 1$. Let P' be a height-one prime ideal of A containing PA. Since $\dim B = 3$, $\operatorname{ht} P > 1$ and $x \notin P' \cap B$, it follows that $P = P' \cap B$. If $p \notin P$, then Proposition 15.9 implies that $A_{P'} = B_P$. Since P' is a height-one prime ideal of A, it follows that P is a height-one prime ideal of B in case $x \notin P$ and $p \notin P$.

Now suppose that $p_i \in P$ for some i. Then p_iR^* is a height-one prime ideal contained in PR^* and so $p_iR^* = PR^*$. Hence P is squeezed between p_iB and

$Q_i = p_iR^* \cap B \neq (x, y)B$. Since $\dim B = 3$, either P has height one or $P = Q_i$ for some i. This completes the proof of item 5.

For item 6, we show that each Q_i is not finitely generated by showing that $f_{n+1} \notin (p_i, f_n)B$ for each $n \geq 0$. We have $f = p\tau$ and thus $f_n = p\tau_n$. It follows that $f_n = xf_{n+1} + pxc_{n+1}$, by Equation 5.4.2. Assume that $f_{n+1} \in (p_i, f_n)B$. Then

$$(p_i, f_n)B = (p_i, xf_{n+1} + pxc_{n+1})B \implies f_{n+1} = ap_i + b(xf_{n+1} + pxc_{n+1}),$$

for some $a, b \in B$. Thus $f_{n+1}(1 - xb) \in p_iB$. Since $1 - xb$ is a unit of B, it follows that $f_{n+1} \in p_iB$, and thus $f_{n+1} \in p_iB_{n+r}$, for some $r \geq 1$. By Equation 5.4.2, we have

$$f_{n+1} = x^{r-1}f_{n+r} + p\alpha,$$

where $\alpha \in R$. Thus $x^{r-1}f_{n+r} \in (p, f_{n+1})B_{n+r}$. Since $f_{n+1} \in p_iB_{n+r}$, we have $x^{r-1}f_{n+r} \in p_iB_{n+r}$. This implies $f_{n+r} \in p_iB_{n+r}$, a contradiction because the ideal $(p_i, f_{n+r})B_{n+r}$ has height two, since f_{n+r} is a variable of the localized polynomial ring B_{n+r}. We conclude that Q_i is not finitely generated.

Since B is a UFD, the height-one primes of B are principal and since the maximal ideal of B is two-generated, every nonfinitely generated prime ideal of B has height two and thus is in the set $\{Q_1, \ldots, Q_m\}$. This completes the proof of item 6.

For item 7, the chain $(0) \subset xB \subset (x, y)B = \mathbf{m}_B$ is saturated and has length two, while the chain $(0) \subset p_1B \subset Q_1 \subset \mathbf{m}_B$ is saturated and has length three. □

Remark 15.11. With Notation 15.2 and the notation of Theorem 15.10, we obtain the following additional details about the prime ideals of B.

(1) If $P \in \operatorname{Spec} B$, $P \neq (0)$ and $P \neq \mathbf{m}_B$, then $\operatorname{ht}(PR^*) = 1$ and $\operatorname{ht}(PA) = 1$. Thus every nonmaximal prime ideal of B is contained in a nonmaximal prime ideal of A.
(2) If $P \in \operatorname{Spec} B$ is such that $P \cap R = (0)$, then $\operatorname{ht}(P) \leq 1$ and P is principal.
(3) If $P \in \operatorname{Spec} B$, $\operatorname{ht} P = 1$ and $P \cap R \neq 0$, then $P = (P \cap R)B$.
(4) Let p_i be one of the prime factors of p. Then p_iB is prime in B. Moreover the ideals p_iB and $Q_i := p_iA \cap B = (p_i, f_1, f_2, \ldots)B$ are the only nonmaximal prime ideals of B that contain p_i. Thus they are the only prime ideals of B that lie over p_iR in R.
(5) The constructed ring B has Noetherian spectrum.

Proof. For the proof of item 1, if $P = Q_i$ for some i, then $PR^* \subseteq p_iR^*$ and $\operatorname{ht} PR^* = 1$. If P is not one of the Q_i, then P is a principal height-one prime and $\operatorname{ht} PR^* = 1$ by Theorem 15.10 parts 5 and 1. Since A is Noetherian and local, R^* is faithfully flat over A and hence $\operatorname{ht} PA = 1$. The proof that $\operatorname{ht}(PR^*) \leq 1$ is contained in the proof of item 5 of Theorem 15.10.

For item 2, $\operatorname{ht} P \leq 1$ because the field of fractions $K(f)$ of B has transcendence degree one over the field of fractions K of R; see Cohen's Theorem 2.20. Since B is a UFD, P is principal.

For item 3, if $x \in P$, then $P = xB$ and the statement is clear. Assume $x \notin P$. By Remark 15.3, $B[1/x]$ is a localization of B_n, and so $\operatorname{ht}(P \cap B_n) = 1$ for all integers $n \geq 0$. Thus $(P \cap R)B_n = P \cap B_n$, for each n, and so $P = (P \cap R)B$.

For item 4, p_iB is prime by Proposition 5.16.2. By Theorem 15.10, $\dim B = 3$ and the Q_i are the only height-two primes of B. Since the ideal $p_iR + p_jR$ is $\mathbf{m}_R$-primary for $i \neq j$, it follows that $p_iB + p_jB$ is $\mathbf{n}$-primary, and hence p_iB and Q_i are the only nonmaximal prime ideals of B that contain p_i.

Item 5 follows from Theorem 15.10, since the prime spectrum is Noetherian if it satisfies the ascending chain condition and if, for each finite set in the spectrum, there are only finitely many prime ideals minimal with respect to containing the given finite set of prime ideals. Thus the proof is complete. □

REMARK 15.12. Rotthaus and Sega prove that the approximation domains B in the setting of Theorems 15.10 and 16.5 are coherent and regular; they show that every finitely generated submodule of a free module over B has a finite free resolution [**137**]. For the ring $B = \bigcup_{n=1}^{\infty} B_n$ of these constructions, it is stated in [**137**] that $B_n[1/x] = B_{n+k}[1/x] = B[1/x]$ and that B_{n+k} is generated over B_n by a single element for all positive integers n and k. This is not correct for the local rings B_n. However, if instead of asserting these statements for the localized polynomial rings B_n and their union B of the construction, one makes the statements for the underlying polynomial rings U_n and their union U defined in Equation 5.4.5, or those defined in Examples 15.1, then one does have that $U_n[1/x] = U_{n+k}[1/x] = U[1/x]$ and that U_{n+k} is generated over U_n by a single element for all positive integers n and k; see Remark 15.3.

We use the following lemma.

LEMMA 15.13. *Assume Notation 15.2 and the notation of Theorem 15.10.*

(1) *For every element $c \in \mathbf{m}_R \setminus xR$ and every $t \in \mathbb{N}$, the element $c + x^t f$ is a prime element of the UFD B.*
(2) *For every fixed element $c \in \mathbf{m}_R \setminus xR$, the set $\{c+x^t f\}_{t\in\mathbb{N}}$ consists of infinitely many nonassociate prime elements of B, and so there exist infinitely many distinct height-one primes of B of the form $(c + x^t f)B$.*

PROOF. For the first item, since $f = p\tau$, Equation 15.4.1 implies that

$$f_r = pc_{r+1}x + xf_{r+1}$$

for each $r \geq 0$. In $B_0 = k[x, y, f]_{(x,y,f)}$, the polynomial $c + x^t f$ is linear in the variable $f = f_0$ and the coefficient x^t of f is relatively prime to the constant term c. Thus $c+x^t f$ is irreducible in B_0. Since $f = f_0 = pc_1x+xf_1$ in $B_1 = k[x, y, f_1]_{(x,y,f_1)}$, the polynomial $c + x^t f = c + x^t pc_1x + x^{t+1}f_1$ is linear in the variable f_1 and the coefficient x^{t+1} of f_1 is relatively prime to the constant term c. Thus $c + x^t f$ is irreducible in B_1. To see that this pattern continues, observe that in B_2, we have

$$\begin{aligned} f &= pc_1x + xf_1 = pc_1x + pc_2x^2 + x^2f_2 \implies \\ c + x^t f &= c + pc_1x^{t+1} + pc_2x^{t+2} + x^{t+2}f_2, \end{aligned}$$

a linear polynomial in the variable f_2. Thus $c + x^t f$ is irreducible in B_2 and a similar argument shows that $c+x^t f$ is irreducible in B_r for each positive integer r. Therefore for each $t \in \mathbb{N}$, the element $c + x^t f$ is prime in B.

For item 2, we prove that $(c+x^t f)B \neq (c+x^m f)B$, for positive integers $t > m$. Assume that $\mathbf{q} := (c+x^t f)B = (c+x^m f)B$ is a height-one prime ideal of B. Then

$$(x^t - x^m)f = x^m(x^{t-m} - 1)f \in \mathbf{q}.$$

Since $c \notin xB$ we have $\mathbf{q} \neq xB$. Thus $x^m \notin \mathbf{q}$. Since B is local, $x^{t-m} - 1$ is a unit of B. It follows that $f \in \mathbf{q}$ and thus $(c, f)B \subseteq \mathbf{q}$. By Remark 15.3, $B[1/x]$ is a localization of $R[f] = S$, and $x \notin \mathbf{q}$ implies that $B_{\mathbf{q}} = S_{\mathbf{q}\cap S}$. This is a contradiction since the ideal $(c, f)S$ has height two.

We conclude that there exist infinitely many distinct height-one primes of the form $(c + x^t f)B$. □

Lemma 15.14 is useful for giving a more precise description of $\operatorname{Spec} B$ for B as in Examples 15.1. For each nonempty finite subset H of $\{Q_1, \ldots, Q_m\}$, we show there exist infinitely many height-one prime ideals contained in each $Q_i \in H$, but not contained in Q_j if $Q_j \notin H$. Recall that "lost" is defined in Definition 15.5.

LEMMA 15.14. *With Notation 15.2 and the notation of Theorem 15.10, let G be a nonempty subset of $\{1, \ldots, m\}$, let $H = \{Q_i \mid i \in G\}$, and let $p_G = \prod\{p_i \mid i \in G\}$. Then we have, for each $t \in \mathbb{N}$:*

(1) *$(p_G + x^t f)B$ is a prime ideal of B that is lost in A; see Definition 15.5.*
(2) *$(p_G^2 + x^t f)B$ is a prime ideal of B that is not lost in A.*

The sets $\{(p_G + x^t f)B\}_{t\in\mathbb{N}}$ and $\{(p_G^2 + x^t f)B\}_{t\in\mathbb{N}}$ are both infinite. Moreover, the prime ideals in both item 1 and item 2 are contained in each Q_i such that $Q_i \in H$, but are not contained in Q_j if $Q_j \notin H$.

PROOF. For item 1, we have

$$(p_G + x^t f)A \cap B = p_G(1 + x^t \tau \prod_{j\notin G} p_j)A \cap B = p_G A \cap B = \bigcap_{i\in G} Q_i. \tag{15.14.1}$$

Thus each prime ideal of B of the form $(p_G + x^t f)B$ is lost in A and R^*. By the second item of Lemma 15.13, there exist infinitely many height-one primes $(p_G + x^t f)B$ of B that are lost in A and R^*.

For item 2, we have

$$\begin{aligned}(p_G^2 + x^t f)A \cap B &= (p_G^2 + x^t p_G(\prod_{j\notin G} p_j)\tau)A \cap B \\ &= p_G(p_G + x^t(\prod_{j\notin G} p_j)\tau)A \cap B \subsetneq p_G A \cap B = \bigcap_{i\in G} Q_i.\end{aligned} \tag{15.14.2}$$

The strict inclusion is because $p_G + x^t(\prod_{j\notin G} p_j)\tau \in \mathbf{m}_A$. This implies that prime ideals of B of form $(p_G^2 + x^t f)B$ are not lost. By Lemma 15.13 there are infinitely many distinct prime ideals of that form.

The "moreover" statement for the prime ideals in item 1 follows from Equation 15.14.1. Equation 15.14.2 implies that the prime ideals in item 2 are contained in each $Q_i \in H$. For $j \notin G$, if $p_G^2 + x^t f \in Q_j$, then $p_j + x^t f \in Q_j$ implies that $p_G^2 - p_j \in Q_j$ by subtraction. Since $p_j \in Q_j$, this would imply that $p_G^2 \in Q_j$, a contradiction. This completes the proof of Lemma 15.14. □

REMARK 15.15. With Notation 15.2, consider the birational inclusion $B \hookrightarrow A$ and the faithfully flat map $A \hookrightarrow R^*$. The following statements hold concerning the inclusion maps $R \hookrightarrow B \hookrightarrow A \hookrightarrow R^*$, and the associated maps in the opposite direction of their spectra: (See Discussion 3.22 for information concerning the spectral maps.)

(1) The map $\operatorname{Spec} R^* \to \operatorname{Spec} A$ is surjective, since every prime ideal of A is contracted from a prime ideal of R^*, while the maps $\operatorname{Spec} R^* \to \operatorname{Spec} B$ and $\operatorname{Spec} A \to \operatorname{Spec} B$ are not surjective. All the induced maps to $\operatorname{Spec} R$ are surjective since the map $\operatorname{Spec} R^* \to \operatorname{Spec} R$ is surjecive.

(2) By Lemma 15.14, each of the prime ideals Q_i of B contains infinitely many height-one primes of B that are the contraction of prime ideals of A and infinitely many that are not.

An ideal contained in a finite union of prime ideals is contained in one of the prime ideals; see [**11**, Prop. 1.11, page 8] or [**103**, Ex. 1.6, page 6]. Thus there are infinitely many non-associate prime elements of the UFD B that are not contained in the union $\bigcup_{i=1}^{m} Q_i$. We observe that for each prime element q of B with $q \notin \bigcup_{i=1}^{m} Q_i$ the ideal qA is contained in a height-one prime $\mathbf{q}$ of A and $\mathbf{q} \cap B$ is properly contained in $\mathbf{m}_B$ since $\mathbf{m}_A$ is the unique prime ideal of A lying over $\mathbf{m}_B$. Hence $\mathbf{q} \cap B = qB$. Thus each qB is contracted from A and R^*.

In the four-dimensional example B of Theorem 16.5, each height-one prime of B is contracted from R^*, but there are infinitely many height-two primes of B that are lost in R^*, in the sense of Definition 15.5; see Section 16.2.

(3) Among the prime ideals of the domain B of Examples 15.1 that are not contracted from A are the p_iB. Since $p_iA \cap B = Q_i$ properly contains p_iB, the prime ideal p_iB is lost in A.

(4) Since x and y generate the maximal ideals of B and A, and since B is integrally closed, a version of Zariski's Main Theorem [**125**], [**40**], implies that A is not essentially finitely generated as a B-algebra. ("Essentially finitely generated" is defined in Section 2.1.)

Using the information above, we display below a picture of $\operatorname{Spec}(B)$ in the case $m = 2$.

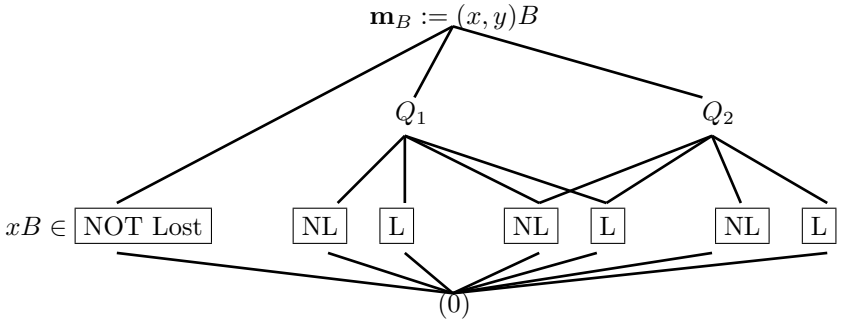


Diagram 15.15.0

Comments on Diagram 15.15.0. Here we have $Q_1 = p_1R^* \cap B$ and $Q_2 = p_2R^* \cap B$, and each box represents an infinite set of height-one prime ideals. We label a box "NL" for "not lost" and "L" for "lost". An argument similar to that given for the Type I primes in Example 15.4 shows that the height-one primes q such that $q \notin Q_1 \cup Q_2$ are not lost. That the other boxes are infinite follows from Lemma 15.14.

Exercises

(1) Let $R = k[x,y]_{(x,y)}$ be the localized polynomial ring in the variables x, y over a field k. Consider the local quadratic transformation $S := R[\frac{y}{x}]_{x,\frac{y}{x})R[\frac{y}{x}]}$ of the 2-dimensional RLR R. Using the terminology of Definition 15.5
   (a) Prove that there are infinitely many height-one primes of R that are lost in S.
   (b) Prove that there are infinitely many height-one primes of R that are not lost in S.
   (c) Describe precisely the height-one primes of R that are lost in S, and the prime ideals of R that are not lost in S.

(2) Prove the assertion in Remark 15.15 that each of the prime ideals Q_i of B contains infinitely many height-one primes of B that are the contraction of prime ideals of A and infinitely many that are not, i.e., there exist infinitely many height-one primes of B contained in Q_i that are lost in A and infinitely many that are not lost in A.

   Suggestion: A solution for this exercise can be patterned along the lines of the arguments given in Example 15.4. Since $A[1/x]$ is a localization of the polynomial ring $R[\tau]$, for every nonzero element $c \in (x,y)R$, the ideal $(\tau - c)A$ is a height-one prime in A, and $a\tau - ac$ is a nonzero element in each of the prime ideals Q_i of B. Since p_iA is the only prime ideal of A lying over Q_i in B, the ideal $(\tau - c)A \cap B$ is a height-one prime of B. Also consider elements of the form $p_i + x^n f \in B$.

(3) In connection with Remarks 15.7.2, let $(R, \mathbf{m})$ be a local domain with principal maximal ideal $\mathbf{m} = aR$.
   (a) Prove that $\bigcap_{n=1}^{\infty} \mathbf{m}^n = P$, where P is a prime ideal properly contained in $\mathbf{m}$.
   (b) Prove that every prime ideal of R properly contained in $\mathbf{m}$ is contained in P.
   (c) Prove that R/P is a DVR.
   (d) Prove that $P = PR_P$.
   (e) Prove that R is a valuation domain if and only if R_P is a valuation domain [**79**, Prop. 3.5(iv)].
   (f) Construct an example of a local domain $(R, \mathbf{m})$ with principal maximal ideal $\mathbf{m}$ such that R is not a valuation domain.

   Suggestion: To construct an example for part f, let x, y be indeterminates over a field k, let $U = k(x)[y]$, let W be the DVR U_{yU}, and let $P := yW$ denote the maximal ideal of W. Then $W = k(x) + P$. Let $R = k[x^2]_{(x^2k[x^2])} + P$. This is an example of a "$D + M$" construction, as outlined in Remark 16.13.

CHAPTER 16

Non-Noetherian insider examples with $\dim \geq 4$

In this chapter we extend the methods of Chapter 15 to construct a four-dimensional local domain that is not Noetherian, but is very close to being Noetherian. We use Insider Construction 10.1 of Section 10.1. This four-dimensional non-catenary non-Noetherian local unique factorization domain has exactly one prime ideal Q of height three; the ideal Q is not finitely generated.

Section 16.1 contains a description of the example. In Section 16.2 we verify that the example has the stated properties.

16.1. A 4-dimensional prime spectrum

In Example 16.1, we present a four-dimensional example analogous to Example 15.4.

EXAMPLE 16.1. Let k be a field, let x, y and z be indeterminates over k. Set

$$R := k[x,y,z]_{(x,y,z)} \qquad \text{and} \qquad R^* := k[y,z]_{(y,z)}[[x]],$$

and let $\mathbf{m}_R$ and $\mathbf{m}_{R^*}$ denote the maximal ideals of R and R^*, respectively. The power series ring R^* is the xR-adic completion of R. Consider τ and σ in $xk[[x]]$

$$\tau := \sum_{n=1}^{\infty} c_n x^n \quad \text{and} \quad \sigma := \sum_{n=1}^{\infty} d_n x^n,$$

where the c_n and d_n are in k and τ and σ are algebraically independent over $k(x)$. Define

$$f := y\tau + z\sigma \quad \text{and} \quad A := A_f = R^* \cap k(x,y,z,f),$$

that is, A is the intersection domain associated with f. For each integer $n \geq 0$, let τ_n and σ_n be the n^{th} endpieces of τ and σ as in Equation 15.1.a. Then the n^{th} endpiece of f is $f_n = y\tau_n + z\sigma_n$. As in Equation 15.1.b, we have

$$\tau_n = x\tau_{n+1} + xc_{n+1} \quad \text{and} \quad \sigma_n = x\sigma_{n+1} + xd_{n+1},$$

where c_{n+1} and d_{n+1} are in the field k. Therefore

$$\begin{aligned} f_n = y\tau_n + z\sigma_n &= yx\tau_{n+1} + yxc_{n+1} + zx\sigma_{n+1} + zxd_{n+1} \\ &= xf_{n+1} + yxc_{n+1} + zxd_{n+1}. \end{aligned} \tag{16.1.1}$$

The approximation domains U_n, B_n, U and B for A are as follows:

$$\begin{aligned} &\text{For } n \geq 0, \quad U_n := k[x,y,z,f_n] \qquad B_n := k[x,y,z,f_n]_{(x,y,z,f_n)} \\ &U := \bigcup_{n=0}^{\infty} U_n \quad \text{and} \qquad B := B_f = \bigcup_{n=0}^{\infty} B_n. \end{aligned} \tag{16.1.2}$$

Thus B is the directed union of 4-dimensional localized polynomial rings. It follows that $\dim B \leq 4$.

The rings A and B are constructed inside the intersection domain $A_{\tau,\sigma} := R^* \cap k(x,y,z,\tau,\sigma)$. By Localized Prototype Theorem 17.28, the domain $A_{\tau,\sigma}$ is Noetherian and equals the approximation domain $B_{\tau,\sigma}$ associated to τ,σ and is a three-dimensional RLR that is a directed union of 5-dimensional RLRs and the extension $T := R[\tau,\sigma] \hookrightarrow R^*[1/x]$ is flat.

Before we list and establish the other properties of Example 16.1 in Theorem 16.5, we prove the following proposition concerning the Jacobian ideal and flatness in Example 16.1. The Jacobian ideal is defined and discussed in Definition and Remarks 7.12.1.

PROPOSITION 16.2. *With the notation of Example 16.1, we have*

(1) *For the extension $\varphi : S = R[f] \hookrightarrow T = R[\tau,\sigma]$, the Jacobian ideal J is the ideal $(y,z)T$. Thus the nonflat locus F of φ contains J.*
(2) *For every $P \in \operatorname{Spec}(R^*[1/x])$, the ideal $(y,z)R^*[1/x] \not\subseteq P \iff$ the map $B_{P\cap B} \hookrightarrow (R^*[1/x])_P$ is flat. Thus the ideal $F_1 := (y,z)R^*[1/x]$ defines the nonflat locus of the map $B \hookrightarrow R^*[1/x]$.*
(3) *For every height-one prime ideal $\mathbf{p}$ of R^*, we have $\operatorname{ht}(\mathbf{p} \cap B) \leq 1$.*
(4) *For every prime element w of B, $wR^* \cap B = wB$.*

PROOF. For item 1, the Jacobian ideal is the ideal of T generated by the 1×1 minors of the matrix $\begin{pmatrix} y & z \end{pmatrix}$ by (7.12.1), and so $J = (y,z)T$. By Theorem 7.14.2, $(y,z)T \subseteq F$.

For item 2, the two statements are equivalent by the definition of nonflat locus in Definition and Remarks 7.12.2. To compute the nonflat locus of $B \hookrightarrow R^*[1/x]$, we use that $T := R[\tau,\sigma] \hookrightarrow R^*[1/x]$ is flat as noted in Example 16.1. Let $P \in \operatorname{Spec}(R^*[1/x])$ and let $Q := P \cap T$. The map $B \hookrightarrow R^*[1/x]_P$ is flat $\iff$ the composition

$$k[x,y,z,f] \hookrightarrow k[x,y,z,\tau,\sigma] \hookrightarrow R^*[1/x]_P \text{ is flat } \iff$$

$$S := k[x,y,z,f] \overset{\varphi}{\hookrightarrow} T_Q = k[x,y,z,\tau,\sigma]_Q \text{ is flat.}$$

By item 1, the Jacobian ideal of φ is the ideal $J = (y,z)T$. Since $(y,z)T \cap S = (y,z,f)S$ has height 3, φ_Q is not flat for every $Q \in \operatorname{Spec}(T)$ such that $(y,z)T \subseteq Q$. Thus the nonflat locus of $B \hookrightarrow R^*[1/x]$ is defined by $F_1 = (y,z)R^*[1/x]$ as stated in item 2.

Item 3 is clear if $\mathbf{p} = xR^*$. Let $\mathbf{p}$ be a height-one prime of R^* other than xR^*. Since $\mathbf{p}$ does not contain $(y,z)R^*$, the map $B_{\mathbf{p}\cap B} \hookrightarrow (R^*)_{\mathbf{p}}$ is faithfully flat. Thus $\operatorname{ht}(\mathbf{p} \cap B) \leq 1$. This establishes item 3.

Item 4 is clear if $wB = xB$. Assume that $wB \neq xB$ and let $\mathbf{p}$ be a height-one prime ideal of R^* that contains wR^*. Then $\mathbf{p}R^*[1/x] \cap R^* = \mathbf{p}$, and by item 3, $\mathbf{p} \cap B$ has height at most one. We have $\mathbf{p} \cap B \supseteq wR^* \cap B \supseteq wB$. Thus item 4 follows. □

Next we prove a proposition about homomorphic images of the constructed ring B. This result enables us in Corollary 16.4 to relate the ring B of Example 16.1 to the ring B of Example 15.4.

PROPOSITION 16.3. *Assume the notation of Example 16.1, and let w be a prime element of $R = k[x,y,z]_{(x,y,z)}$ with $wR \neq xR$. Let $\pi : R^* \to R^*/wR^*$ be the natural homomorphism, and let $\bar{}$ denote image in R^*/wR^*. Let B' be the approximation*

domain formed by considering $\overline{R}$ and the endpieces $\overline{f}_n$ of $\overline{f}$, defined analogously to Equation 15.1.a. That is, B' is defined by setting

$$U'_n \;=\; \overline{k[x,y]}[\overline{f}_n],\; B'_n \;=\; (U'_n)_{\mathbf{n}'_n},\; U' \;=\; \bigcup_{n=1}^{\infty} U'_n,\; \textit{and}\;\; B' \;=\; \bigcup_{n=1}^{\infty} B'_n,$$

where $\mathbf{n}'_n$ is the maximal ideal of U'_n that contains $\overline{f}_n$ and the image of $\mathbf{m}_R$. Then $B' = \overline{B}$.

Proof. By Proposition 5.16.2, wB is a prime ideal of B. By Proposition 16.2.4, $wR^* \cap B = wB$. Hence $\overline{B} = B/(wR^* \cap B) = B/wB$. We have

$$\overline{R}/x\overline{R} \;=\; \overline{B}/x\overline{B} \;=\; \overline{R^*}/x\overline{R^*},$$

and the ring $\overline{R^*}$ is the $(\overline{x})$-adic completion of $\overline{R}$. Since the ideal $(y,z)R$ has height 2 and the kernel of π has height 1, at least one of $\overline{y}$ and $\overline{z}$ is nonzero. Since τ and σ are algebraically independent over $k(x,y,z)$, the element $\overline{f} = \overline{y}\cdot\overline{\tau} + \overline{z}\cdot\overline{\sigma}$ of the integral domain $\overline{B}$ is transcendental over $\overline{R}$. Similarly the endpieces $\overline{f}_n$ are transcendental over $\overline{R}$. The fact that $\overline{R^*}$ may fail to be an integral domain does not affect the algebraic independence of these elements that are inside the integral domain $\overline{B}$.

By Construction Properties Theorem 5.14.4, with adjustments using Remark 5.15, parts 2 and 3, we have $U_0[1/x] = U_n[1/x] = U[1/x]$, and thus $wU \cap U_n = wU_n$ for each $n \in \mathbb{N}$. Since B_n is a localization of U_n, we also have $wB \cap B_n = wB_n$. Since $wR^* \cap B = wB$, it follows that $wR^* \cap B_n = wB_n$. Thus we have

$$\overline{R} \;\subseteq\; \overline{B_n} \;=\; B_n/wB_n \;\subseteq\; \overline{B} \;=\; B/wB \;\subseteq\; \overline{R^*} \;=\; R^*/wR^*.$$

We conclude that $\overline{B} = \bigcup_{n=0}^{\infty} \overline{B_n}$. Since $B'_n = \overline{B}_n$, we have $B' = \overline{B}$. □

Corollary 16.4. *The homomorphic image B/zB of the ring B of Example 16.1 is isomorphic to the three-dimensional ring B of Example 15.4.*

Proof. Assume the notation of Example 16.1 and Proposition 16.3 and let $w = z$. We show that the ring $B/zB \cong C$, where C is the ring called B in Example 15.4. By Proposition16.3, we have $B' = B/zB$, where B' is the approximation domain over $\overline{R} = R/zR$ using the element $\overline{f}$, transcendental over $\overline{R}$. Let R_C denote the base ring $k[x,y]_{(x,y)}$ for C in Example 15.4, and let $\psi_0 : \overline{R} \to R_C$ denote the k-isomorphism defined by $\overline{x} \mapsto x$ and $\overline{y} \mapsto y$. Then, as in the proof of Proposition 16.3, $\overline{R^*}$ is the $(\overline{x})$-adic completion of $\overline{R}$. Thus ψ_0 extends to an isomorphism $\psi : \overline{R^*} \to (R_C)^*$ that agrees with ψ_0 on $\overline{R}$ and such that $\psi(\overline{\tau}) = \tau$. Furthermore $\psi(\overline{f}) = \psi(\overline{y}\cdot\overline{\tau} + \overline{z}\cdot\overline{\sigma}) = y\tau$, which is the transcendental element f used in the construction of C. Thus ψ is an isomorphism from $\overline{B} = B/zB$ to C, the ring constructed in Example 15.4. □

16.2. Verification of the example

We record in Theorem 16.5 properties of the ring B and its prime spectrum.

Theorem 16.5. *As in Example 16.1, $R := k[x,y,z]_{(x,y,z)}$ with k a field, x, y and z indeterminates, and $R^* := k[y,z]_{(y,z)}[[x]]$, the xR-adic completion of R. Let τ and $\sigma \in xk[[x]]$ be algebraically independent over $k(x)$. Set $f := y\tau + z\sigma$, $A := R^* \cap k(x,y,z,f)$, and $B := \bigcup_{n=0}^{\infty} B_n = \bigcup_{n=0}^{\infty} k[x,y,z,f_n]_{(x,y,z,f_n)}$ as in (16.1.2). Let $Q := (y,z)R^* \cap B$. Then*

(1) *The rings A and B are equal.*
(2) *The ring B is a four-dimensional non-Noetherian local UFD with maximal ideal $\mathbf{m}_B = (x, y, z)B$, and the $\mathbf{m}_B$-adic completion of B is the three-dimensional RLR $k[[x, y, z]]$.*
(3) *The ring $B[1/x]$ is a Noetherian regular UFD, the ring B/xB is a two-dimensional RLR, and, for every nonmaximal prime ideal P of B, the ring B_P is an RLR.*
(4) *The ideal Q is the unique prime ideal of B of height 3.*
(5) *The ideal Q equals $\bigcup_{n=0}^{\infty} Q_n$ where $Q_n := (y, z, f_n)B_n$, Q is a nonfinitely generated prime ideal, and $QB_Q = (y, z, f)B_Q$.*
(6) *There exist infinitely many height-two prime ideals of B not contained in Q and each of these prime ideals is contracted from R^*.*
(7) *For certain height-one primes p contained in Q, there exist infinitely many height-two primes between p and Q that are contracted from R^*, and infinitely many that are not contracted from R^*. Hence the map $\operatorname{Spec} R^* \to \operatorname{Spec} B$ is not surjective.*
(8) *Every saturated chain of prime ideals of B has length either 3 or 4, and there exist saturated chains of prime ideals of lengths both 3 and 4. Thus B is not catenary.*
(9) *Each height-one prime ideal of B is the contraction of a height-one prime ideal of R^*.*
(10) *B has Noetherian spectrum.*

We prove Theorem 16.5 below. First, assuming Theorem 16.5, we display a picture of $\operatorname{Spec}(B)$ and make comments about the diagram.

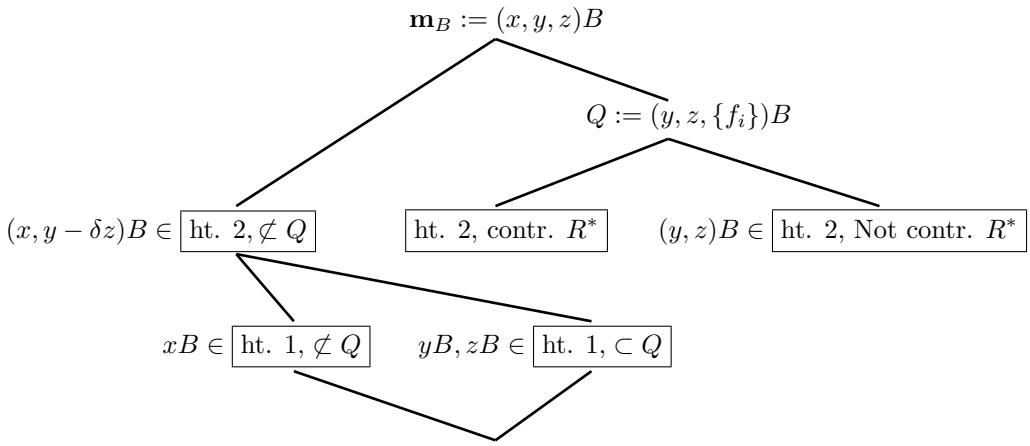


Diagram 16.5.0

Comments on Diagram 16.5.0. A line going from a box at one level to a box at a higher level indicates that every prime ideal in the lower level box is contained in at least one prime ideal in the higher level box. Thus as indicated in the diagram, every height-one prime gB of B is contained in a height-two prime of B that contains x and so is not contained in Q. This is obvious if $gB = xB$ and can be seen by considering minimal primes of $(g, x)B$ otherwise. Thus B has no maximal saturated chain of length 2. We have not drawn any lines from the

lower level righthand box to higher boxes that are contained in Q because we are uncertain about what inclusion relations exist for these primes. We discuss this situation in Remarks 16.12.

PROOF. (of Theorem 16.5.) By Proposition 16.2.1, $(y,z)T \subseteq F$ where F is the nonflat locus F of the extension $S \hookrightarrow T$. Hence $\operatorname{ht}(FR^*[1/x]) > 1$. Since $R[\tau]$ is a UFD, Proposition 10.5 implies equality of the approximation and intersection domains B and A corresponding to the element f of R^*. This completes item 1.

For item 2, since B is a directed union of four-dimensional RLRs, we have $\dim B \leq 4$. By Corollary 16.4 and Theorem 15.10, $\dim(B/zB) = 3$. Thus $\dim B \geq 4$, and so $\dim B = 4$. By Proposition 5.16.5, the ring B is local with maximal ideal $\mathbf{m}_B = (x,y,z)B$. By Krull's Altitude Theorem 2.17, B is not Noetherian. The ring B is a UFD by Theorem 5.17. Since the (x)-adic completion of B is R^*, the $\mathbf{m}_B$-adic completion of B is $k[[x,y,z]]$.

For item 3, by Theorem 5.17, the ring $B[1/x]$ is a Noetherian regular UFD. By Construction Properties Theorem 5.14.2, we have $R/xR = B/xB$. Thus B/xB is a two-dimensional RLR.

For the last part of item 3, if $x \notin P$, then B_P is a localization of $B[1/x]$, which is Noetherian and regular, and so B_P is a regular local ring. In particular, this proves that B_Q is a regular local ring. If $x \in P$ and $\operatorname{ht} P = 1$, then $P = (x)$ and B_{xB} is a DVR. If $x \in P$ and $\operatorname{ht}(P) = 2$, the ideal P is finitely generated since B/xB is an RLR. Since B is a UFD from item 2, it follows that B_P is a local UFD of dimension 2 with finitely generated maximal ideal. Thus B_P is Noetherian by Cohen's Theorem 2.19. This, combined with B/xB a regular local ring, implies that B_P is a regular local ring. Since $\operatorname{ht} P \leq 2$ for every nonmaximal prime ideal P of R with $x \in P$, this completes the proof of item 3.

For item 4, since $(y,z)R^*$ is a prime ideal of R^*, the ideal $Q = (y,z)R^* \cap B$ is prime. By Proposition 5.16.2, the ideals yB and $(y,z)B$ are prime. Consider the chain of prime ideals

$$(0) \ \subset yB \ \subset \ (y,z)B \ \subset \ Q \ \subset \ \mathbf{m}_B.$$

The list y, z, f, x shows that each of the inclusions is strict; for example, we have $f \in Q \setminus (y,z)B$ since $f \notin (y,z)B_n$ for every $n \in \mathbb{N}$. By item 2 we have $\operatorname{ht} \mathbf{m}_B = 4$. Thus $\operatorname{ht} Q = 3$. This also implies that $(y,z)B$ is a height-two prime ideal of B.

For the uniqueness in item 4, let P be a nonmaximal prime ideal of B. We first consider the case that $x \notin P$. Then, by Proposition 5.16.3, $x^n \notin PR^*$ for each positive integer n. Hence $PR^*[1/x] \neq R^*[1/x]$. Let P_1 be a prime ideal of $R^*[1/x]$ such that $P \subseteq P_1$. If both y and z are in P_1, then $(y,z)R^*[1/x] \subseteq P_1$. Since $(y,z)R^*[1/x]$ is maximal, we have $(y,z)R^*[1/x] = P_1$. Therefore, $P \subseteq (y,z)R^*[1/x] \cap B = Q$, and so either $\operatorname{ht}(P) \leq 2$ or $P = Q$.

Next suppose that $x \notin P$ and y or z is not in P_1. Then the map $\psi : B \to R^*[1/x]_{P_1}$ is flat by Proposition 16.2.2. Since $\dim R^*[1/x] = 2$ we have $\operatorname{ht}(P_1) \leq 2$. Flatness of ψ implies $\operatorname{ht}(P_1 \cap B) \leq 2$, by Remark 2.3110. Hence $\operatorname{ht} P \leq 2$.

To complete the proof of item 4, we consider the case that $x \in P$. We have $\operatorname{ht} P \leq 3$, since $\dim B = 4$ and P is not maximal. If $\operatorname{ht} P \geq 3$, there exists a chain of primes of the form

$$(0) \subsetneq P_1 \subsetneq P_2 \subsetneq P \subsetneq (x,y,z)B. \tag{16.5.1}$$

By Construction Properties Theorem 5.14.2, $B/xB \cong R/xR$; thus $\dim(B/xB) = 2$. If $x \in P_2$, then $\operatorname{ht} P_2 \geq 2$ implies that $(0) \subsetneq xB \subsetneq P_2 \subsetneq P \subsetneq (x,y,z)B$, a contradiction to $\dim(B/xB) = 2$. Thus $x \notin P_2$. Since $x \in P$ and P is nonmaximal, we have that y or z is not in P. Hence y or z is not in P_2.

By Theorem 5.14.2, P corresponds to a nonmaximal prime ideal P' of R^* containing PR^*. Let P_2' be a prime ideal of R^* inside P' that is minimal over P_2R^*. If both y and z are in P_2', then, $(x,y,z)R^* \subseteq P'$, a contradiction to P' nonmaximal. By Proposition 5.16.4, P_2' does not contain x. Thus $P_2' \subsetneq P' \subsetneq (x,y,z)R^*$. Also $P_2' = P_2'' \cap R^*$, where P_2'' is a prime ideal of $R^*[1/x]$, and one of y and z is not an element of P_2''.

Since the Jacobian ideal of $\varphi : S \to T[1/x]$ is $(y,z)T[1/x]$, Proposition 16.2.2 implies the map $\psi : B \to R^*[1/x]_{P_2''}$ is flat. This implies $\operatorname{ht}(P_2'') \geq \operatorname{ht}(P_2'' \cap B) \geq \operatorname{ht} P_2 \geq 2$; that is, $\operatorname{ht}(P_2'') \geq 2$. Also P_2'' intersects R^* in P_2', and so $\operatorname{ht} P_2' \geq 2$. Thus in R^* we have a chain of primes $P_2' \subsetneq P' \subsetneq (x,y,z)R^*$, where $\operatorname{ht} P_2' \geq 2$, a contradiction, since R^*, a localization of $k[y,z][[x]]$, has dimension 3. This proves item 4.

For item 5, let $Q' = \bigcup_{n=0}^{\infty} Q_n$, where each $Q_n = (y,z,f_n)B_n$. Each Q_n is a prime ideal of height 3 in the 4-dimensional RLR B_n. Therefore Q' is a prime ideal of B of height ≤ 3 that is contained in Q. The ideal $(y,z)B$ is a prime ideal of height 2 strictly contained in Q by the proof of item 3. Hence $\operatorname{ht}(Q') = 3$ and we have $Q' = Q$.

To show the ideal Q is not finitely generated, we show for each positive integer n that $f_{n+1} \notin (y,z,f_n)B$. By Equation 16.1.1, $f_n = xf_{n+1} + yxc_{n+1} + zxd_{n+1}$. If $f_{n+1} \in (y,z,f_n)B$, then $f_{n+1} = ay + bz + c(xf_{n+1} + yxc_{n+1} + zxd_{n+1})$, where $a,b,c \in B$. This implies $f_{n+1}(1-cx)$ is in the ideal $(y,z)B$. By Proposition 5.16.1, $x \in \mathcal{J}(B)$, and so $1-cx$ is a unit of B. This implies $f_{n+1} \in (y,z)B \cap B_{n+1}$. By Proposition 5.16.2, we have $(y,z)B \cap B_{n+1} = (y,z)B_{n+1}$ Thus $f_{n+1} \in (y,z)B_{n+1}$. Since the ring $B_{n+1} = k[x,y,z,f_{n+1}]_{(x,y,z,f_{n+1})}$, where x,y,z and f_{n+1} are algebraically independent variables over k, this is a contradiction. We conclude that Q is not finitely generated.

We show above for item 3 that B_Q is a three-dimensional regular local ring. Since $Q = (y,z,f,f_1,f_2,\dots)B$ and, since x is a unit of B_Q, it follows from Remark 15.3 that $QB_Q = (y,z,f)B_Q$. This establishes item 5.

For item 6, since $x \notin Q$ and $B/xB \cong R/xR$ is a Noetherian ring of dimension two, there are infinitely many height-two primes of B containing xB; see Exercise 5 of Chapter 2. This proves there are infinitely many height-two primes of B not contained in Q. If P is a height-two prime of B not contained in Q, then $\operatorname{ht}(\mathbf{m}_B/P) = 1$, by item 4 above, and so, by Proposition 5.16.5, P is contracted from R^*. This completes the proof of item 6.

For item 7 we show that $p = zB$ has the stated properties. By Corollary 16.4, the ring B/zB is isomorphic to the ring called B in Example 15.4. For convenience we relabel the ring of Example 15.4 as B'. By Theorem 15.10, B' has exactly one non-finitely generated prime ideal, which we label Q', and $\operatorname{ht} Q' = 2$. It follows that $Q/zB = Q'$. By Discussion 15.6, there are infinitely many height-one primes contained in Q' of Type II (that is, primes that are contracted from R^*/zR^*) and infinitely many height-one primes contained in Q' of Type III (that is, primes that are not contracted from R^*/zR^*). The preimages in R^* of these primes are height-two primes of B that are contained in Q and contain zB. It follows that there are

infinitely many contracted from R^* and there are infinitely many not contracted from R^*, as desired for item 7.

For item 8, we have a saturated chain of prime ideals

$$(0) \subset xB \subset (x,y)B \subset (x,y,z)B = \mathbf{m}_B$$

of length 3 since $B/xB = R/xR$ by Theorem 5.14.2. We have a saturated chain of prime ideals

$$(0) \subset yB \subset (y,z)B \subset Q \subset \mathbf{m}_B$$

of length 4 from the proof of item 4. Hence B is not catenary. By item 2, $\dim B = 4$, and so there is no saturated chain of prime ideals of B of length greater than 4. By Comments 16.5.0, there is no saturated chain of prime ideals of B of length less than 3.

For item 9, since R^* is a Krull domain and $B = A = \mathcal{Q}(B) \cap R^*$, it follows that B is a Krull domain and each height-one prime of B is the contraction of a height-one prime of R^*. Item 10 follows since B/xB and $B[1/x]$ are Noetherian [**58**]. □

Remarks 16.6. Let the notation be as in Theorem 16.5.

(1) It follows from Theorem 16.5 that the localization $B[1/x]$ has a unique maximal ideal $QB[1/x] = (y,z,f)B[1/x]$ of height three and has infinitely many maximal ideals of height two. We observe that $B[1/x]$ has no maximal ideal of height one. To show this last statement it suffices to show for each irreducible element p of B with $pB \neq xB$ there exists $P \in \operatorname{Spec} B$ with $pB \subsetneq P$ and $x \notin P$. Assume there does not exist such a prime ideal P. Consider the ideal $(p,x)B$. This ideal has height two and has only finitely many minimal primes since B/xB is Noetherian. Let g be an element of $\mathbf{m}_B$ not contained in any of the minimal primes of $(p,x)B$. Every prime ideal of B that contains $(g,p)B$ also contains x and hence has height greater than two. Since $x \notin Q$, it follows that $(g,p)B$ is $\mathbf{m}_B$-primary, and hence that $(g,p)R^*$ is $\mathbf{m}_{R^*}$-primary. Since R^* is Noetherian and $\operatorname{ht} \mathbf{m}_{R^*} = 3$, this contradicts Krull's Altitude Theorem 2.17.

2) Let I be an ideal of B. Then IR^* is $\mathbf{m}_{R^*}$-primary $\iff$ I is $\mathbf{m}_B$-primary, by Proposition 5.16.5.

(3) Define

$$C_n := \frac{B_n}{(y,z)B_n} \quad \text{and} \quad C := \frac{B}{(y,z)B}.$$

We have $C = \bigcup_{n=0}^{\infty} C_n$ by item 1. We show that C is a rank 2 valuation domain with principal maximal ideal generated by the image of x. For each positive integer n, let $g_n \in C_n$ denote the image in C_n of the element $f_n \in B_n$ and let x denote the image of x. Then $C_n = k[x,g_n]_{(x,g_n)}$ is a 2-dimensional RLR. By (16.1.1), $f_n = xf_{n+1} + x(c_n y + d_n z)$. It follows that $g_n = xg_{n+1}$ for each $n \in \mathbb{N}$. Thus C is an infinite directed union of quadratric transformations of 2-dimensional regular local rings. Thus C is a valuation domain of dimension at most 2 by [**2**]. By items 2 and 4 of Theorem 16.5, $\dim C \geq 2$, and therefore C is a valuation domain of rank 2. The maximal ideal of C is xC.

By Corollary 16.4, $B/zB \cong D$, where D is the ring B of Example 15.4. By an argument similar to that of Proposition 16.3 and by Corollary 16.4, we see that the above ring C is isomorphic to D/yD.

QUESTION 16.7. For the ring B constructed as in Example 16.1, we ask: Is Q the only prime ideal of B that is not finitely generated?

Theorem 16.5 implies that the only possible nonfinitely generated prime ideals of B other than Q have height two. We do not know whether every height-two prime ideal of B is finitely generated. We show in Corollary 16.10 and Theorem 16.11 that certain of the height-two primes of B are finitely generated.

We recall Lemma 6.2, which was the key to the proof of Theorem 17.13. For convenience we repeat two parts of the lemma that are useful in this chapter:

LEMMA 16.8. *Let S be a subring of a ring T and let $b \in S$ be a regular element of both S and T. Assume that $bS = bT \cap S$ and $S/bS = T/bT$. Then*

1. *$T[1/b]$ is flat over $S \iff T$ is flat over S.*
2. *If T and $S[1/b]$ are both Noetherian and T is flat over S, then S is Noetherian*

The following theorem shows that the nonflat locus of the map $\varphi : B \to R^*[1/a]$ yields flatness for certain homomorphic images of B, if R, a, R^* and B are as in the general construction outlined in Inclusion Construction 5.3.

THEOREM 16.9. *Let R be a Noetherian integral domain with field of fractions $K := \mathcal{Q}(R)$, let $a \in R$ be a nonzero nonunit, and let R^* denote the (a)-adic completion of R. Let s be a positive integer and let $\underline{\tau} = \{\tau_1, \ldots, \tau_s\}$ be a set of elements of R^* that are algebraically independent over K, so that $R[\underline{\tau}]$ is a polynomial ring in s variables over R. Define $A = A_{inc} := K(\underline{\tau}) \cap R^*$, as in Inclusion Construction 5.3. Let U_n, B_n, B and U be defined as in Equations 5.4.4 and 5.4.5. Assume that F is an ideal of $R^*[1/a]$ that defines the nonflat locus of the map $\varphi : B \to R^*[1/a]$. Let I be an ideal in B such that $IR^* \cap B = I$ and a is regular on R^*/IR^*.*

1. *If $IR^*[1/a] + F = R^*[1/a]$, then $\varphi \otimes_B (B/I)$ is flat.*
2. *If $R^*[1/a]/IR^*[1/a]$ is flat over B/I, then R^*/IR^* is flat over B/I.*
3. *If $\varphi \otimes_B (B/I)$ is flat, then B/I is Noetherian.*

PROOF. For item 1, φ_P is flat for each $P \in \operatorname{Spec} R^*[1/a]$ with $I \subseteq P$ by hypothesis. Hence for each such P we have $\varphi_P \otimes_B (B/I)$ is flat. Since flatness is a local property, it follows that $\varphi \otimes_B (B/I)$ is flat.

For items 2 and 3, apply Lemma 16.8 with $S = B/I$ and $T = R^*/IR^*$; the element b of Lemma 16.8 is the image in B/IB of the element a from the setting of Theorem 6.3. Since $IR^* \cap B = I$, the ring B/I embeds into R^*/IR^*, and since $B/aB = R^*/aR^*$, the ideal $a(R^*/IR^*) \cap (B/I) = a(B/I)$. Thus item 2 and item 3 of Theorem 16.9 follow from item 1 and item 2, respectively, of Lemma 16.8. □

COROLLARY 16.10. *Assume the notation of Example 16.1. Let w be a prime element of B. Then B/wB is Noetherian if and only if $w \notin Q$. Thus every every nonfinitely generated ideal of B is contained in Q.*

PROOF. If $w \in Q$, then B/wB is not Noetherian since Q is not finitely generated. Assume $w \notin Q$. Since B/xB is known to be Noetherian, we may assume that $wB \neq xB$. By Proposition 16.2.1, $QR^*[1/x] = (y, z)R^*[1/x]$ defines the nonflat locus of $\varphi : B \to R^*[1/x]$. Since $wR^*[1/x] + (y, z)R^*[1/x] = R^*[1/x]$, Theorem 16.9 with $I = wB$ and $a = x$ implies that B/wB is Noetherian.

For the second statement, we use that every nonfinitely generated ideal is contained in an ideal maximal with respect to not being finitely generated and the latter ideal is prime. Thus it suffices to show every prime ideal P not contained in Q is finitely generated. If $P \not\subseteq Q$, then, since B is a UFD, there exists a prime element $w \in P \setminus Q$. By the first statement, B/wB is Noetherian, and so P is finitely generated. □

THEOREM 16.11. *Assume the notation of Example 16.1. Thus*

$$R \;:=\; k[x,y,z]_{(x,y,z)} \quad \text{and} \quad R^* \;:=\; k[y,z]_{(y,z)}[[x]],$$

where k is a field, and x, y and z are indeterminates over k. Also τ and σ are elements of $xk[[x]]$ that are algebraically independent over $k(x)$, and $f = y\tau + z\sigma$. Thus the approximation domain B formed by the procedure outlined above is a non-Noetherian four-dimensional UFD with exactly one height-three prime ideal Q and Q is not finitely generated. We have:

$$R = k[x,y,z]_{(x,y,z)} \subseteq R[f] \subseteq B \subseteq R^* \cap k(x,y,z,f) \subseteq R^* :=\; k[y,z]_{(y,z)}[[x]].$$

Let w be a prime element of R with $w \in (y,z)k[x,y,z]$. If w is linear in either y or z, then Q/wB is the unique nonfinitely generated prime ideal of B/wB. Thus Q is the unique nonfinitely generated prime ideal of B that contains w.

PROOF. Let $^-$ denote image under the canonical map $\pi : R^* \to R^*/wR^*$. We may assume that w is linear in z, that the coefficient of z is 1 and therefore that $w = z - yg(x,y)$, where $g(x,y) \in k[x,y]$. Thus $\overline{R} \cong k[x,y]_{(x,y)}$. By Proposition 16.3, $\overline{B}$ is the approximation domain over $\overline{R}$ with respect to the transcendental element

$$\overline{f} = \overline{y} \cdot \overline{\tau} + \overline{z} \cdot \overline{\sigma} = \overline{y} \cdot \overline{\tau} + \overline{y} \cdot \overline{g(x,y)} \cdot \overline{\sigma}.$$

The setting of Theorem 5.17 applies with B replaced by $\overline{B}$, the underlying ring R replaced by $\overline{R}$, and $z = \overline{x}$. Thus the ring $\overline{B}$ is a UFD, and so every height-one prime ideal of $\overline{B}$ is principal. Since $w \in Q$ and Q is not finitely generated, it follows that $\operatorname{ht}(\overline{Q}) = 2$ and that $\overline{Q}$ is the unique nonfinitely generated prime ideal of $\overline{B}$. Hence the theorem holds. □

REMARKS 16.12. It follows from Proposition 5.16.5 that every height two prime of B that is not contained in Q is contracted from a prime ideal of R^*. As we state in item 7 of Theorem 16.5, there are infinitely many height-two prime ideals of B that are contained in Q and are contracted from R^* and there are infinitely many height-two prime ideals of B that are contained in Q and are *not* contracted from R^*. In particular infinitely many of each type exist between zB and Q by Corollary 16.4, and similarly infinitely many of each type exist between yB and Q.

Since B_Q is a three-dimensional regular local ring, for each height-one prime p of B with $p \subset Q$, the set

$$\mathcal{S}_p \;=\; \{P \in \operatorname{Spec} B \mid p \subset P \subset Q \text{ and } \operatorname{ht} P = 2\}$$

is infinite. The infinite set $\mathcal{S}_p$ is the disjoint union of the sets $\mathcal{S}_{pc}$ and $\mathcal{S}_{pn}$, where the elements of $\mathcal{S}_{pc}$ are contracted from R^* and the elements of $\mathcal{S}_{pn}$ are not contracted from R^*.

We do not know whether there exists a height-one prime p contained in Q having the property that one of the sets $\mathcal{S}_{pc}$ or $\mathcal{S}_{pn}$ is empty. Furthermore if one of these sets is empty, which one is empty? If there are some such height-one primes p

with one of the sets $\mathcal{S}_{pc}$ or $\mathcal{S}_{pn}$ empty, which height-one primes are they? It would be interesting to know the answers to these questions.

REMARK 16.13. A natural question related to Example 16.1 is to ask how it compares to a ring constructed using the three-dimensional ring of Example 15.4 and applying the popular "$D + M$" technique of multiplicative ideal theory; see for example the work of Gilmer in [**45**, p. 95], [**46**] or the paper of Brewer and Rutter [**19**]. The "$D + M$" construction involves an integral domain D and a prime ideal M of an extension domain E of D such that $D \cap M = (0)$. Then $D + M = \{a + b \mid a \in D,\ b \in M\}$. Moreover, for $a, a' \in D$ and $b, b' \in M$, if $a + b = a' + b'$, then $a = a'$ and $b = b'$. Since D embeds in E/M, the ring $D + M$ may be regarded as a pullback as in the paper of Gabelli and Houston [**47**] or the book of Leuschke and R. Wiegand [**91**, p. 42].

In Example 16.14, we consider a "$D + M$" construction that contrasts nicely with Example 16.1.

EXAMPLE 16.14. Let $(B, \mathbf{m}_B)$ be the ring of Example 15.4. Thus k is a coefficient field of B and $B = k + \mathbf{m}_B$. Assume the field k is the field of fractions of a DVR V, and let t be a generator of the maximal ideal of V. Define

$$C \ := \ V \ + \ \mathbf{m}_B \ = \ \{\ a \ + \ b \mid a \ \in \ V,\ b \ \in \ \mathbf{m}_B\ \}.$$

The integral domain C has the following properties:

(1) The maximal ideal $\mathbf{m}_B$ of B is also a prime ideal of C, and $C/\mathbf{m}_B \cong V$.
(2) C has a unique maximal ideal $\mathbf{m}_C$; moreover, $\mathbf{m}_C = tC$.
(3) $\mathbf{m}_B = \bigcap_{n=1}^{\infty} t^n C$, and $B = C_{\mathbf{m}_B} = C[1/t]$.
(4) Each $P \in \operatorname{Spec} C$ with $P \neq \mathbf{m}_C$ is contained in $\mathbf{m}_B$; thus $P \in \operatorname{Spec} B$.
(5) $\dim C = 4$ and C has a unique prime ideal of height h, for $h = 2, 3$ or 4.
(6) $\mathbf{m}_C$ is the only nonzero prime ideal of C that is finitely generated. Indeed, every nonzero proper ideal of B is an ideal of C that is not finitely generated.

Thus C is a non-Noetherian non-catenary four-dimensional local domain.

PROOF. Since C is a subring of B, $\mathbf{m}_B \cap V = (0)$ and $V\mathbf{m}_B = \mathbf{m}_B$, item 1 holds. We have $C/(tV + \mathbf{m}_B) = V/tV$. Thus $tV + \mathbf{m}_B$ is a maximal ideal of C. Let $b \in \mathbf{m}_B$. Since $1 + b$ is a unit of the local ring B, we have

$$\frac{1}{1+b} \ = \ 1 - \frac{b}{1+b} \quad \text{and} \quad \frac{b}{1+b} \ \in \ \mathbf{m}_B.$$

Hence $1+b$ is a unit of C. Let $a+b \in C \setminus (tV+\mathbf{m}_B)$, where $a \in V \setminus tV$ and $b \in \mathbf{m}_B$. Then a is a unit of V and thus a unit of C. Moreover, $a^{-1}(a + b) = 1 + a^{-1}b$ and $a^{-1}b \in \mathbf{m}_B$. Therefore $a + b$ is a unit of C. We conclude that $\mathbf{m}_C := tV + \mathbf{m}_B$ is the unique maximal ideal of C. Since t is a unit of B, we have $\mathbf{m}_B = t\mathbf{m}_B$. Hence $\mathbf{m}_C = tV + \mathbf{m}_B = tC$. This proves item 2.

For item 3, since t is a unit of B, we have $\mathbf{m}_B = t^n\mathbf{m}_B \subseteq t^nC$ for all $n \in \mathbb{N}$. Thus $\mathbf{m}_B \subseteq \bigcap_{n=1}^{\infty} t^nC$. If $a + b \in \bigcap_{n=1}^{\infty} t^nC$ with $a \in V$ and $b \in \mathbf{m}_B$, then

$$b \ \in \ \bigcap_{n=1}^{\infty} t^nC \ \implies \ a \ \in \ (\bigcap_{n=1}^{\infty} t^nC) \cap V \ = \ \bigcap_{n=1}^{\infty} t^nV \ = \ (0).$$

Hence $\mathbf{m}_B = \bigcap_{n=1}^{\infty} t^nC$. Again using $t\mathbf{m}_B = \mathbf{m}_B$, we obtain

$$C[1/t] \ = \ V[1/t] \ + \ \mathbf{m}_B \ = \ k \ + \ \mathbf{m}_B \ = \ B.$$

Since $t \notin \mathbf{m}_B$, we have $B = C[1/t] \subseteq C_{\mathbf{m}_B} \subseteq B_{\mathbf{m}_B} = B$. This proves item 3.

For P as in item 4, we have $P \subsetneq tC$. Since P is a prime ideal of C, it follows that $P = t^n P$ for each $n \in \mathbb{N}$. By item 3, $P \subseteq \mathbf{m}_B$, and it follows that $P \in \operatorname{Spec} B$. Item 5 now follows from item 4 and the structure of $\operatorname{Spec} B$.

For item 6, let J be a nonzero proper ideal of B. Since t is a unit of B, we have $J = tJ$. This implies by Nakayama's Lemma that J as an ideal of C is not finitely generated; see [**19**, Lemma 1]. Thus item 6 follows from item 4.

By item 6, C is non-Noetherian. Since $(0) \subsetneq xB \subsetneq \mathbf{m}_B \subsetneq tC$ is a saturated chain of prime ideals of C of length 3, and $(0) \subsetneq yB \subsetneq Q \subsetneq \mathbf{m}_B \subsetneq tC$ is a saturated chain of prime ideals of C of length 4, the ring C is not catenary. □

REMARK 16.15. An integral domain R is said to be a *finite conductor domain* if for elements a, b in the field of fractions of R the R-module $aR \cap bR$ is finitely generated. This concept is considered in the paper of McAdam [**104**].

A ring R is said to be *coherent* if every finitely generated ideal of R is finitely presented. By a theorem of Chase [**26**, Theorem 2.2], this condition is equivalent to each of the following:

(1) For each finitely generated ideal I and element a of R, the ideal $(I :_R a) = \{b \in R \mid ba \in I\}$ is finitely generated.
(2) For each $a \in R$ the ideal $(0 :_R a) = \{b \in R \mid ba = 0\}$ is finitely generated, and the intersection of two finitely generated ideals of R is again finitely generated.

Thus a coherent integral domain is a finite conductor domain. Examples of finite conductor domains that are not coherent are given by Glaz in [**49**, Example 4.4] and by Olberding and Saydam in [**122**, Proposition 3.7].

As noted in Remark 15.12, the rings of Examples 15.4 and 16.1 are coherent. On the other hand, by a result of Brewer and Rutter [**19**, Prop. 2], the ring of Example 16.14 is not a finite conductor domain and thus is not coherent.

16.3. Non-Noetherian examples in higher dimension

We show in Theorem 16.16 that the rings constructed in Examples 10.9 have many of the properties of Examples 15.1 and 16.1.

THEOREM 16.16. *Let k be a field, let d be a positive integer, and let $x, y_1, \ldots, y_d$ be indeterminates over k. For every positive integer m, there exists a non-Noetherian local integral domain $(B, \mathbf{n})$ with*

$$R := k[x, y_1, \ldots, y_d]_{(x, y_1, \ldots, y_d)} \subset B \subset R^* := k[y_1, \ldots, y_d]_{(y_1, \ldots, y_d)}[[x]]$$

having the following properties:

(1) $\dim B = d + 2$.
(2) *The maximal ideal $\mathbf{n}$ of B is generated by $x, y_1, \ldots, y_d$, and the $\mathbf{n}$-adic completion of B is the formal power series ring $k[[x, y_1, \ldots, y_d]]$, a regular local domain of dimension $d + 1$.*
(3) *The ring B has exactly m prime ideals $Q_1, \ldots, Q_m$ of height $d + 1$.*
(4) *Each Q_i is not finitely generated.*
(5) *The ring $B[1/x]$ is a regular Noetherian UFD.*
(6) *The ring B is a UFD that is not catenary.*
(7) *The local ring $(B, \mathbf{n})$ birationally dominates a localized polynomial ring in $d + 2$ variables over the field k.*

PROOF. We first define the extension domain B of $R = k[x, y_1, \dots, y_d]_{(x,y_1,\dots,y_d)}$. Let $\tau_1, \dots, \tau_d \in xk[[x]]$ be algebraically independent over R. For $1 \le i \le m$, let $p_i = y_1 - x^i$. Set $q = p_1 p_2 \dots p_m$, and consider the element

$$f = q\tau_1 + y_2\tau_2 + \dots y_d\tau_d,$$

and let f_n denote the n^{th}-endpiece of f as in Equation 15.1.a. Define B to be the nested union of localized polynomial rings of dimension $d+2$:

$$B = \bigcup_{n=1}^{\infty} B_n, \quad \text{where} \quad B_n = k[x, y_1, \dots, y_d, f_n]_{(x,y_1,\dots,y_d,f_n)}.$$

Thus B is local, $\dim B \le d+2$, and $\mathbf{n} = \bigcup_{n=1}^{\infty}(x, y_1, \dots, y_d, f_n)B_n$. We have $\mathbf{n} = (x, y_1, \dots, y_d)B$ because $f_n \in (x, y_1, \dots, y_d)B$ for each n. By Construction Properties Theorem 5.14.3, the (x)-adic completion of B is R^*. Hence the $\mathbf{n}$-adic completion of B is the same as the $\mathbf{m}$-adic completion of R, that is $\widehat{B} = k[[x, y_1, \dots, y_d]]$. This proves item 2.

The inclusion map:

$$\varphi : \ S \ := \ R[f] \ \hookrightarrow \ T \ := \ R[\tau_1, \dots, \tau_d]$$

is not flat because the prime ideal $P = (p_1, y_2, y_3, \dots, y_d)T$ has height d, while $P \cap S$ has height $d+1$; see Remark 2.31.10. Thus by Theorem 10.3.2 the ring B is non-Noetherian.

For item 1, it remains to show $\dim B \ge d+2$. Define $Q_i = (p_i, y_2, \dots, y_d)R^* \cap B$, for i with $1 \le i \le m$. Since $(p_i, y_2, \dots, y_d)R^*$ is a prime ideal of R^*, the ideal Q_i is prime. By Proposition 5.16.2, the ideals p_iB and $(p_i, y_2, \dots, y_j)B$ are prime, for every j with $2 \le j \le d$. The inclusions in the chain of prime ideals

$$(0) \ \subset p_iB \ \subset (p_i, y_2)B \ \subset \cdots \subset (p_i, y_2, \dots, y_d)B \subset \ Q_i \ \subset \ \mathbf{n}.$$

are strict because the contractions to B_n are strict for each n; to verify this, consider the list $p_i, y_2, y_3, \dots, y_d, f, x$, and use that $f \in Q_i \setminus (p_i, y_2, \dots, y_d)B_n$ for each n. Thus $\dim B = d+2$, each Q_i has height $d+1$, and $(p_i, y_2, \dots, y_d))B$ has height d for each i. This proves item 1 and part of item 3.

To complete the proof of item 3, we show that $Q_1, \dots, Q_m$ are the only prime ideals of B of height $d+1$. Let P be a nonmaximal prime ideal of B. We first consider the case where $x \notin P$. By Proposition 5.16.3, $x^n \notin PR^*$ for each positive integer n. Hence $PR^*[1/x] \ne R^*[1/x]$. Let P^* be a prime ideal of $R^*[1/x]$ such that $P \subseteq P^*$. If $q, y_2, \dots, y_d$ are all in P^*, then for some i with $1 \le i \le m$ we have $(p_i, y_2, \dots, y_d)R^*[1/x] \subseteq P^*$. Since $(p_i, y_2, \dots, y_d)R^*[1/x]$ is a maximal ideal, we have $(p_i, y_2, \dots, y_d)R^*[1/x] = P^*$. Therefore, $P \subseteq (p_i, y_2, \dots, y_d)R^*[1/x] \cap B = Q_i$, and so either $\operatorname{ht}(P) \le d$ or $P = Q_i$.

By Theorem 10.7.2, the nonflat locus of $\beta : B \hookrightarrow R^*[1/x]$ is defined by $LR^*[1/x]$, where $L = (q, y_2, \dots, y_d)R$. Hence if $x \notin P$ and some element of $\{q, y_2, \dots, y_d\}$ is not in P^*, then the map $\beta_{P^*} : B \to R^*[1/x]_{P^*}$ is flat. Since $\dim(R^*[1/x]) = d$ we have $\operatorname{ht}(P^*) \le d$. Flatness of β_{P^*} implies $\operatorname{ht}(P^* \cap B) \le d$, by Remark 2.31.10. Hence $\operatorname{ht} P \le d$. This completes the proof of item 3 in the case where $x \notin P$.

Assume $x \in P$. We have $\operatorname{ht} P \le d+1$, since $\dim B = d+2$ and P is not maximal. Suppose $\operatorname{ht} P = d+1$. Then there exists a saturated chain of prime ideals of B:

$$(0) \subsetneq P_1 \subsetneq P_2 \dots \subsetneq P_d \ \subsetneq \ P_{d+1} = P \ \subsetneq (x, y_1, y_2, \dots, y_d)B = \mathbf{m}_B. \tag{16.5.1}$$

By Construction Properties Theorem 5.14.2, we have $R/xR = B/xB = R^*/xR^*$. It follows that $d = \dim(R/xR) = \dim(B/xB) = \dim(R^*/xR^*)$. If $x \in P_1$, then $P_1 = xB$, and we get a chain of prime ideals in B/xB of length $d+1$, a contradiction.

Thus we have $x \notin P_1$, and there exists an integer i with $1 \leq i \leq d$ such that $x \in P_{i+1} \setminus P_i$. Using $B/xB = R^*/xR^*$ and $x \in P_{i+1}$, the righthand part of Equation 16.5.1 extends to a chain of prime ideals

$$P_{i+1}R^* \subsetneq \cdots \subsetneq P_{d+1}R^* = PR^* \subsetneq \mathbf{m}_{R^*}$$

of R^*. Also $P_iR^* \subseteq P_{i+1}R^*$, and so there exists a prime ideal $P_i' \in \operatorname{Spec} R^*$ such that $P_i' \subseteq P_{i+1}R^*$ and P_i' is minimal over P_iR^*. By Proposition 5.16.4, P_i' does not contain x. Thus $P_i' \subsetneq P_{i+1}R^*$. Moreover $x \notin P_i'$ implies $P_i'R^*[1/x]$ is a prime ideal of $R^*[1/x]$, and $P_i'R^*[1/x] \cap B = P_i' \cap B = P_i$.

Since $x \in P$ and $P \subsetneq \mathbf{m}_B$, there exists an integer j with $1 \leq j \leq d$ such that $y_j \notin P$; thus $y_j \notin P_i$. It follows that $y_j \notin P_i'' := P_i'R^*[1/x]$, and so P_i'' does not contain $LR^*[1/x] = (q, y_1, \ldots, y_d)R^*[1/x]$. By Theorem 10.7.2 again, the map $\beta_{P_i''} : B \hookrightarrow R^*[1/x]_{P''}$ is flat. Thus $P_i'' \cap B = P_i' \cap B = P_i$ and $\operatorname{ht}(P_i) = i$ together imply that $\operatorname{ht} P_i' = \operatorname{ht} P_i'' \geq \operatorname{ht} P_i = i$. We now have the following chain of prime ideals of R^* of length $d + 2 - i$:

$$P_i' \subsetneq P_{i+1}R^* \subsetneq \cdots \subsetneq P_{d+1}R^* \subsetneq \mathbf{m}_{R^*},$$

as well as the information that $\operatorname{ht} P_i' \geq i$, a contradiction to $\dim R^* = d + 1$. This contradiction completes the proof of item 3 of Theorem 16.16.

For item 4, we first show for each i with $1 \leq i \leq m$:

$$Q_i = Q_i' := \bigcup_{n\in\mathbb{N}} Q_{in}, \quad \text{where} \quad Q_{in} = (p_i, y_2, \ldots, y_d, f_n)B_n.$$

Each Q_{in} is a prime ideal of height $d + 1$ in the $(d + 2)$-dimensional RLR B_n. Thus Q_i' is a prime ideal of B of height $\leq d + 1$ that is contained in Q_i. Since $f \in Q_i' \setminus (p_i, y_2, \ldots, y_d)B$ and $(p_i, y_2, \ldots, y_d)B$ has height d, we have $\operatorname{ht} Q_i' = d$, and hence $Q_i' = Q_i$.

To show Q_i is not finitely generated, define $P_i := (p_i, y_2, \ldots, y_d)R$. It suffices to show for each $n \in \mathbb{N}$ that $f_{n+1} \notin P_iB + f_nB$. Suppose

$$f_{n+1} = \alpha + f_n\beta, \tag{16.5.2}$$

where $\alpha \in P_iB$ and $\beta \in B$. By definition $f = q\tau_1 + y_2\tau_2 + \ldots + y_d\tau_d \in P_iR[\tau_1, \ldots, \tau_d]$. Since each of the τ_i has an expression as a "power series" in x where the coefficients are in R, it follows that there is an expression for f as a "power series" in x where the coefficients are in P_i. Thus, with $f = \gamma$ in Equation 5.4.2, we have $f_n = ax + xf_{n+1}$, where $a \in P_i$. By replacing f_n in Equation 16.5.2, we get

$$f_{n+1} = \alpha + ax\beta + xf_{n+1}\beta \implies f_{n+1}(1 - \beta x) \in P_iB.$$

Since Proposition 5.16.1 implies that x is in the Jacobson radical of B, that is, $1 - x\beta$ is a unit of B, we have

$$f_{n+1} \in P_iB \cap B_{n+1} = P_iB_{n+1},$$

where the last equality is by Proposition 5.16.2. This is a contradiction, since the set $\{p_i = y_1 - x^i, x, y_2, \ldots, y_d, f_{n+1}\}$ is a minimal generating set for the maximal ideal of the regular local ring B_{n+1}. This contradiction shows that Q_i is not finitely generated.

Theorem 5.17 implies item 5, and the assertion in item 5 that B is a UFD. There always exists a saturated chain of prime ideals of B between (0) and $\mathbf{n}$ that contains the height-one prime xB and $B/xB = R^*/xR^*$ implies that this chain has length equal to $\dim R^* = d+1$. Since $\dim B > \dim R^*$, there also exists a saturated chain of prime ideals in B of length $d+2 = \dim B$. Hence B is not catenary. Thus item 5 holds. Item 6 follows from the construction of B. □

QUESTION 16.17. For the ring B constructed in Theorem 16.16, are the prime ideals

$$Q_i \ := \ (p_i, y_2, \ldots, y_{d-1}, f_1, \ldots, f_i, \ldots)B$$

the only prime ideals of B that are not finitely generated?

Exercises

(1) Let K denote the field of fractions of the integral domain B of Example 15.4, let t be an indeterminate over K and let V denote the DVR $K[t]_{(t)}$. Let M denote the maximal ideal of V. Thus $V = K + M$. Define $C := B + M$. Show that the integral domain C has the following properties:
   (a) $\mathbf{m}_B C$ is the unique maximal ideal of C, and is generated by two elements.
   (b) For every nonzero element $a \in \mathbf{m}_B$, we have $M \subset aC$.
   (c) M is the unique prime ideal of C of height one; moreover M is not finitely generated as an ideal of C.
   (d) $\dim C = 4$ and C has a unique prime ideal of height h, for $h = 1, 3$ or 4.
   (e) For each $P \in \operatorname{Spec} C$ with $\operatorname{ht} P \geq 2$, the ring C_P is not Noetherian.
   (f) C has precisely two prime ideals that are not finitely generated.
   (g) C is non-catenary.

(2) Let $C = V + \mathbf{m}_B$ be as in Example 16.14. Assume that V has a coefficient field L, and that L is the field of fractions of a DVR V_1. Define $C_1 := V_1 + tC$. Let s be a generator of V_1. Show that the integral domain C_1 has the following properties:
   (a) The maximal ideal $\mathbf{m}_C$ of C is also a prime ideal of C_1, and $C_1/\mathbf{m}_C \cong V_1$.
   (b) The principal ideal sC_1 is the unique maximal ideal of C_1.
   (c) $\mathbf{m}_C = \bigcap_{n=1}^{\infty} s^n C_1$, and $C = C_1[1/s]$.
   (d) Each $P \in \operatorname{Spec} C_1$ with $P \neq sC_1$ is contained in $\mathbf{m}_C$; thus $P \in \operatorname{Spec} C$.
   (e) $\dim C_1 = 5$.
   (f) C_1 has a unique prime ideal of height h for $h = 2, 3, 4$, or 5.
   (g) The maximal ideal of C_1 is the only nonzero prime ideal of C_1 that is finitely generated. Indeed, every nonzero proper ideal of C is an ideal of C_1 that is not finitely generated.
   (h) C_1 is non-catenary.

   Comment: For item h, exhibit two saturated chains of prime ideals from (0) to sC_1 of different lengths.

CHAPTER 17

The Homomorphic Image Construction (constrhomim)

In the first section of this chapter we describe Homomorphic Image Construction 17.2 for integral domains. This version of Basic Construction 1.5 is different from Inclusion Construction 5.3, and leads to different types of examples.

- As described in Chapter 5, *Inclusion Construction* 5.3 defines an intersection domain $A = A_{\rm inc} := R^* \cap L$ included in R^*, where R^* is an ideal-adic completion of an integral domain R and L is a subfield of the total quotient ring $\mathcal{Q}(R)$ of R^*.
- *Homomorphic Image Construction* 17.2 is an intersection $A = A_{\rm hom}$ of a homomorphic image of an ideal-adic completion R^* of a Noetherian integral domain R with the field of fractions of R.

In Section 17.2 we construct an Approximation Domain for Homomorphic Image Construction 17.2. In Sections 17.3-17.5, we prove Construction Properties Theorem 17.11, Noetherian Flatness Theorem 17.13 and Weak Flatness Theorem 17.19; these are Homomorphic Image versions of theorems proved in earlier chapters for Inclusion Construction 5.3. Theorem 17.17 extends the range of applications of Homomorphic Image Construction 17.2. In Section 17.6, we show that Inclusion Construction 5.3 can be naturally identified with a special case of Homomorphic Image Construction 17.2. Under this identification, Approximation Domains for Inclusion Construction 5.3 correspond to Approximation Domains fitting the Homomorphic Image format of Section 17.2. We also consider a Homomorphic Image form of the Prototypes, the polynomial rings or localized polynomial rings over DVRs defined in Definitions 4.27 and 9.3.These Prototypes are useful for the non-catenary examples of Chapter 18. They are excellent if the underlying DVR is excellent.

17.1. Two construction methods and a picture

Our Setting 17.1 for Homomorphic Image Construction 17.2 is the same as Setting 5.1, in order to facilitate comparison with Inclusion Construction 5.3.

SETTING 17.1. Let R be an integral domain with field of fractions $K := \mathcal{Q}(R)$. Assume $z \in R$ is a nonzero nonunit such that $\bigcap_{n\geq 1} z^n R = (0)$, the (z)-adic completion R^* is Noetherian, and z is a regular element of R^*.

We present the Homomorphic Image Construction 17.2.

CONSTRUCTION 17.2. *Homomorphic Image Construction*: With R, z and R^* as in Setting 17.1, let I be an ideal of R^* such that $P \cap R = (0)$ for each $P \in \operatorname{Spec} R^*$ that is associated to I. Define the *Intersection Domain* $A = A_{\rm hom} := K \cap (R^*/I)$.

The ring A_{hom} is contained in a homomorphic image of R^* and is a birational extension of R.

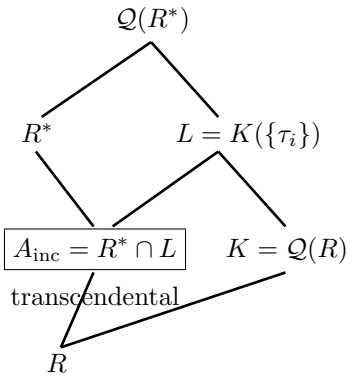
NOTE 17.3. The condition in (17.2), that $P \cap R = (0)$ for every prime ideal P of R^* that is associated to I, implies that the field of fractions K of R embeds in the total quotient ring $\mathcal{Q}(R^*/I)$ of R^*/I. To see this, observe that the canonical map $R \to R^*/I$ is an injection and that regular elements of R remain regular as elements of R^*/I. In this connection see Exercise 1 of this chapter.

We briefly summarize Inclusion Construction 5.3, relabeled as Construction 17.4, for easy reference and comparison to Homomorphic Image Construction 17.2.

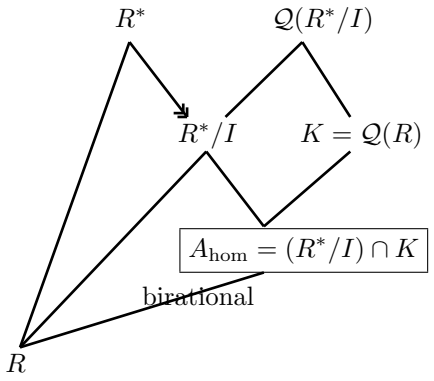
CONSTRUCTION 17.4. (Inclusion Construction 5.3): Assume Setting 17.1. Let $\tau_1, \ldots, \tau_s \in zR^*$ be algebraically independent elements over R such that $K(\tau_1, \ldots, \tau_s) \subseteq \mathcal{Q}(R^*)$. The *Intersection Domain* $A = A_{\text{inc}} := K(\tau_1, \ldots, \tau_s) \cap R^*$.

In Construction 17.2, the Intersection Domain A_{hom} is an integral domain that is birational over R and is contained in a homomorphic image of a power series extension of R. The Intersection Domain A_{inc} associated with Inclusion Construction 17.4 is an integral domain that is not algebraic over R and is contained in a power series extension of R.

PICTURE 17.5. The diagram below shows the relationships among these rings.



(17.4) $A := L \cap R^*$



(17.2) $A := K \cap (R^*/I)$

REMARKS 17.6. Homomorphic Image Construction 17.2 is widely applicable. If a Noetherian local domain R is essentially finitely generated over a field, then there often exist ideals I in the completion $\widehat{R}$ of R such that the intersection domain $\mathcal{Q}(R) \cap (\widehat{R}/I)$ is a Noetherian local domain that birationally dominates R; see Theorem 4.2. Construction 17.2 may be used to describe Example 4.14 of Nagata, Christel's Example 4.16, and other examples given by Brodmann and Rotthaus, Heitmann, Ogoma and Weston, [**20**], [**21**], [**81**], [**123**], [**124**], and [**156**].

While Inclusion Construction 17.4 does appear simpler, Homomorphic Image Construction 17.2 has more flexibility and yields examples that are not possible with Construction 17.4. Construction 17.4 is not sufficient to obtain certain types of rings such as Ogoma's celebrated example [**123**] of a normal non-catenary Noetherian

local domain. As Theorem 6.21 shows, the universally catenary property holds for every Noetherian ring constructed using Inclusion Construction 17.4 over a Noetherian universally catenary local domain R.

Remark 17.16 and Example 18.13 show that examples constructed with Homomorphic Image Construction 17.2 may result in a non-catenary Noetherian local domain even if the base domain is universally catenary, Noetherian and local. In Example 18.13, we construct a Noetherian local domain with geometrically regular formal fibers that is not universally catenary.

17.2. Approximations for the Homomorphic Image Construction

The approximation methods in this chapter describe a subring B inside the constructed Intersection Domain A of Construction 17.2. This subring is useful for describing A.

The Approximation Domain B for Construction 17.2 is a nested union of birational extensions of R that are essentially finitely generated R-algebras. As with the Approximation Domain for Inclusion Construction 17.4 from Definition 5.7, we approach A using a sequence of "approximation rings" over R. We use the *frontpieces* of the power series involved, rather than the endpieces that are used for the approximations in Inclusion Construction 17.4. The Approximation Domains that are so obtained are not localizations of polynomial rings over R.

A goal of these computations is to prove Noetherian Flatness Theorem 17.13 for Homomorphic Image Construction 17.2.

FRONTPIECE NOTATION 17.7. Let R be an integral domain with field of fractions $K := \mathcal{Q}(R)$. Let $z \in R$ be a nonzero nonunit such that $\bigcap_{n\geq 1} z^n R = (0)$, the (z)-adic completion R^* is Noetherian, and z is a regular element of R^*. Let I be an ideal of R^* such that $P \cap R = (0)$, for each $P \in \operatorname{Spec} R^*$ that is associated to R^*/I. As in Construction 17.2, define $A = A_{\text{hom}} := K \cap (R^*/I)$.

Since $I \subset R^*$, each $\gamma \in I$ has an expansion as a power series in z over R,

$$\gamma \ := \ \sum_{i=0}^{\infty} a_i z^i, \quad \text{where} \quad a_i \in R.$$

For each positive integer n we define the n^{th} *frontpiece* γ_n of γ with respect to this expansion:

$$\gamma_n \ := \ \sum_{j=0}^{n} \frac{a_j z^j}{z^n}.$$

Thus, if $I := (\sigma_1, \ldots, \sigma_t)R^*$, then for each σ_i we have

$$\sigma_i \ := \ \sum_{j=0}^{\infty} a_{ij} z^j, \qquad \text{where the} \quad a_{ij} \ \in \ R,$$

and the n^{th} frontpiece σ_{in} of σ_i is

$$\sigma_{in} \ := \ \sum_{j=0}^{n} \frac{a_{ij} z^j}{z^n} \ \in \ K. \tag{17.7.1}$$

For the Homomorphic Image Construction 17.2, we obtain approximating rings as follows: We define

$$U_n := R[\sigma_{1n}, \ldots, \sigma_{tn}], \qquad \text{and } B_n := (1 + zU_n)^{-1} U_n. \tag{17.7.2}$$

The rings U_n and B_n are subrings of K. We observe in Proposition 17.9 that they may also be considered to be subrings of R^*/I. First we show in Proposition 17.8 that the approximating rings U_n and B_n are nested.

PROPOSITION 17.8. *With the setting of Frontpiece Notation 17.7, for each integer $n \geq 0$ and for each integer i with $1 \leq i \leq t$, we have*

(1) $\sigma_{in} = -za_{i,n+1} + z\sigma_{i,n+1}$.
(2) $(z, \sigma_i)R^* = (z, a_{i0})R^*$ *and hence* $(z, I)R^* = (z, a_{10}, \ldots, a_{t0})R^*$.
(3) $(z, \sigma_i)R^* = (z, z^n\sigma_{in})R^*$ *and hence* $(z, I)R^* = (z, z^n\sigma_{1n}, \ldots, z^n\sigma_{tn})R^*$.

Thus $R \subseteq U_0$ and we have $U_n \subseteq U_{n+1}$ and $B_n \subseteq B_{n+1}$ for each positive integer n.

PROOF. For item 1, by Definition 17.7.1, we have $\sigma_{i,n+1} := \sum_{j=0}^{n+1} \frac{a_{ij}z^j}{z^{n+1}}$. Thus

$$z\sigma_{i,n+1} = \sum_{j=0}^{n+1} \frac{a_{ij}z^{j+1}}{z^{n+1}} = \sum_{j=0}^{n} \frac{a_{ij}z^j}{z^n} + za_{i,n+1} = \sigma_{in} + za_{i,n+1}.$$

For item 2, by definition

$$\sigma_i := \sum_{j=0}^{\infty} a_{ij}z^j = a_{i0} + z\big(\sum_{j=1}^{\infty} a_{ij}z^{j-1}\big).$$

For item 3, we have the following equation in R^*:

$$\sigma_i \;=\; \sum_{j=0}^{\infty} a_{ij}z^j \;=\; z^n\sigma_{in} + z^{n+1}\big(\sum_{j=n+1}^{\infty} a_{ij}z^{j-n-1}\big),$$

since $z^n\sigma_{in} \in R$. The asserted inclusions follow from this equation. □

Even though they appear different, Proposition 17.9 shows for each of the power series σ_i generating the ideal I that the n^{th} approximation in Frontpiece Notation 17.7 is, modulo the ideal I, the negative of the n^{th} approximation in Endpiece Notation 5.4.

PROPOSITION 17.9. *Assume the setting of Frontpiece Notation 17.7 and let n be a positive integer. As an element of the total quotient ring of R^*/I, the n^{th} frontpiece σ_{in} is the negative of the n^{th} endpiece of σ_i defined in Endpiece Notation 5.4, that is, for $\sigma_i := \sum_{j=0}^{\infty} a_{ij}z^j$, where each $a_{ij} \in R$,*

$$\sigma_{in} = -\sum_{j=n+1}^{\infty} \frac{a_{ij}z^j}{z^n} = -\sum_{j=n+1}^{\infty} a_{ij}z^{j-n} \qquad (\text{mod } I).$$

It follows that $\sigma_{in} \in K \cap (R^*/I)$, and so U_n and B_n are subrings of A and of R^*/I.

PROOF. Let π denote the natural homomorphism from R^* onto R^*/I. Using that the restriction of π to R is the identity map on R, we have

$$\begin{aligned}
\sigma_i = z^n\sigma_{in} + \sum_{j=n+1}^{\infty} a_{ij}z^j \;&\Longrightarrow\; z^n\sigma_{in} = \sigma_i - \sum_{j=n+1}^{\infty} a_{ij}z^j \\
&\Longrightarrow\; \pi(z^n\sigma_{in}) = \pi(\sigma_i) - \pi\big(\sum_{j=n+1}^{\infty} a_{ij}z^j\big) \\
&\Longrightarrow\; z^n\sigma_{in} = -z^n\pi\big(\sum_{j=n+1}^{\infty} a_{ij}z^{j-n}\big).
\end{aligned}$$

Therefore $z^n\sigma_{in} \in z^n\pi(R^*) = z^n(R^*/I)$. Since z is a regular element of R^*/I, we have $\sigma_{in} = -\pi(\sum_{j=n+1}^{\infty} a_{ij}z^{j-n})$ is an element of R^*/I. □

DEFINITION 17.10. Assume the setting of Frontpiece Notation 17.7. We define the nested union U, the *Approximation Domain* B and the *Intersection Domain* A:

$$U := \bigcup_{n=1}^{\infty} U_n, \qquad B := \bigcup_{n=1}^{\infty} B_n = (1+zU)^{-1}U, \qquad A := K \cap (R^*/I). \tag{17.10.1}$$

By Remark 3.2.1, the element z is in the Jacobson radical of R^*. By Proposition 17.9, $B \subseteq A$. Construction 17.2 is said to be *limit-intersecting* if $B = A$.

17.3. Basic properties of the Approximation Domains

Construction Properties Theorem 17.11 relates to the analysis of the Homomorphic Image Construction. The proof uses Lemma 5.12 to establish relationships among rings that arise in the Homomorphic Image Construction 17.2 and the approximations in Section 17.2.

CONSTRUCTION PROPERTIES THEOREM 17.11. (Homomorphic Image Version) *Let R be an integral domain with field of fractions $K := \mathcal{Q}(R)$. Let $z \in R$ be a nonzero nonunit such that $\bigcap_{n\geq 1} z^nR = (0)$, the (z)-adic completion R^* is Noetherian, and z is a regular element of R^*. Let I be an ideal of R^* such that $P \cap R = (0)$ for each $P \in \operatorname{Spec} R^*$ that is associated to R^*/I. With the notation of Frontpiece Notation 17.7 and Definition 17.10, we have for each positive integer n:*

(1) *The ideals of R containing z^n are in one-to-one inclusion preserving correspondence with the ideals of R^* containing z^n. In particular, we have $(I,z)R^* = (a_{10},\dots,a_{t0},z)R^*$ and*

$$(I,z)R^* \cap R = (a_{10},\dots,a_{t0},z)R^* \cap R = (a_{10},\dots,a_{t0},z)R.$$

(2) *The ideal $(a_{10},\dots,a_{t0},z)R$ equals $z(R^*/I)\cap R$ under the identification of R as a subring of R^*/I, and the element z is in the Jacobson radical of both R^*/I and B.*

(3) $z^n(R^*/I)\cap A = z^nA, \qquad z^n(R^*/I)\cap U = z^nU, \qquad z^n(R^*/I)\cap B = z^nB.$

(4) *$U/z^nU = B/z^nB = A/z^nA = R^*/(z^nR^* + I)$. The rings A, U and B all have (z)-adic completion R^*/I, that is, $A^* = U^* = B^* = R^*/I$.*

(5) *$R[1/z] = U[1/z]$, $U = R[1/z]\cap B = R[1/z]\cap A = R[1/z]\cap(R^*/I)$ and the integral domains R, U, B and A all have the same field of fractions K.*

PROOF. The first assertion of item 1 follows because R/z^nR is canonically isomorphic to R^*/z^nR^*. The next assertion of (1) follows from part 2 of Proposition 17.8. If $\gamma = \sum_{i=1}^{t}\sigma_i\beta_i + z\tau \in (I,z)R^*\cap R$, where $\tau, \beta_i \in R^*$, then write each $\beta_i = b_i + z\beta_i'$, where $b_i \in R$, $\beta_i' \in R^*$. Thus $\gamma - \sum_{i=1}^{t} a_{i0}b_i \in zR^*\cap R = zR$, and so $\gamma \in (a_{10},\dots,a_{t0},z)R$.

Since $z(R^*/I) = (z,I)R^*/I$, we have $(a_{10},\dots,a_{t0},z)R \subseteq z(R^*/I)\cap R$. The reverse inclusion in item 2 follows from $(I,z)R^* = (a_{10},\dots,a_{t0},z)R^*$. For the last part of item 2, we have that $z \in \mathcal{J}(R^*)$ and so $1+az$ is outside every maximal ideal of R^* for every $a \in R^*$. Thus $z \in \mathcal{J}(R^*/I)$. By the definition of B in Equation 17.10.1, $z \in \mathcal{J}(B)$.

The first assertion of item 3 follows from the definition of A as $(R^*/I) \cap K$. To see that $z(R^*/I) \cap U \subseteq zU$, let $g \in z(R^*/I) \cap U$. Then $g \in U_n$, for some n, implies $g = r_0 + g_0$, where $r_0 \in R$, $g_0 \in (\sigma_{1n}, \ldots, \sigma_{tn})U_n$. Also $\sigma_{in} = -za_{i,n+1} + z\sigma_{i,n+1}$, and so $g_0 \in zU_{n+1} \subseteq z(R^*/I)$. Now $r_0 \in (z, \sigma_1, \ldots, \sigma_t)R^* = (I, z)R^*$. Thus by item 1, $r_0 \in (a_{10}, \ldots, a_{t0}, z)R$. Also each $a_{i0} = \sigma_i - z\sum_{j=1}^{\infty} a_{ij}z^{j-1} \in zU$ because $a_{i0} = z\sigma_{i1} - za_{i1}$.

Thus $r_0 \in zU$, as desired. This proves that $z^n(R^*/I) \cap U = z^nU$. Since $B = (1 + zU)^{-1}U$, we also have $z^n(R^*/I) \cap B = z^nB$.

With $S = A$, $T = R^*/I$ and $x = z$, condition 4 of Lemma 5.12 holds since $A = A[1/z] \cap (R^*/I)$ and $(R^*/I)[1/z] = A[1/z] + (R^*/I)$. Thus item 4 follows from item 3 and Lemma 5.12.

For item 5: if $g \in U$, then $g \in U_n$, for some n. Clearly each $\sigma_{in} \in R[1/z]$, and so $g \in R[1/z]$. To see that $U = R[1/z] \cap B$, apply Lemma 5.12 with $S = U$, $B = T$. Similarly we see that $U = R[1/z] \cap A$, since

$$R[1/z] \cap A = R[1/z] \cap (\mathcal{Q}(R) \cap (R^*/I)) = R[1/z] \cap (R^*/I).$$

It is clear that the integral domains R, U, B and A all have the same field of fractions K. □

Remark 17.12. We note the following implications from Theorem 17.11 .

(1) Item 5 of Theorem 17.11 implies that the definitions in (17.10.1) of B and U are independent of
   (a) the choice of generators for I, and
   (b) the representation of the generators of I as power series in z.
(2) Item 5 of Theorem 17.11 implies that the rings $U = R[1/z] \cap (R^*/I)$ and $B = (1 + zU)^{-1}U$ are uniquely determined by z and the ideal I of R^*.
(3) Since z is in the Jacobson radical of B, item 4 of Theorem 17.11 implies that if $b \in B$ is a unit of A, then b is already a unit of B.
(4) The diagram below displays the relationships among these rings.

$$\begin{array}{ccccccccc}
\mathcal{Q}(R) & = & \mathcal{Q}(U) & = & \mathcal{Q}(B) & = & \mathcal{Q}(A) & \xrightarrow{\subseteq} & \mathcal{Q}(R^*/I) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
R[1/z] & = & U[1/z] & \xrightarrow{\subseteq} & B[1/z] & \xrightarrow{\subseteq} & A[1/z] & \xrightarrow{\subseteq} & (R^*/I)[1/z] \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
R & \xrightarrow{\subseteq} & U = \cup\, U_n & \xrightarrow{\subseteq} & B & \xrightarrow{\subseteq} & A & \xrightarrow{\subseteq} & R^*/I.
\end{array}$$

17.4. Noetherian Flatness for homomorphic images

Noetherian Flatness Theorem 17.13 (Homomorphic Image Version) gives precise conditions for the Approximation Domain B of Homomorphic Image Construction 17.2 to be Noetherian.

Noetherian Flatness Theorem 17.13. (Homomorphic Image Version) *Let R be an integral domain with field of fractions $K := \mathcal{Q}(R)$. Let $z \in R$ be a nonzero nonunit such that $\bigcap_{n\geq 1} z^nR = (0)$, the (z)-adic completion R^* is Noetherian, and z is a regular element of R^*. Let I be an ideal of R^* having the property that*

$\mathbf{p} \cap R = (0)$ *for each* $\mathbf{p} \in \operatorname{Ass}(R^*/I)$. *As in Frontpiece Notation 17.7.2 and Definition 17.10.1, let*

$$U := \bigcup_{n=1}^{\infty} U_n, \quad B := \bigcup_{n=1}^{\infty} B_n = (1+zU)^{-1}U, \quad \text{and} \quad A := K \cap (R^*/I).$$

The following statements are equivalent:

(1) *The extension* $R \hookrightarrow (R^*/I)[1/z]$ *is flat.*
(2) *The ring* B *is Noetherian.*
(3) *The extension* $B \hookrightarrow R^*/I$ *is faithfully flat.*
(4) *The ring* $A := K \cap (R^*/I)$ *is Noetherian and* $A = B$.
(5) *The ring* U *is Noetherian*
(6) *The ring* A *is both Noetherian and a localization of a subring of* $R[1/z]$.

PROOF. For (1) $\implies$ (2), if $R \hookrightarrow (R^*/I)[1/z]$ is flat, by factoring through $U[1/z] = R[1/z] \hookrightarrow (R^*/I)[1/z]$, we see that $U \hookrightarrow (R^*/I)[1/z]$ and $B \hookrightarrow (R^*/I)[1/z]$ are flat. By Lemma 6.2.2, where we let $S = U$ and $T = R^*/I$, the ring B is Noetherian.

For (2) $\implies$ (3), $B^* = R^*/I$ is flat over B, by Theorem 17.11.4 and Remark 3.2.3. By Proposition 5.16.1, $z \in \mathcal{J}(B)$, and so, using Remark 3.2.4, we have $B^* = R^*/I$ is faithfully flat over B.

For (3) $\implies$ (4), again Theorem 17.11.4 yields $B^* = R^*/I$, and so B^* is faithfully flat over B. Then

$$B = \mathcal{Q}(B) \cap R^* = \mathcal{Q}(A) \cap R^* = K \cap R^* = A$$

by Remark 2.31.9 and Theorem 5.14.2. By Remark 2.31.8, A is Noetherian.

For (4) $\implies$ (5), the composite embedding

$$U \hookrightarrow B = A \hookrightarrow B^* = A^* = R^*/I$$

is flat because B is a localization of U and A is Noetherian; see Remark 3.2.3. By Remark 3.2.4 again, A^* is faithfully flat over A. Thus by Lemma 6.2, parts 1 and 3, where again we let $S = U$ and $T = R^*/I$, we have $S[1/z] = U[1/z] = R[1/z]$ is Noetherian, and hence U is Noetherian by Lemma 6.2.4.

If U is Noetherian, then the localization B of U is Noetherian, and as above $B = A$. Hence A is a localization of U, a subring of $R[1/z]$. Thus (5) $\implies$ (6).

For (6) $\implies$ (1): since A is a localization of a subring D of $R[1/z]$, we have $A := \Gamma^{-1}D$, where Γ is a multiplicatively closed subset of D. Now

$$R \subseteq A = \Gamma^{-1}D \subseteq \Gamma^{-1}R[1/z] = \Gamma^{-1}A[1/z] = A[1/z],$$

and so $A[1/z]$ is a localization of R. That is, to obtain $A[1/z]$ we localize R by the elements of Γ and then localize by the powers of $1/z$. Since A is Noetherian, $A \hookrightarrow A^* = R^*/I$ is flat by Remark 3.2.2. Thus $A[1/z] \hookrightarrow (R^*/I)[1/z]$ is flat. It follows that $R \hookrightarrow (R^*/I)[1/z]$ is flat. □

COROLLARY 17.14. *Let* R, I *and* z *be as in Noetherian Flatness Theorem 17.13 (Homomorphic Image version). If* $\dim(R^*/I) = 1$, *then* $\varphi : R \hookrightarrow W := (R^*/I)[1/z]$ *is flat and therefore the equivalent conditions of Theorem 17.13 all hold.*

PROOF. We have z is in the Jacobson radical of R^*/I by Construction Properties Theorem 17.11.2. Thus $\dim(R^*/I) = 1$ implies that $\dim W = 0$. The hypothesis on the ideal I implies that every prime ideal P of W contracts to (0) in

R. Hence

$$\varphi_P : R_{P\cap R} = R_{(0)} = K \hookrightarrow W_P.$$

Thus W_P is a K-module and so a vector space over K. By Remark 2.31.2, φ_P is flat. Since flatness is a local property by Remark 2.311, the map φ is flat. □

REMARKS 17.15. With R, I, z, A and B as in Noetherian Flatness Theorem 17.13:

(1) We show in Section 17.6 that the Intersection Domain and Approximation Domain of Inclusion Construction 5.3 (Construction 17.4) are the same as the domains constructed in Homomorphic Image Construction 17.2 under change of base ring. Thus, by Remark 6.8, there are examples using Construction 17.2 such that the Intersection Domain A is Noetherian, but the Approximation Domain $B \neq A$, and other examples where $A = B$ is non-Noetherian.
(2) A necessary and sufficient condition that $A = B$ is that A is a localization of $R[1/z] \cap A$. Indeed, Theorem 17.11.5 implies that $R[1/z] \cap A = U$ and, by Definition 17.10.1, $B = (1 + zU)^{-1}U$. Therefore the condition is sufficient. On the other hand, if $A = \Gamma^{-1}U$, where Γ is a multiplicatively closed subset of U, then by Remark 17.12.3, each $y \in \Gamma$ is a unit of B, and so $\Gamma^{-1}U \subseteq B$ and $A = B$. See also Theorem 17.19 for more discussion of when $A = B$,
(3) We discuss in Chapter 9 a family of Prototype examples where the conditions of the Inclusion version of Noetherian Flatness Theorem 6.3 hold, in a rather trivial way. Under the identifications of Diagram 17.20.1 below, these examples become examples using Homomorphic Image Construction 17.2 where $R \hookrightarrow (R^*/I)[1/z]$ is flat; see Theorem 17.25.

REMARK 17.16. By Theorem 6.21, the universal catenary property is preserved by Inclusion Construction 5.3. In contrast, consider the constructed domains A and B of Homomorphic Image Construction 17.2, for $(R, \mathbf{m})$ a universally catenary Noetherian local domain, $z \in \mathbf{m}$ an appropriate nonzero element and I an ideal of the (z)-adic completion R^* of R. Then A and B are local and

$$A^* \;=\; B^* \;=\; R^*/I, \quad \text{and so} \quad \widehat{A} \;=\; \widehat{B} \;=\; \widehat{R}/I\widehat{R},$$

by Construction Properties Theorem 17.11.4. Even if $A = B$ and is Noetherian as in Noetherian Flatness Theorem 17.13, it is not necessarily true that $\widehat{R}/I\widehat{R}$ is equidimensional. In Example 18.13, with base ring R a localized polynomial ring in 3 variables over a field, so that R is certainly universally catenary, we construct a Noetherian local domain A that is not universally catenary by using Homomorphic Image Construction 17.2.

Theorem 17.17 extends the range of applications of Homomorphic Image Construction 17.2.

THEOREM 17.17. *Let R be a Noetherian integral domain with field of fractions K. Let z be a nonzero nonunit of R and let R^* denote the (z)-adic completion of R. Let I be an ideal of R^* having the property that $\mathbf{p}\cap R = (0)$ for each $\mathbf{p} \in \mathrm{Ass}(R^*/I)$. Assume that I is generated by a regular sequence of R^*. If $R \hookrightarrow (R^*/I)[1/z]$ is flat, then for each $n \in \mathbb{N}$ we have*

(1) $\mathrm{Ass}(R^*/I^n) = \mathrm{Ass}(R^*/I)$,

(2) *R canonically embeds in R^*/I^n, and*
(3) *$R \hookrightarrow (R^*/I^n)[1/z]$ is flat.*

PROOF. Let $I = (\sigma_1, \ldots, \sigma_r)R^*$, where $\sigma_1, \ldots, \sigma_r$ is a regular sequence in R^*. Then the sequence $\sigma_1, \ldots, \sigma_r$ is quasi-regular in the sense of [**103**, Theorem 16.2, page 125]; that is, the associated graded ring of R^* with respect to I, which is the direct sum $R^*/I \oplus I/I^2 \oplus \ldots$, is a polynomial ring in r variables over R^*/I. For each positive integer n, the component I^n/I^{n+1} is a free (R^*/I)-module generated by the monomials of total degree n in these variables. Thus $\mathrm{Ass}(I^n/I^{n+1}) = \mathrm{Ass}(R^*/I)$; that is, a prime ideal P of R^* annihilates a nonzero element of R^*/I if and only if P annihilates a nonzero element of I^n/I^{n+1}.

For item 1 we proceed by induction: assume $\mathrm{Ass}(R^*/I^n) = \mathrm{Ass}(R^*/I)$ and $n \in \mathbb{N}$. Consider the exact sequence

$$0 \to I^n/I^{n+1} \hookrightarrow R^*/I^{n+1} \to R^*/I^n \to 0. \tag{17.17.0}$$

Then $\mathrm{Ass}(R^*/I) = \mathrm{Ass}(I^n/I^{n+1}) \subseteq \mathrm{Ass}(R^*/I^{n+1})$. Also

$$\mathrm{Ass}(R^*/I^{n+1}) \ \subseteq \ \mathrm{Ass}(I^n/I^{n+1}) \ \cup \ \mathrm{Ass}(R^*/I^n) \ = \mathrm{Ass}(R^*/I)$$

by [**103**, Theorem 6.3, p. 38], and so it follows that $\mathrm{Ass}(R^*/I^{n+1}) = \mathrm{Ass}(R^*/I)$. Thus R canonically embeds in R^*/I^n for each $n \in \mathbb{N}$.

That $R \hookrightarrow (R^*/I^n)[1/z]$ is flat for every $n \in \mathbb{N}$ now follows by induction on n and by considering the exact sequence obtained by tensoring over R the short exact sequence (17.17.0) with $R[1/z]$. □

EXAMPLE 17.18. Let $R = k[x, y]$ be the polynomial ring in the variables x and y over a field k and let $R^* = k[y][[x]]$ be the (x)-adic completion of R. Fix an element $\tau \in xk[[x]]$ such that x and τ are algebraically independent over k, and define the $k[[x]]$-algebra homomorphism $\theta : k[y][[x]] \to k[[x]]$, by setting $\theta(y) = \tau$. Then $\ker(\theta) = (y - \tau)R^*$. Set $I := (y - \tau)R^*$. Notice that $\theta(R) = k[x, \tau] \cong R$ because x and τ are algebraically independent over k. Hence $I \cap R = (0)$. Also I is a prime ideal generated by a regular element of R^*, and $(I, x)R^* = (y, x)R^*$ is a maximal ideal of R^*. Corollary 17.14 and Theorem 17.17 imply that for each positive integer n, the Intersection Domain $A_n := (R^*/I^n) \cap k(x, y)$ is a one-dimensional Noetherian local domain having (x)-adic completion R^*/I^n. Since x generates an ideal primary for the unique maximal ideal of R^*/I^n, the ring R^*/I^n is also the completion of A_n with respect to the powers of the unique maximal ideal $\mathbf{n}_n$ of A_n. The ring A_1 is a DVR since R^*/I is a DVR by Remark 2.1. For $n > 1$, the completion of A_n has nonzero nilpotent elements, and hence the integral closure of A_n is not a finitely generated A_n-module by Remarks 3.12. The inclusion $I^{n+1} \subsetneq I^n$ and the fact that A_n has completion R^*/I^n imply that $A_{n+1} \subsetneq A_n$ for each $n \in \mathbb{N}$. Hence the rings A_n form a strictly descending chain

$$A_1 \ \supset \ A_2 \ \supset \cdots \supset \ A_n \ \supset \cdots$$

of one-dimensional local birational extensions of $R = k[x, y]$.

17.5. Weak Flatness for the Homomorphic Image Construction

In Theorem 17.19, we present a version of Weak Flatness Theorem 8.7 that applies to Homomorphic Image Construction 17.2.

WEAK FLATNESS THEOREM 17.19. (Homomorphic Image Version) *Let R be a normal Noetherian integral domain and let $z \in R$ be a nonzero nonunit. Let R^* denote the (z)-adic completion of R and let I be an ideal of R^* having the property that $P \cap R = (0)$ for each associated prime ideal P of I. Let the rings A and B be as defined in Section 17.10. Assume that B is a Krull domain. Then*

(1) *If the extension $R \hookrightarrow (R^*/I)[1/z]$ is weakly flat, then $A = B$, that is, the construction is limit-intersecting as in Definition 17.10.*
(2) *If R^*/I is a normal integral domain, then the following statements are equivalent:*
   (a) *$A = B$.*
   (b) *$R \hookrightarrow (R^*/I)[1/z]$ is weakly flat.*
   (c) *The extension $B \hookrightarrow (R^*/I)[1/z]$ is weakly flat.*
   (d) *The extension $B \hookrightarrow R^*/I$ is weakly flat.*

PROOF. Theorem 17.11.3 implies that each height-one prime of B containing zB is contracted from R^*/I. Using Frontpiece Notation 17.7, Definition 17.10 and Theorem 17.11, we have $B[1/z]$ is a localization of $R[1/z] = U[1/z]$. Since $R \hookrightarrow (R^*/I)[1/z]$ is weakly flat, it follows that $B \hookrightarrow (R^*/I)[1/z]$ is weakly flat by Remark 8.5.b. Therefore $B \hookrightarrow R^*/I$ is weakly flat. By Proposition 8.3.1, we have $B = (R^*/I) \cap \mathcal{Q}(B) = A$. This proves item 1.

For item 2, since R^*/I is a normal integral domain, $A = (R^*/I) \cap \mathcal{Q}(R)$ is a Krull domain. As noted in the proof of item 1, Theorem 17.11 implies that each height-one prime of B containing zB is contracted from R^*/I and $B[1/z]$ is a localization of $R[1/z] = U[1/z]$. It follows that (b), (c) and (d) are equivalent. By Proposition 8.3.3, (a) $\implies$ (d), and by Proposition 8.3.1, (d) $\implies$ (a). □

17.6. Inclusion Constructions are Homomorphic Images

For this section we revise our notation so that R denotes the base ring for Inclusion Construction 17.4.

REVISED NOTATION 17.20. Let R, z, and R^* be as in Setting 17.1. As in Construction 17.4, let $\tau_1, \ldots, \tau_s \in zR^*$ be algebraically independent elements over R such that $K(\tau_1, \ldots, \tau_s) \subseteq \mathcal{Q}(R^*)$. We define A to be the Intersection Domain $A = A_{\text{inc}} := K(\tau_1, \ldots, \tau_s) \cap R^*$, a subring of R^* that is not algebraic over R. Thus

$$\tau_i \;:=\; \sum_{j=1}^{\infty} r_{ij} z^j \quad \text{where} \quad r_{ij} \;\in\; R.$$

Let $t_1, \ldots, t_s$ be indeterminates over R, define $S := R[t_1, \ldots, t_s]$, let S^* be the (z)-adic completion of S, and let I denote the ideal $(t_1 - \tau_1, \ldots, t_s - \tau_s)S^*$. Notice that $S^*/I \cong R^*$ implies that $P \cap S = (0)$ for each prime ideal $P \in \text{Ass}(S^*/I)$. Thus we are in the setting of the Homomorphic Image Construction where we define the Intersection Domain $D := A_{\text{hom}} := K(t_1, \ldots, t_s) \cap (S^*/I)$. Let $\sigma_i := t_i - \tau_i$, for each i with $1 \le i \le s$. For each $n \in \mathbb{N}_0$ and each i with $1 \le i \le s$, the element τ_{in} of R^* is the n^{th} endpiece of τ_i and the element $\sigma_{in} \in S^*$ is the n^{th} frontpiece of σ_i.

17.6.1. Matching up Intersection Domains. Consider Diagram 17.20.1, where λ is the R-algebra isomorphism of S into R^* that maps $t_i \to \tau_i$ for $i = 1, \ldots, s$. Here $D := A_{\text{hom}} := \mathcal{Q}(S) \cap (S^*/I)$; that is, A_{hom} is the Intersection Domain of Construction 17.2, if R and R^* there are replaced by S and S^*. The map λ

naturally extends to a homomorphism of S^* onto R^*, and the ideal I is the kernel of this extension. Thus there is an induced isomorphism of S^*/I onto R^* that we also label λ.

(17.20.1)
$$\begin{array}{ccccc} S := R[t_1,\dots,t_s] & \longrightarrow & D := K(t_1,\dots,t_s)\cap(S^*/I) & \longrightarrow & S^*/I \\ \lambda\downarrow & & \lambda\downarrow & & \lambda\downarrow \\ R \longrightarrow S' := R[\tau_1,\dots,\tau_s] & \longrightarrow & A := K(R)(\tau_1,\dots,\tau_s)\cap R^* & \longrightarrow & R^*. \end{array}$$

Then λ maps D isomorphically onto A via $\lambda(t_i) = \tau_i$, for every i with $1 \le i \le s$.

PROPOSITION 17.21. *Inclusion Construction 5.3 is a special case, up to isomorphism, of Homomorphic Image Construction 17.2. That is, under the identifications of Diagram 17.20.1, the Intersection Domain of Inclusion Construction 17.2 fits the description of the Intersection Domain of Homomorphic Image Construction 17.2.*

PROOF. Since λ maps $D = A_{\text{hom}}$ isomorphically onto $A = A_{\text{inc}}$, Construction 17.2 includes Construction 5.3 as a special case. □

17.6.2. Matching up Approximation Domains. By Proposition 17.22, the identifications of Diagram 17.20.1 transform the Approximation Domain for Inclusion Construction 5.3 into the Approximation Domain of Homomorphic Image Construction 17.2. That is, the formula given in Equation 5.4.5 of Section 5.2 using endpieces becomes the formula given in Definition 17.10 defined on S and S^*/I using frontpieces.

PROPOSITION 17.22. *Assume the setting of Revised Notation 17.20. As in Frontpiece Notation 17.7, define σ_{in} to be the n^{th} frontpiece for σ_i over S. Denote by V_n, C_n, V, C the rings constructed in Frontpiece Notation 17.7 and Equation 17.10.1 with S as the base ring, as shown in Equations 17.22.1. Define U_n, B_n, U, B using Endpiece Notation 5.4 and Equations 5.4.4 and 5.4.5 over R. Thus*

(17.22.1)
$$\begin{aligned} V_n &:= S[\sigma_{1n},\dots,\sigma_{sn}] = R[t_1,\dots,t_s][\sigma_{1n},\dots,\sigma_{sn}], \\ C_n &:= (1+zV_n)^{-1}V_n, \\ V &:= \bigcup_{n=1}^{\infty} V_n, \qquad C := \bigcup_{n=1}^{\infty} C_n = (1+zV)^{-1}V \\ U &:= \bigcup_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty} R[\tau_{1n},\dots,\tau_{sn}] \quad \textit{and} \\ B &:= \bigcup_{n=1}^{\infty} B_n = (1+zU)^{-1}U. \end{aligned}$$

Then the R-algebra isomorphism λ has the following properties:

$$\lambda(D) = A, \quad \lambda(\sigma_{in}) = \tau_{in}, \quad \lambda(V_n) = U_n, \quad \lambda(C_n) = B_n, \quad \lambda(V) = U, \quad \lambda(C) = B,$$

for all i with $1 \le i \le s$ and all $n \in \mathbb{N}$.

PROOF. We have elements $r_{ij} \in R$ so that

$$\begin{aligned}
\tau_i &:= \sum_{j=1}^{\infty} r_{ij} z^j, \qquad \tau_{in} := \sum_{j=n+1}^{\infty} r_{ij} z^{j-n} \\
\sigma_i &:= t_i - \tau_i = t_i - \sum_{j=1}^{\infty} r_{ij} z^j, \qquad \sigma_{in} := \frac{t_i - \sum_{j=1}^{n} r_{ij} z^j}{z^n} \\
&\Longrightarrow \lambda(\sigma_{in}) = \frac{\tau_i - \sum_{j=1}^{n} r_{ij} z^j}{z^n} \\
&= \tau_{in}.
\end{aligned}$$

The remaining statements of Proposition 17.22 now follow. □

REMARK 17.23. With the setting of Revised Notation 17.20, Proposition 17.22 implies that each V_n is a polynomial ring over R in the variables $\sigma_{1n}, \ldots, \sigma_{sn}$, since each U_n is a polynomial ring over R in the variables $\tau_{1n}, \ldots \tau_{sn}$. Thus

$$V_n := S[\sigma_{1n}, \ldots, \sigma_{sn}] = R[t_1, \ldots, t_s][\sigma_{1n}, \ldots, \sigma_{sn}] \overset{\lambda}{\cong} R[\sigma_{1n}, \ldots, \sigma_{sn}],$$

where λ is defined as in Diagram 17.20.1; that is, $\lambda(t_i) = \tau_i$, for each i.

17.6.3. Making the Prototype a Homomorphic Image. We apply the identifications of Diagram 17.20.1 to the Prototypes and Local Prototypes of Definitions 9.3.and 4.27 so that they have the form of Homomorphic Image Construction 17.2. All Prototypes satisfy the conditions of Noetherian Flatness Theorem 17.13 (Homomorphic Image version), and so they are equal to their Approximation Domains B.

The "Homomorphic Image" Prototypes are used to produce Homomorphic Image examples of noncatenary local Noetherian domains in Chapter 18.

We expand the Prototype Setting of Definition 9.3 to fit the Homomorphic Image Construction as in Revised Notation 17.20.

SETTING AND NOTATION 17.24. Let x be an indeterminate over a field k. Let r be a nonnegative integer and s a positive integer. Assume that $\tau_1, \ldots, \tau_s \in xk[[x]]$ are algebraically independent over $k(x)$ and let $y_1, \ldots, y_r$ and $t_1, \ldots, t_s$ be additional indeterminates. We define the following rings:

(17.24.a)
$$R := k[x, y_1, \ldots, y_r],\ R^* = k[y_1, \ldots, y_r][[x]],\ V = k(x, \tau_1, \ldots, \tau_s) \cap k[[x]].$$

Notice that R^* is the (x)-adic completion of R and V is a DVR by Remark 2.1.

By Prototype Theorem 9.2, the Prototype D of Definition 9.3 satisfies these equations:

(17.24.b)
$$\begin{aligned}
D := A_{\text{incl}} &:= k(x, y_1, \ldots, y_r, \tau_1, \ldots \tau_s) \cap R^* \\
&= (1 + xV[y_1, \ldots, y_r])^{-1} V[y_1, \ldots, y_r] \\
&= B_{\text{incl}} := (1 + xU_{\text{incl}})^{-1} U_{\text{incl}},
\end{aligned}$$

where $U_{\text{incl}} := \bigcup_{n \in \mathbb{N}} R[\tau_{1n}, \ldots \tau_{sn}]$, each τ_{in} is the n^{th} endpiece of τ_i and each $\tau_{in} \in R^*$, for $1 \leq i \leq s$. By Construction Properties Theorem 5.14.3, the ring R^* is the (x)-adic completion of each of the rings $A_{\text{incl}}, B_{\text{incl}}$ and U_{incl}.

Set $S := k[x, y_1, \ldots, y_r, t_1, \ldots, t_s]$, let S^* be the (x)-adic completion of S, and let $\sigma_i := t_i - \tau_i$, for each i. We define $I := (t_1 - \tau_1, \ldots, t_s - \tau_s)S^* = (\sigma_1, \ldots, \sigma_s)S^*$. Using Homomorphic Image Construction 17.2 and Section 17.2, we have:
(17.24.c)

$$A_{\text{hom}} := k(x, y_1, \ldots, y_r, t_1, \ldots, t_s) \cap (S^*/I), \quad B_{\text{hom}} := (1 + xU_{\text{hom}})^{-1}U_{\text{hom}},$$

where $U_{\text{hom}} := \bigcup_{n\in\mathbb{N}} S[\sigma_{1n}, \ldots, \sigma_{sn}]$, each σ_{in} is the n^{th} frontpiece of σ_i and each $\sigma_{in} \in \mathcal{Q}(S) \cap (S^*/I)$, for $1 \leq i \leq s$, by Proposition 17.9. The ideal $I := (t_1 - \tau_1, \ldots, t_s - \tau_s)S^*$ is a prime ideal of S^* and $S^*/I \cong k[y_1, \ldots, y_r][[x]]$. The fact that $\tau_1, \ldots, \tau_s$ are algebraically independent over $k(x)$ implies that $I \cap S = (0)$. By Construction Properties Theorem 17.11.4, the ring S^*/I is the (x)-adic completion of each of the rings $A_{\text{hom}}, B_{\text{hom}}$ and U_{hom}.

We state and prove a Homomorphic Image version of Prototype theorem 9.2. The proof uses that Prototypes constructed using Inclusion Construction notation, as in Equation 17.24.b, are isomorphic to Prototypes using Homomorphic Image notation, as in Equation 17.24.c.

PROTOTYPE THEOREM 17.25. (Homomorphic Image Version) *Assume Setting and Notation 17.24, with A_{hom} and B_{hom} defined as in Equation 17.24.c. Then*

1. *B_{hom} is a directed union of localizations of polynomial rings in $r + s + 1$ variables over k.*
2. *$B_{hom} = A_{hom}$ is Noetherian of dimension $r+1$ and is a localization of the polynomial ring $V[y_1, \ldots, y_r]$ over the DVR $V := k(x, \tau_1, \ldots, \tau_s) \cap k[[x]]$. Thus A_{hom} is a regular integral domain.*
3. *The canonical map $\alpha : S \hookrightarrow (S^*/I)[1/x]$ is flat.*
4. *If k has characteristic zero, then $B_{hom} = A_{hom}$ is excellent.*

PROOF. Proposition 17.22 implies $\lambda(B_{\text{hom}}) = B_{\text{incl}}$. By Prototype Theorem 9.2, B_{incl} is a directed union of localizations of polynomial rings in $r + s + 1$ variables over k. Since the map λ is an isomorphism, item 1 holds.

For item 2, Proposition 17.22 implies $\lambda(A_{\text{hom}}) = A_{\text{incl}}$ and $\lambda(B_{\text{hom}}) = B_{\text{incl}}$. By Theorem 9.2, $A_{\text{incl}} = B_{\text{incl}}$ and B_{incl} is a localization of $V[y_1, \ldots, y_r]$. Item 2 follows.

Item 3 follows from item 1 of Theorem 9.2 because the map $\alpha \hookrightarrow R^*$ corresponds to the map $S \hookrightarrow S^*/I$ under the identifications of Diagram 17.20.1. Item 4 follows from item 4 of Theorem 9.2. □

This leads to the following definition:

DEFINITION 17.26. Let r be a nonnegative integer and s a positive integer and let $x, y_1, \ldots, y_r$ and $t_1, \ldots, t_s$ be indeterminates over a field k. Set

$$S := k[x, y_1, \ldots, y_r, t_1, \ldots, t_s],$$

and let S^* be the (x)-adic completion of S. Assume $\tau_1, \ldots, \tau_s \in xk[[x]]$ are algebraically independent over $k(x)$, set $\sigma_i := t_i - \tau_i$, for each i and set

$$I := (t_1 - \tau_1, \ldots, t_s - \tau_s)S^* = (\sigma_1, \ldots, \sigma_s)S^*.$$

Then $A_{\text{hom}} := k(x, y_1, \ldots, y_r, t_1, \ldots, t_s) \cap (S^*/I)$ is the *Homomorphic Image Prototype* corresponding to $k, x, \{y_i\}_{i=1}^r, \{t_j\}_{j=1}^s$ and $\{\tau_j\}_{j=1}^s$.

By adjusting the Local Prototype of Definition 4.27 in the same way, we have a local version.

DEFINITION 17.27. Let r be a nonnegative integer and s a positive integer and let $x, y_1, \ldots, y_r$ and $t_1, \ldots, t_s$ be indeterminates over a field k. Assume that $\tau_1, \ldots, \tau_s \in xk[[x]]$ are algebraically independent over $k(x)$ and set

$$S := k[x, y_1, \ldots, y_r, t_1, \ldots, t_s]_{(x, y_1, \ldots, y_r, t_1, \ldots, t_s)}.$$

Let S^* be the (x)-adic completion of S, let $\sigma_i := t_i - \tau_i$, for each i, and set $I := (t_1 - \tau_1, \ldots, t_s - \tau_s)S^* = (\sigma_1, \ldots, \sigma_s)S^*$. Then

$$A_{\text{hom}} := k(x, y_1, \ldots, y_r, t_1, \ldots, t_s) \cap (S^*/I)$$

is the *Localized Homomorphic Image Prototype* corresponding to k and the variables $x, \{y_i\}_{i=1}^r, \{t_j\}_{j=1}^s$ and $\{\tau_j\}_{j=1}^s$.

LOCAL PROTOTYPE THEOREM 17.28. Localized Homomorphic Image version. *With the setting of Definition17.27, let $A = A_{hom}$ be the Localized Prototype. Then:*

(1) $A_{hom} = B_{hom} \cong V[y_1, \ldots, y_r]_{(x, y_1, \ldots, y_r)}$, *where* $V = k(x, \tau_1, \ldots, \tau_s) \cap k[[x]]$. *Thus A is an RLR.*
(2) *The canonical map*

$$\alpha : S = k[x, y_1, \ldots, y_r, t_1, \ldots, t_s]_{(x, y_1, \ldots, y_r, t_1, \ldots, t_s)} \hookrightarrow (S^*/I)[1/x]$$

is flat.
(3) *A is a nested union of localized polynomial rings*
(4) *If V is excellent, then A is excellent.*

In Remark 9.5 below, we note that V is not always excellent.

REMARK 17.29. There exists a one-dimensional Prototype A that fits Definition 17.27 (a Homomorphic Image Prototype) that is Noetherian but not excellent. To exhibit this ring, let k be a perfect field with characteristic $p > 0$, let $s = 1$, let $r \in \mathbb{N}$ and let $\tau = \tau_1$. This example corresponds to the example of Remark 9.5 formulated with Inclusion Construction 5.3 under the identifications of Diagram 17.20.1. As in Remark 9.5, $V = k(x, \tau) \cap D[[x]]$ is not excellent, and the Localized Prototype is also not excellent.

EXAMPLE 17.30. Let S be as in Localized Prototype Theorem 17.28. We have $t_1 - \tau_1, \ldots, t_s - \tau_s$ is a regular sequence in S^*, defined in Chapter 2. Let $I = (t_1 - \tau_1, \ldots, t_s - \tau_s)S^*$, as in Localized Prototype Theorem 17.28. Then Theorem 17.17 implies that $S \hookrightarrow (S^*/I^n)[1/x]$ is flat for each positive integer n. Using I^n in place of I, Theorem 17.28.2 implies the existence for every r and n in $\mathbb{N}$ of a Noetherian local domain A having dimension $r+1$ such that the (x)-adic completion A^* of A has nilradical $\mathbf{n}$ with $\mathbf{n}^{n-1} \neq (0)$.

Here are some more specific examples to which Prototype Theorems 17.25, 9.2 and 17.28 apply. Example 17.31 shows that the dimension of U can be greater than the dimension of B_{hom}.

EXAMPLES 17.31. Assume Setting and Notation 17.24. The identifications of Proposition 17.21 transform Inclusion Construction Prototypes into analogous Prototypes in the format of Homomorphic Image Construction 17.2.

(1) Let $S := k[x, t_1, \ldots, t_s]$, that is, there are no y variables, and let S^* denote the (x)-adic completion of S. Then $I = (t_1 - \tau_i, \ldots, t_s - \tau_s)$ and, by Theorem 17.25,

$$\begin{aligned} V := (S^*/I) \cap \mathcal{Q}(S) &= (k[t_1, \ldots, t_s][[x]]/(t_1 - \tau_i, \ldots, t_s - \tau_s)) \cap k(x, t_1, \ldots, t_s) \\ &= A_{\rm hom} = B_{\rm hom} = k(x, \tau_1, \ldots, \tau_s) \cap k[[x]]. \end{aligned}$$

The DVR V is also obtained by localizing $U = U_{\rm hom} = \bigcup_{n\in\mathbb{N}} S[\sigma_{1n}, \ldots, \sigma_{sn}]$ at the prime ideal xU; each σ_{in} is the $n^{\rm th}$ frontpiece of σ_i. In this example $S[1/x] = U[1/x]$ has dimension $s + 1$ and so $\dim U = s + 1$, while

$$\dim(S^*/I) = \dim A_{\rm hom} = \dim B_{\rm hom} = 1.$$

(2) Essentially the same example as in item 1 can be obtained by using Theorem 9.2 as follows. Let $R = k[x]$. Then $R^* = k[[x]]$, and

$$A_{\rm incl} = k(x, \tau_1, \ldots, \tau_s) \cap k[[x]] \quad \text{and} \quad A_{\rm incl} = B_{\rm incl},$$

by Theorem 9.2. In this case $U_{\rm incl}$ is a directed union of polynomial rings over k,

$$U_{\rm incl} \ = \ \bigcup_{n=1}^{\infty} k[x][\tau_{1n}, \ldots, \tau_{sn}],$$

where the τ_{in} are the $n^{\rm th}$ endpieces of the τ_i as in Section 5.2. By Proposition 17.9, the endpieces are related to the frontpieces of the homomorphic image construction.

(3) By taking $S = k[t_1, \ldots, t_s, x]_{(t_1,\ldots,t_s,x)}$, an $(s+1)$-dimensional regular local domain, and applying Localized Prototype Theorem 17.28, we obtain a modification of Example 17.31.1. In this case $S[1/x] = U[1/x]$ has dimension s, while we still have $S^*/I \cong k[[x]]$. Thus $\dim(S^*/I) = 1 = \dim A_{\rm hom} = \dim B_{\rm hom}$ whereas $\dim U = s + 1$.

One can also obtain a local version of Example 17.31.2 using the inclusion construction with $R = k[x]_{(x)}$ and applying Theorem 9.2. We again have $R^* = k[[x]]$.

With S as in either Example 17.31.1 or 17.31.3, the domains B_n constructed from S as in Section 17.2 of Chapter 17 are $(s + 1)$-dimensional regular local domains dominated by $k[[x]]$ and having k as a coefficient field. In either case, since $(S^*/I)[1/x]$ is a field, the extension $S \hookrightarrow (S^*/I)[1/x]$ is flat. Thus by Theorem 17.13 the family $\{B_n\}_{n\in\mathbb{N}}$ is a directed union of $(s + 1)$-dimensional regular local domains whose union B is Noetherian, and is, in fact a DVR.

(4) Assuming Setting and Notation 17.24 with the adjustment of Localized Prototype Theorem 17.28, let $r = 1$ and $y_1 = y$. Thus

$$S = k[x, y, t_1, \ldots, t_s]_{(x,y,t_1,\ldots,t_s)}.$$

Then $S^*/I \cong k[y]_{(y)}[[x]]$. By Theorem 17.28.2, the extension $S \hookrightarrow (S^*/I)[1/x]$ is flat. Let $V = k[[x]] \cap k(x, \tau_1, \ldots, \tau_s)$. Then V is a DVR and

$$(S^*/I) \cap \mathcal{Q}(S) \cong V[y]_{(x,y)}$$

is a 2-dimensional regular local domain that is the directed union of $(s + 2)$-dimensional regular local domains.

Exercises

(1) Let A be an integral domain and let $A \hookrightarrow B$ be an injective map to an extension ring B. For an ideal I of B, prove that the following are equivalent:
   (i) The induced map $A \to B/I$ is injective, and each nonzero element of A is regular on B/I.
   (ii) The field of fractions $\mathcal{Q}(A)$ of A naturally embeds in the total quotient ring $\mathcal{Q}(B/I)$ of B/I.

   If B is Noetherian, prove that conditions (i) and (ii) are also equivalent to the following condition:

   (iii) For each prime ideal P of B that is associated to I we have $P \cap A = (0)$.

(2) Let A be an integral domain and let $A \hookrightarrow B$ be an injective map to an extension ring B. Let I be an ideal of B having the property that $I \cap A = (0)$ and every nonzero element of A is a regular element on B/I. Let $C := \mathcal{Q}(A) \cap (B/I)$.
   (i) Prove that $C = \{a/b \mid a, b \in A,\ b \neq 0 \text{ and } a \in I + bB\ \}$.
   (ii) Assume that $J \subseteq I$ is an ideal of B having the property that every nonzero element of A is a regular element on B/J. Let $D := \mathcal{Q}(A) \cap (B/J)$. Prove that $D \subseteq C$.

   Suggestion: Item ii is immediate from item i. To see item i, observe that $bC = b(B/I) \cap \mathcal{Q}(A)$, and $a \in bC \iff a \in b(B/I) \iff a \in I + bB$.

(3) Assume the setting of Frontpiece Notation 17.7 and Definition 17.10. If J is a proper ideal of B, prove that JB^* is a proper ideal of B^*, where B^* is the (z)-adic completion of B.

(4) Assume the setting of Frontpiece Notation 17.7, and let W denote the set of elements of R^* that are regular on R^*/I. Prove that the natural homomorphism $\pi : R^* \to R^*/I$ extends to a homomorphism $\overline{\pi} : W^{-1}R^* \to W^{-1}(R^*/I)$.

(5) Describe Example 17.31.4 in terms of Inclusion Construction 5.3. In particular, determine the appropriate base ring R for this construction.

CHAPTER 18

Catenary local rings with geometrically normal formal fibers,

In this chapter, we consider the catenary property in a Noetherian local ring $(R, \mathbf{m})$ having geometrically normal formal fibers.[1] Recall that a ring R is *catenary* if, for every pair of comparable prime ideals $P \subset Q$ of R, every saturated chain of prime ideals from P to Q has the same length.The ring R is *universally catenary* if every finitely generated R-algebra is catenary. From Definition 3.29, the ring R has *geometrically normal*, respectively, *geometrically regular, formal fibers* if, for each prime P of R and for each finite algebraic extension k' of the field $k(P) := R_P/PR_P$, the ring $\widehat{R} \otimes_R k'$ is normal, respectively, regular. By Remark 3.32, regular fibers are normal.

If $(R, \mathbf{m})$ has geometrically normal formal fibers, we prove that the Henselization R^h of R is universally catenary,[2] and we relate the catenary and universally catenary properties of R to the fibers of the map $R \hookrightarrow R^h$. We present for each integer $n \geq 2$ an example of a catenary Noetherian local integral domain of dimension n that has geometrically regular formal fibers and is not universally catenary. We thank M. Brodmann and R. Sharp for raising a question on catenary and universally catenary rings that motivated our work in this chapter.

18.1. History, terminology and summary

Krull proves in [**88**] that every integral domain that is a finitely generated algebra over a field is catenary. Cohen proves in [**28**] that every complete Noetherian local ring is catenary. These results motivated the question of whether every Noetherian ring (or equivalently every Noetherian local integral domain) is catenary. Nagata answers this question by giving an example of a family of non-catenary Noetherian local domains in [**114**]; see also [**117**, Example 2, pages 203-205]. Each domain in this family is not integrally closed and has the property that its integral closure is catenary and Noetherian.

These examples of Nagata motivated the question of whether the integral closure of a Noetherian local domain is catenary. Work on this question continued for over 20 years with Ratliff being a leading researcher in this area, [**127**], [**128**].

[1]The material in this chapter comes from a paper we wrote that is included in a volume dedicated to Shreeram S. Abhyankar in celebration of his seventieth birthday. In his mathematical work Ram has opened up many avenues. In this chapter we are pursuing one of these related to power series and completions.

[2]The terms "Henselization" and "Henselian" are defined in Remarks 2.15.1 and Definition 2.13.

In 1980, T. Ogoma resolved this question by establishing the existence of a 3-dimensional Henselian Nagata local domain that is integrally closed but not catenary [**123**]. Heitmann in [**84**] gives a simplified presentation of Ogoma's example.

Heitmann in [**82**] obtains the following notable characterization of the complete Noetherian local rings that are the completion of a UFD. He proves that every complete Noetherian local ring $(T, \mathbf{n})$ that has depth at least two [3] and has the property that no element in the prime subring of T is a zerodivisor on T is the completion of a Noetherian local UFD. Let x, y, z, w be indeterminates over a field k, and let $T := k[[x, y, z, w]]/(xy, xz)$. Heitmann uses his result to establish the existence of a 3-dimensional Noetherian local UFD $(R, \mathbf{m})$ having completion T. It follows that R is catenary but not universally catenary [**82**, Theorem 9].

In Section 18.2 we present conditions for a Noetherian local ring $(R, \mathbf{m})$ to be universally catenary. Theorem 18.6 asserts that R is universally catenary if and only if the set Γ_R is empty, where

$$\Gamma_R := \{P \in \operatorname{Spec}(R^h) \mid \dim(R^h/P) < \dim(R/(P \cap R))\}.$$

We also prove that the subset Γ_R of $\operatorname{Spec} R^h$ is *stable under generalization* in the sense that, if $Q \in \Gamma_R$ and $P \in \operatorname{Spec} R^h$ is such that $P \subseteq Q$, then $P \in \Gamma_R$. Thus Γ_R satisfies a "strong" Going-down property.

In Theorem 18.7 we prove that a Noetherian local domain R having geometrically normal formal fibers is catenary but is not universally catenary if and only if the set Γ_R is nonempty and $\dim(R^h/P) = 1$ for each prime ideal P in Γ_R. We show in this case that Γ_R is a subset of the minimal primes of R^h. Since R^h is Noetherian, Γ_R is finite. Thus, as we observe in Corollary 18.8, if R is catenary but not universally catenary, then there exists a minimal prime $\widehat{q}$ of the $\mathbf{m}$-adic completion $\widehat{R}$ of R such that $\dim(\widehat{R}/\widehat{q}) = 1$. If R is catenary, each minimal prime $\widehat{q}$ of $\widehat{R}$ such that $\dim(\widehat{R}/\widehat{q}) \neq \dim(R)$ must have $\dim(\widehat{R}/\widehat{q}) = 1$.

Theorem 18.10 gives conditions such that the flatness and Noetherian properties for the integral domains associated with ideals $I_1, \ldots, I_n$ of an ideal-adic completion R^* in Homomorphic Image Construction 17.2 transfer to the integral domain associated with their intersection $I = I_1 \cap \cdots \cap I_n$. Similarly, in Theorem 18.12, we give conditions so that geometrically regular formal fibers for the constructed ring of ideals $I_1, \ldots, I_n$ transfer to rings constructed using the intersection $I = I_1 \cap \cdots \cap I_n$ have geometrically regular formal fibers. In Section 18.5, we use Theorem 18.10 to produce Noetherian local domains that are not universally catenary. In Section 18.6 we examine the depths of the constructed rings.

18.2. Geometrically normal formal fibers and the catenary property

Throughout this section $(R, \mathbf{m})$ is a Noetherian local ring with $\mathbf{m}$-adic completion $\widehat{R}$. The ring R is *formally equidimensional*, or in other terminology *quasi-unmixed*, provided $\dim(\widehat{R}/\widehat{q}) = \dim \widehat{R}$ for every minimal prime $\widehat{q}$ of $\widehat{R}$. Ratliff's Equidimension Theorem 3.19, that R is universal catenary if and only if R is formally equidimensional, is crucial for our work. We use Theorem 3.19 to prove:

THEOREM 18.1. *Let $(R, \mathbf{m})$ be a Henselian Noetherian local ring having geometrically normal formal fibers. Then:*

[3] See Definition 3.25.

(1) *For each prime ideal P of R, the extension $P\widehat{R}$ to the* $\mathbf{m}$*-adic completion of R is also a prime ideal.*
(2) *The ring R is universally catenary.*

PROOF. Item 2 follows from item 1 and Theorem 3.19. In order to prove item 1, observe that the completion of R/P is $\widehat{R}/P\widehat{R}$, and R/P is a Noetherian Henselian local integral domain having geometrically normal formal fibers. By passing from R to R/P, we see that for item 1 it suffices to prove: If R is a Henselian Noetherian local integral domain having geometrically normal formal fibers, then the completion $\widehat{R}$ of R is an integral domain.

For this, assume that R as above is an integral domain and let U be the nonzero elements of R. Since R has normal formal fibers, the ring $U^{-1}\widehat{R}$ is a normal Noetherian ring. Hence $U^{-1}\widehat{R}$ is a finite product of normal Noetherian domains by Remark 2.2, and so $U^{-1}\widehat{R}$ is reduced. Every element of U is a regular element of $\widehat{R}$ by the flatness of $\widehat{R}$ over R, and so $U^{-1}\widehat{R}$ has the same total quotient ring as $\widehat{R}$. Thus $\widehat{R}$ is reduced, and so the integral closure $\overline{R}$ of R is a finitely generated R-module by Remark 3.12.4b. Moreover, since R is Henselian, $\overline{R}$ is local; see Remark 2.15.5. Since $\overline{R}$ is an integrally closed integral domain, $\overline{R}$ is normal.

The completion $\widehat{\overline{R}}$ of $\overline{R}$ is $\widehat{R} \otimes_R \overline{R}$ by [**117**, (17.8)]. We show that the formal fibers of $\overline{R}$ are normal: Let $\overline{P} \in \operatorname{Spec} \overline{R}$ and let $P = \overline{P} \cap R$. Since $\overline{R}$ is a finite R-module, $k(\overline{P}) = \overline{R}_{\overline{P}}/\overline{P}R_{\overline{P}}$ is a finite $k(P)$-module, where $k(P) = R_P/PR_P$. Thus $k(\overline{P})$ is a finite field extension of $k(P)$. Since R has generically normal formal fibers,

$$\widehat{R} \otimes_R k(P) \otimes_{k(P)} k(\overline{P}) = \widehat{R} \otimes_R k(\overline{P}) = \widehat{R} \otimes_R \overline{R} \otimes_{\overline{R}} k(\overline{P}) = \widehat{\overline{R}} \otimes_{\overline{R}} k(\overline{P})$$

is a normal ring. That is, for each $\overline{P} \in \operatorname{Spec} \overline{R}$, the fiber ring of the map $\varphi : \overline{R} \to \widehat{\overline{R}}$ over $\overline{P}$ is normal. Since $\overline{R}$ is a normal ring and φ is a flat local homomorphism with normal fibers, it follows that $\widehat{\overline{R}}$ is normal by Theorem 3.23.3. Since $\widehat{\overline{R}}$ is local, $\widehat{\overline{R}}$ is an integral domain, by Remark 2.2. Also $\widehat{R}$ is a flat R-module, and so $\widehat{R} = \widehat{R} \otimes_R R$ is a subring of $\widehat{\overline{R}} = \widehat{R} \otimes_R \overline{R}$. Therefore $\widehat{R}$ is an integral domain, as desired for the completion of the proof of Theorem 18.1. □

REMARK 18.2. Let $(R, \mathbf{m})$ be a Noetherian local ring. An interesting result proved by Nagata establishes the existence of a one-to-one correspondence between the minimal primes of the Henselization R^h of R and the maximal ideals of the integral closure $\overline{R}$ of R; see Remarks 3.16.2. Moreover, if a maximal ideal $\overline{\mathbf{m}}$ of $\overline{R}$ corresponds to a minimal prime q of R^h, then the integral closure of the Henselian local domain R^h/q is the Henselization of $\overline{R}_{\overline{\mathbf{m}}}$; see [**117**, Ex. 2, page 188], [**111**]. Therefore $\operatorname{ht}(\overline{\mathbf{m}}) = \dim(R^h/q)$.

REMARK 18.3. Let $(R, \mathbf{m})$ be a Noetherian local ring, let $\widehat{R}$ denote the $\mathbf{m}$-adic completion of R, and let R^h denote the Henselization of R. The canonical map $R \hookrightarrow R^h$ is a regular map with zero-dimensional fibers by Remarks 13.28.2, and $\widehat{R}$ is also the completion of R^h with respect to its unique maximal ideal $\mathbf{m}^h = \mathbf{m}R^h$ by Remarks 2.15.1.

We prove that the following statements are equivalent:

(1) The map $R \hookrightarrow \widehat{R}$ has (geometrically) normal fibers.
(2) The map $R^h \hookrightarrow \widehat{R}$ has (geometrically) normal fibers.

Let P be a prime ideal of R and let $U = R \setminus P$. Then $PR^h = P_1 \cap \cdots \cap P_n$, where the P_i are the minimal prime ideals of PR^h. Then $P\widehat{R} = (\cap_{i=1}^n P_i)\widehat{R}$. Since $\widehat{R}$ is faithfully flat over R^h, finite intersections distribute over this extension, and $P\widehat{R} = \cap_{i=1}^n (P_i\widehat{R})$. Let $S = U^{-1}(\widehat{R}/P\widehat{R})$ denote the fiber over P in $\widehat{R}$ and let $q_i = P_iS$. The ideals $q_1, \ldots, q_n$ of S intersect in (0) and are pairwise comaximal because for $i \neq j$, $(P_i + P_j) \cap U \neq \emptyset$. Therefore $S \cong \prod_{i=1}^n (S/q_i)$. By Remark 2.2, a Noetherian ring is normal if and only if it is a finite product of normal Noetherian domains. Thus the fiber over P in $\widehat{R}$ is normal if and only if the fiber over each of the P_i in $\widehat{R}$ is normal.

COROLLARY 18.4. *Let R be a Noetherian local domain having geometrically normal formal fibers. Then*

(1) *The Henselization R^h of R is universally catenary.*
(2) *If the integral closure $\overline{R}$ of R is again local, then R is universally catenary.*

In particular, if R is a normal Noetherian local domain having geometrically normal formal fibers, then R is universally catenary.

PROOF. For item 1, the Henselization R^h of R is a Noetherian local ring having geometrically normal formal fibers by Remark 18.3, and so Theorem 18.1 implies that R^h is universally catenary. For item 2, if the integral closure of R is local, then, by Remark 18.2, the Henselization R^h has a unique minimal prime. Since R^h is universally catenary, the completion $\widehat{R}$ is equidimensional by Ratliff's Equidimensional Theorem 3.18, and hence R is universally catenary. □

Theorem 18.5 relates the catenary property of R to the height of maximal ideals in the integral closure of R.

THEOREM 18.5. *Let $(R, \mathbf{m})$ be a Noetherian local domain of dimension d and let $\overline{R}$ denote the integral closure of R. If $\overline{R}$ contains a maximal ideal $\overline{\mathbf{m}}$ with $\operatorname{ht}(\overline{\mathbf{m}}) = r \notin \{1, d\}$, then there exists a saturated chain of prime ideals in R of length $\leq r$. Hence in this case R is not catenary.*

PROOF. Since $\overline{R}$ has only finitely many maximal ideals [**117**, (33.10)], there exists $b \in \overline{\mathbf{m}}$ such that b is in no other maximal ideal of $\overline{R}$. Let $R' = R[b] \subseteq \overline{R}$ and let $\mathbf{m}' = \overline{\mathbf{m}} \cap R'$. Notice that $\overline{\mathbf{m}}$ is the unique prime ideal of $\overline{R}$ that contains $\mathbf{m}'$. We show that $\operatorname{ht} \mathbf{m}' = r$: Let $S = R' \setminus \mathbf{m}'$. The extension $R'_{\mathbf{m}'} \hookrightarrow S^{-1}\overline{R}$ is integral. Since $b \in \mathbf{m}'$ and the only maximal ideal of $\overline{R}$ that contains b is $\overline{\mathbf{m}}$, the ring $S^{-1}\overline{R}$ is local with maximal ideal $\overline{\mathbf{m}}(S^{-1}\overline{R})$. Since $S \subseteq \overline{R} \setminus \overline{\mathbf{m}}$ and $S^{-1}\overline{R}$ is integrally closed, we have $S^{-1}\overline{R} = \overline{R}_{\overline{\mathbf{m}}}$. Thus $R'_{\mathbf{m}'} \hookrightarrow \overline{R}_{\overline{\mathbf{m}}}$ is integral, and so $\dim(R'_{\mathbf{m}'}) = \dim(R_{\overline{\mathbf{m}}})$, by the Going-up Theorem [**103**, (9.3)]. Therefore $\operatorname{ht} \mathbf{m}' = r$.

Since R' is a finitely generated R-module and is birational over R, there exists a nonzero element $a \in \mathbf{m}$ such that $aR' \subseteq R$. It follows that $R[1/a] = R'[1/a]$. The maximal ideals of $R[1/a]$ have the form $PR[1/a]$, where $P \in \operatorname{Spec} R$ is maximal with respect to not containing a. For $P \in \operatorname{Spec} R$ such that $PR[1/a]$ is maximal in $R[1/a]$, there are no prime ideals strictly between P and $\mathbf{m}$ by Theorem 2.17. If $\operatorname{ht} P = h$, then there exists a saturated chain $(0) \subsetneq \cdots \subsetneq P \subsetneq \mathbf{m}$ of prime ideals of R of length $h + 1$. Thus, to show R is not catenary, it suffices to establish the existence of a maximal ideal of $R[1/a]$ having height different from $d - 1$. Since $R[1/a] = R'[1/a]$, the maximal ideals of $R[1/a]$ correspond to the prime ideals P' in

R' maximal with respect to not containing a. Since $\operatorname{ht} \mathbf{m}' > 1$, there exists $c \in \mathbf{m}'$ such that c is not in any minimal prime of aR' nor in any maximal ideal of R' other than $\mathbf{m}'$. Hence there exist prime ideals of R' containing c and not containing a. Let $P' \in \operatorname{Spec}(R')$ be maximal with respect to $c \in P'$ and $a \notin P'$. Then $P' \subset \mathbf{m}'$, so $\operatorname{ht} P' \leq r - 1 < d - 1$. It follows that there exists a saturated chain of prime ideals of R of length $\leq r$, and hence R is not catenary. □

THEOREM 18.6. *Let $(R, \mathbf{m})$ be a Noetherian local integral domain having geometrically normal formal fibers and let R^h denote the Henselization of R. Consider the set*

$$\Gamma_R \;:=\; \{P \in \operatorname{Spec}(R^h) \mid \dim(R^h/P) < \dim(R/(P \cap R))\}.$$

Then the following statements hold.

1. *For $\mathbf{p} \in \operatorname{Spec}(R)$, the ring $R/\mathbf{p}$ is not universally catenary if and only if there exists $P \in \Gamma_R$ such that $\mathbf{p} = P \cap R$.*
2. *The set Γ_R is empty if and only if R is universally catenary.*
3. *If $Q \in \Gamma_R$, then each prime ideal P of R^h such that $P \subseteq Q$ is also in Γ_R, that is, the subset Γ_R of $\operatorname{Spec} R^h$ is stable under generalization.*
4. *If $\mathbf{p} \subset \mathbf{q}$ are prime ideals in R and if there exists $Q \in \Gamma_R$ with $Q \cap R = \mathbf{q}$, then there also exists $P \in \Gamma_R$ with $P \cap R = \mathbf{p}$ and $P \subseteq Q$.*

PROOF. The map of $R/\mathbf{p}$ to its $\mathbf{m}$-adic completion $\widehat{R}/\mathbf{p}\widehat{R}$ factors through $R^h/\mathbf{p}R^h$. Since $R \hookrightarrow \widehat{R}$ has geometrically normal fibers, so does the map $R^h \hookrightarrow \widehat{R}$ by Remark 18.3. Theorem 18.1 implies that each prime ideal P of R^h extends to a prime ideal $P\widehat{R}$. Therefore, by Theorem 3.19, the ring $R/\mathbf{p}$ is universally catenary if and only if $R^h/\mathbf{p}R^h$ is equidimensional if and only if there does not exist $P \in \Gamma_R$ with $P \cap R = \mathbf{p}$. This proves items 1 and 2.

For item 3, let $P \in \operatorname{Spec} R^h$ be such that $P \subset Q$, and let $\operatorname{ht}(Q/P) = n$. Since the fibers of the map $R \hookrightarrow R^h$ are zero-dimensional, the contraction to R of an ascending chain of primes

$$P \;=\; P_0 \;\subsetneq\; P_1 \;\subsetneq\; \cdots \;\subsetneq P_n \;=\; Q$$

of R^h is a strictly ascending chain of primes from $\mathbf{p} := P \cap R$ to $\mathbf{q} := Q \cap R$. Hence $\operatorname{ht}(\mathbf{q}/\mathbf{p}) \geq n$. Since R^h is catenary, we have

$$\dim(R^h/P) \;=\; n + \dim(R^h/Q) \;<\; n + \dim(R/\mathbf{q}) \;\leq\; \dim(R/\mathbf{p}),$$

where the strict inequality is because $Q \in \Gamma_R$. Therefore $P \in \Gamma_R$.

It remains to prove item 4. The extension $R \hookrightarrow R^h$ is fairhfully flat, and so the extension satisfies the Going-down property, by Remark 2.31.10. Thus there exists a prime ideal P of R^h such that $P \subseteq Q$ and $P \cap R = \mathbf{p}$. By item 3, we have $P \in \Gamma_R$. □

Recall that the *dimension* of a prime ideal $\mathbf{p}$ of a ring R refers to the Krull dimension of the factor ring, that is, the dimension of $\mathbf{p}$ is $\dim(R/\mathbf{p})$.

THEOREM 18.7. *Let $(R, \mathbf{m})$ be a Noetherian local integral domain having geometrically normal formal fibers and let Γ_R be defined as in Theorem 18.6. The ring R is catenary but not universally catenary if and only if*

(i) *the set Γ_R is nonempty, and*
(ii) $\dim(R^h/P) = 1$, *for each prime ideal $P \in \Gamma_R$.*

If these conditions hold, then each $P \in \Gamma_R$ is a minimal prime of R^h, and Γ_R is a finite nonempty open subset of $\operatorname{Spec} R^h$.

PROOF. Assume that R is catenary but not universally catenary. By Theorem 18.6, the set Γ_R is nonempty and there exist minimal primes P of R^h such that $\dim(R^h/P) < \dim(R^h)$. By Remark 18.2, if a maximal ideal $\overline{\mathbf{m}}$ of $\overline{R}$ corresponds to a minimal prime P of R^h, then $\operatorname{ht}(\overline{\mathbf{m}}) = \dim(R^h/P)$. Since R is catenary, Theorem 18.5 implies that the height of each maximal ideal of the integral closure $\overline{R}$ of R is either one or $\dim(R)$. Therefore $\dim(R^h/P) = 1$ for each minimal prime P of R^h for which $\dim(R^h/P) \neq \dim(R^h)$. Item 4 of Theorem 18.6 implies each $P \in \Gamma_R$ is a minimal prime of R^h and $\dim(R^h/P) = 1$.

For the converse, assume that Γ_R is nonempty and each prime ideal $W \in \Gamma_R$ has dimension one. Then R is not universally catenary by item 2 of Theorem 18.6. By item 3 of Theorem 18.6, if $W \in \Gamma_R$ and $V \in \operatorname{Spec}(R^h)$ with $V \subsetneq W$, then $V \in \Gamma_R$. But then $\dim(R^h/W) = 1 = \dim(R^h/V)$ is a contradiction. Therefore every element of Γ_R is a minimal prime ideal of R^h; by item 4 of Theorem 18.6 every element of Γ_R lies over (0) in R.

To show R is catenary, it suffices to show for each nonzero nonmaximal prime ideal $\mathbf{p}$ of R that $\operatorname{ht}(\mathbf{p}) + \dim(R/\mathbf{p}) = \dim(R)$ [**103**, Theorem 31.4]. Let P be a minimal prime ideal of $\mathbf{p}R^h$ in R^h. Since R^h is flat over R with zero-dimensional fibers, $\operatorname{ht}(\mathbf{p}) = \operatorname{ht}(P)$. Thus P is nonzero and non-maximal. Let Q be a minimal prime of R^h with $Q \subseteq P$. Then $Q \cap R = (0)$. We show $Q \notin \Gamma_R$: If $Q \in \Gamma_R$, then $\dim(R^h/Q) = 1$ by assumption. Thus $0 \neq \dim(R^h/P \leq \dim(R^h/Q) = 1$, and so $Q = P$. But $P \cap R = \mathbf{p}$, which is nonzero, and $Q \cap R = (0)$, a contradiction. Therefore $Q \notin \Gamma_R$. Hence $\dim(R^h/Q) = \dim(R^h)$. Since R^h is catenary, it follows that $\operatorname{ht}(P) + \dim(R^h/P) = \dim(R^h)$. We also have that $P \notin \Gamma_R$, since $P \cap R \neq (0)$. Therefore $\dim(R/\mathbf{p}) = \dim(R^h/P)$, and so $\operatorname{ht}(\mathbf{p}) + \dim(R/\mathbf{p}) = \dim(R)$. Thus R is catenary. □

COROLLARY 18.8. *If R has geometrically normal formal fibers and is catenary but not universally catenary, then there exist minimal prime ideals q of the $\mathbf{m}$-adic completion $\widehat{R}$ of R such that* $\dim(\widehat{R}/q) = 1$.

PROOF. By Theorem 18.7, each prime ideal $Q \in \Gamma_R$ has dimension one and is a minimal prime of R^h. Moreover, $Q\widehat{R} := q$ is a minimal prime of $\widehat{R}$. Since $\dim(R^h/Q) = 1$, we have $\dim(\widehat{R}/q) = 1$. □

18.3. Flatness for the intersection of finitely many ideals

We assume the setting and notation of Homomorphic Image Construction 17.2 and Noetherian Flatness Theorem 17.13:

SETTING AND NOTATION 18.9. Let R be an integral domain with field of fractions $K := \mathcal{Q}(R)$. Let $z \in R$ be a nonzero nonunit such that $\bigcap_{n\geq 1} z^n R = (0)$, the (z)-adic completion R^* is Noetherian, and z is a regular element of R^*. Let I be an ideal of R^* having the property that $\mathbf{p} \cap R = (0)$ for each $\mathbf{p} \in \operatorname{Ass}(R^*/I)$. As in Frontpiece Notation 17.7.2 and Definition 17.10.1, let

$$U := \bigcup_{n=1}^{\infty} U_n, \quad B := \bigcup_{n=1}^{\infty} B_n = (1 + zU)^{-1}U, \quad \text{and} \quad A := K \cap (R^*/I).$$

As shown in Noetherian Flatness Theorem 17.13, flatness of a certain map is equivalent to $B = A$ and B is Noetherian, for the ring B of Setting 18.9. In Theorem 18.10, we give conditions for this flatness and the Noetherian property to transfer to an integral domain associated with an intersection of ideals.

THEOREM 18.10. *We assume Setting and Notation 18.9 for each of n ideals of the base ring R; thus R is an integral domain with field of fractions $K := \mathcal{Q}(R)$, the element $z \in R$ be a nonzero nonunit such that $\bigcap_{n\geq 1} z^n R = (0)$, the (z)-adic completion R^* is Noetherian, and z is a regular element of R^*, and $I_1, \ldots, I_n$ are ideals of R^* such that, for each $i \in \{1, \ldots, n\}$, each associated prime of R^*/I_i intersects R in (0). Also assume the map $R \hookrightarrow (R^*/I_i)[1/z]$ is flat for each i and that the localizations at z of the I_i are pairwise comaximal; that is, for all $i \neq j$, $(I_i + I_j)R^*[1/z] = R^*[1/z]$. Let $I := I_1 \cap \cdots \cap I_n$, $A := K \cap (R^*/I)$ and, for $i \in \{1, 2, \ldots n\}$, let $A_i := K \cap (R^*/I_i)$. Then*

1. *Each associated prime of R^*/I intersects R in (0).*
2. *The map $R \hookrightarrow (R^*/I)[1/z]$ is flat, and so the ring A is Noetherian and is equal to its associated approximation ring B. The (z)-adic completion A^* of A is R^*/I, and the (z)-adic completion A_i^* of A_i is R^*/I_i, for $i \in \{1, \ldots, n\}$.*
3. *The ring $A^*[1/z] \cong A_1^*[1/z] \times \cdots \times A_n^*[1/z]$. If $Q \in \operatorname{Spec}(A^*)$ and $z \notin Q$, then A_Q^* is a localization of precisely one of the A_i^*.*
4. *We have $A \subseteq A_1 \cap \cdots \cap A_n$ and $\cap_{i=1}^n A_i[1/z] \subseteq A_P$ for each $P \in \operatorname{Spec} A$ with $z \notin P$. Thus we have $A[1/z] = \cap_{i=1}^n A_i[1/z]$.*

PROOF. By Construction Properties Theorem 17.11.4, the (z)-adic completion A_i^* of A_i is R^*/I_i. Since $\operatorname{Ass}(R^*/I) \subseteq \bigcup_{i=1}^n \operatorname{Ass}(R^*/I_i)$, the condition on associated primes of Noetherian Flatness Theorem 17.13 holds for the ideal I; that is, item 1 holds.

For item 2, the natural R-algebra homomorphism $\pi : R^* \to \bigoplus_{i=1}^n (R^*/I_i)$ has kernel I. Further, the localization of π at z is onto because $(I_i + I_j)R^*[1/z] = R^*[1/z]$ for all $i \neq j$. Thus $(R^*/I)[1/z] \cong \bigoplus_{i=1}^n (R^*/I_i)[1/z] = \bigoplus_{i=1}^n (A_i^*)[1/z]$ is flat over R. Therefore A is Noetherian and is equal to its associated approximation ring B, by Noetherian Flatness Theorem 17.13, and $A^* = R^*/I$ is the (z)-adic completion of A, by Theorem 17.11.4.

For item 3, if $Q \in \operatorname{Spec}(A^*)$ and $z \notin Q$, then A_Q^* is a localization of

$$A^*[1/z] \cong A_1^*[1/z] \oplus \ \cdots \ \oplus \ A_n^*[1/z].$$

Every prime ideal of $\bigoplus_{i=1}^n A_i^*[1/z]$ has the form $Q_i A_i^*[1/z] \oplus \bigoplus_{j\neq i} A_j^*[1/z]$, where $Q_i \in \operatorname{Spec}(A_i^*)$ for a unique $i \in \{1, \ldots, n\}$. It follows that A_Q^* is a localization of A_i^* for precisely this i. That is, $A_Q^* = (A_i)_{Q_i}$.

Since R^*/I_i is a homomorphic image of R^*/I, we have that $A \subseteq A_i$, for each i. Let $P \in \operatorname{Spec} A$ with $z \notin P$. Since $A^* = R^*/I$ is faithfully flat over A, there exists $P^* \in \operatorname{Spec}(A^*)$ with $P^* \cap A = P$. Then $z \notin P^*$ implies $A_{P^*}^* = (A_i^*)_{P_i^*}$, where $P_i^* \in \operatorname{Spec}(A_i^*)$ for some $i \in \{1, \ldots, n\}$. Let $P_i = P_i^* \cap A_i$. Since $A_P \hookrightarrow A_{P^*}^*$ and $(A_i)_{P_i} \hookrightarrow (A_i^*)_{P_i^*}$ are faithfully flat, we have

$$A_P = A_{P^*}^* \cap K = (A_i^*)_{P_i^*} \cap K = (A_i)_{P_i} \supseteq (A_i)[1/z],$$

by Remark 2.31.9. It follows that $\bigcap_{i=1}^n A_i[1/z] \subseteq A_P$. Thus we have

$$\bigcap_{i=1}^{n} A_i[1/z] \;\subseteq\; \bigcap\{A_P \mid P \in \operatorname{Spec} A \text{ and } z \notin P\} \;=\; A[1/z].$$

Since $A[1/z] \subseteq A_i[1/z]$, for each i, we have $A[1/z] = \bigcap_{i=1}^n A_i[1/z]$. □

18.4. Regular maps and geometrically regular formal fibers

Proposition 18.11 shows that certain regularity conditions on the base ring R and the extension $R \hookrightarrow R^*/I$ in Noetherian Flatness Theorem 17.13 (Homomorphic Image Version) yield geometrically regular formal fibers for the constructed ring A.

PROPOSITION 18.11. *Let $R, z,\ R^*, A, B$ and I be as in Setting and Notation 18.9. Assume that the map $\psi_P : R_{P\cap R} \hookrightarrow (R^*/I)_P$ is regular, for each $P \in \operatorname{Spec}(R^*/I)$ with $z \notin P$. Then $A = B$ and moreover:*

1. *A is Noetherian and the map $A \longrightarrow A^* = R^*/I$ is regular.*
2. *If R is Noetherian semilocal with geometrically regular formal fibers and z is in the Jacobson radical of R, then A has geometrically regular formal fibers.*

PROOF. Since flatness is a local property by (2.31.1), and regularity of a map includes flatness, the map $\psi_z : R \hookrightarrow (R^*/I)[1/z]$ is flat. By Theorem 17.13, the intersection ring A is Noetherian with (z)-adic completion $A^* = R^*/I$. Hence $A \longrightarrow A^*$ is flat.

Let $Q \in \operatorname{Spec}(A)$, let $\mathbf{q} = Q \cap R$, let $k(Q)$ denote the field of fractions of A/Q, and let $A^*_{QA^*} = (A \setminus Q)^{-1}A^*$.

Case 1: $z \in Q$. Then $R/\mathbf{q} = A/Q = A^*/QA^*$. By Equation 3.22.0, we have

$$A^* \otimes_A k(Q) \;=\; \frac{A^*_{QA^*}}{QA^*_{QA^*}} \;=\; \frac{A_Q}{QA_Q} \;=\; k(Q).$$

Thus regularity holds in this case.

Case 2: $z \notin Q$. Let L be a finite algebraic field extension of $k(Q)$. We show the ring $A^* \otimes_A L$ is regular. There is a natural embedding $A^* \otimes_A k(Q) \hookrightarrow A^* \otimes_A L$. Let $W \in \operatorname{Spec}(A^* \otimes_A L)$ and let $W' = W \cap (A^* \otimes_A k(Q))$. We have maps

$$\operatorname{Spec}(A^* \otimes_A k(Q)) \overset{\theta,\cong}{\to} \operatorname{Spec}\big(\frac{A^*_{QA^*}}{QA^*_{QA^*}}\big) \quad \text{and} \quad \operatorname{Spec}\big(\frac{A^*_{QA^*}}{QA^*_{QA^*}}\big) \overset{\rho}{\to} \operatorname{Spec} A^*,$$

since $A^*_{QA^*}/QA^*_{QA^*} = A^* \otimes_A k(Q)$ by Equation 3.22.0, and $A^* \to A^*_{QA^*}/QA^*_{QA^*}$. Let P be the prime ideal $P := \rho(\theta(W')) \in \operatorname{Spec}(A^*)$; then $P \cap A = Q$.

By assumption the map

$$R_{\mathbf{q}} \hookrightarrow (R^*/I)_P = A^*_P$$

is regular. Since $z \notin Q$, it follows that $R_{\mathbf{q}} = U_{Q\cap U} = A_Q$ and that $k(\mathbf{q}) = k(Q)$. Thus the ring $A^*_P \otimes_{A_Q} L$ is regular. Therefore $(A^* \otimes_A L)_W$, which is a localization of this ring, is regular.

For item 2, we use a theorem of Rotthaus [**133**, (3.2), p. 179]: If R is a Noetherian semilocal ring with geometrically regular formal fibers and I_0 is an ideal of R contained in the Jacobson radical of R, then the I_0-adic completion of R also has geometrically regular formal fibers; see also [**103**, Remark 2, p. 260]. Thus R^* has

geometrically regular formal fibers. Since the formal fibers of R^*/I are a subset of the formal fibers of R^*, the map $A^* = R^*/I \longrightarrow \widehat{A} = \widehat{(R^*/I)}$ is regular. By item 1, the map $A \to A^*$ is regular. The composition of two regular maps is regular [**103**, Thm. 32.1 (i)]. Therefore A has geometrically regular formal fibers, that is, the map $A \longrightarrow \widehat{A}$ is regular. □

In Theorem 18.12, we give conditions so that the property of regularity of formal fibers for a Noetherian ring $A = B$ of Setting 18.9 transfers to an integral domain associated with an intersection of ideals.

THEOREM 18.12. *Let n be a positive integer, let R be a Noetherian integral domain with field of fractions K, let z be a nonzero nonunit of R, and let R^* denote the (z)-adic completion of R. Let $I_1, \ldots, I_n$ be ideals of R^* and let $I := I_1 \cap \cdots \cap I_n$. Assume that*

1. *For each each associated prime ideal P of R^*/I, we have $P \cap R = (0)$.*
2. *R is semilocal with geometrically regular formal fibers and z is in the Jacobson radical of R.*
3. *Each $(R^*/I_i)[1/z]$ is a flat R-module and, for each $i \neq j$, the ideals $I_i R^*[1/z]$ and $I_j R^*[1/z]$ are comaximal in $R^*[1/z]$.*
4. *For $i = 1, \ldots, n$, $A_i := K \cap (R^*/I_i)$ has geometrically regular formal fibers.*

Then $A := K \cap (R^*/I)$ is equal to its approximation domain B, and has geometrically regular formal fibers.

PROOF. Since R has geometrically regular formal fibers, it suffices to show, for $W \in \operatorname{Spec}(R^*/I)$ with $z \notin W$ and $W_0 := W \cap R$, that $R_{W_0} \longrightarrow (R^*/I)_W$ is regular, by Proposition 18.11.2. As in Theorem 18.10, we have

$$(R^*/I)[1/z] \;=\; (R^*/I_1)[1/z] \;\oplus \cdots \oplus\; (R^*/I_n)[1/z].$$

It follows that $(R^*/I)_W$ is a localization of R^*/I_i for some $i \in \{1, \ldots, n\}$. If $(R^*/I)_W = (R^*/I_i)_{W_i}$, where $W_i \in \operatorname{Spec}(R^*/I_i)$, then $R_{W_0} = (A_i)_{W_i \cap A_i}$ and $(A_i)_{W_i \cap A_i} \longrightarrow (R^*/I_i)_{W_i}$ is regular. Thus $R_{W_0} \longrightarrow (R^*/I)_W$ is regular. □

18.5. Examples that are not universally catenary

In this section we present non-excellent examples obtained using Prototypes in the terminology of Homomorphic Image Construction 17.2 as in Definition 17.27.

The ring A of Example 18.13 is a two-dimensional Noetherian local domain such that A birationally dominates a three-dimensional regular local domain, A has geometrically regular formal fibers, and A is not universally catenary. This example is obtained via an intersection of two ideals.

EXAMPLE 18.13. Let k be a field of characteristic zero, and let x, y and z be indeterminates over k. Let $R = k[x, y, z]_{(x,y,z)}$, let K denote the field of fractions of R, and let $\tau_1, \tau_2, \tau_3 \in xk[[x]]$ be algebraically independent over $k(x, y, z)$. Let R^* denote the (x)-adic completion of R. As in Definition 17.27 of Localized Homomorphic Image Prototype, we consider the two prime ideals $Q := (z - \tau_1, y - \tau_2)R^*$, which has height 2, and $P := (z - \tau_3)R^*$, which has height 1. Then R^*/P and R^*/Q are examples of the form considered in Examples 17.31. By Localized Prototype Theorem 17.28, $(R^*/P)[1/x]$ and $(R^*/Q)[1/x]$ are both flat over R. Here $R^*/P \cong k[y]_{(y)}[[x]]$ and $R^*/Q \cong k[[x]]$. The ring $V := k[[x]] \cap k(x, \tau_3)$ is a DVR, and

the Intersection Domain $A_1 := (R^*/P) \cap K \cong V[y]_{(x,y)}$ is a two-dimensional regular local domain that is a directed union of three-dimensional RLRs. The Intersection Domain $A_2 := (R^*/Q) \cap K$ is a DVR. By Theorem 17.28.4 and the characteristic zero assumption, the intersection rings A_1 and A_2 are excellent.

Since $\tau_1, \tau_3 \in xk[[x]]$, the ideal $(z - \tau_1, z - \tau_3)R^*$ has radical $(x, z)R^*$. Hence the ideal $P + Q$ is primary for the maximal ideal $(x, y, z)R^*$, and so, in particular, P is not contained in Q. If we take the ideal I to be the intersection of P and Q, then the representation $I = P \cap Q$ is irredundant and $\operatorname{Ass}(R^*/I) = \{P, Q\}$. Since $P \cap R = Q \cap R = (0)$, the ring R injects into R^*/I. Let $A := K \cap (R^*/I)$.

By Theorem 18.10.1, the inclusion $R \hookrightarrow (R^*/I)[1/x]$ is flat, the ring A is Noetherian, A equals its Approximation Domain B and A is a localization of a subring of $R[1/x]$. The map $A \hookrightarrow \widehat{A}$ of A into its completion factors through the map $A \hookrightarrow A^* = R^*/I$. Since R^*/I has minimal primes P/I and Q/I with $\dim R^*/P = 2$ and $\dim R^*/Q = 1$, and since $\widehat{A}$ is faithfully flat over $A^* = R^*/I$, we see that the ring $\widehat{A}$ is not equidimensional. It follows that A is not universally catenary by Ratliff's Equidimension Theorem 3.18. By Remark 3.20, every homomorphic image of a regular local ring, or even of a Cohen-Macaulay local ring, is universally catenary; thus A is not a homomorphic image of a regular local ring.

Finally we show that the ring $A = B$ of Example 18.13 has geometrically regular formal fibers; that is, the map $\phi : A \hookrightarrow \widehat{A}$ is regular. By the definition of R and the observations above, $A = B$ and A_1 and A_2 are excellent. Thus the hypotheses of Theorem 18.12 are satisfied, and so A has geometrically regular formal fibers.

REMARKS 18.14. The completion $\widehat{A}$ of the ring A of Example 18.13 has two minimal primes, one of dimension one and one of dimension two. As we observe above, A is not universally catenary by Ratliff's Equidimension Theorem 3.19. Another example of a Noetherian local domain that is not universally catenary but has geometrically regular formal fibers is given by Grothendieck in [**51**, (18.7.7), page 144] using a gluing construction; also see Greco's article[**50**, (1.1)]. We obtain rings similar to the ring A of Example 18.13 that have any finite number of minimal prime ideals and that are not universally catenary in Examples 18.16-18.18.

NOTES 18.15. We outline the general procedure used for the remaining examples of this section and give some justification here. Let $n \in \mathbb{N}$ and let R be a localized polynomial ring R over a field in $n + 1$ variables, where x is one of the variables. We use Definition 17.27 of Localized Homomorphic Image Prototype to obtain, for each i with $1 \leq i \leq n$, a suitable ideal I_i of the x-adic completion R^* of R and an integral domain A_i inside R^* associated to I_i so that the I_i and the A_i fit the hypotheses of Theorem 18.10, and so that the ring A of Theorem 18.10 associated to the intersection $I = \bigcap_{i=1}^{n} I_i$ has the desired properties. By Construction Properties Theorem 17.11.4, the (x)-adic completion A_i^* of A_i is R^*/I_i. If the dimensions of the A_i are not the same, we show in Examples 18.16-18.18 that the completion of A is not catenary.

If $\operatorname{char} k = 0$, the rings A_i are excellent by Theorem 17.28.5. Thus the A_i have generically regular formal fibers if $\operatorname{char} k = 0$. By Theorem 18.12, A has geometrically regular formal fibers. On the other hand, If k is a perfect field with $\operatorname{char} k \neq 0$, it follows from Remark 9.5 that each A_i is not a Nagata ring, and is not excellent.

We construct in Example 18.16 a two-dimensional Noetherian local domain having geometrically regular formal fibers such that the completion has any desired finite number of minimal primes of dimensions one and two.

EXAMPLE 18.16. Let r and s be positive integers and let R be the localized polynomial ring in three variables $R := k[x, y, z]_{(x,y,z)}$, where k is a field of characteristic zero and the field of fractions of R is $K := k(x, y, z)$. Then the (x)-adic completion of R is $R^* := k[y, z]_{(y,z)}[[x]]$. Let $\tau_1, \ldots, \tau_r, \beta_1, \beta_2, \ldots, \beta_s, \gamma \in xk[[x]]$ be algebraically independent power series over $k(x)$. Define, as in Definition 17.27,

$$Q_i \ := \ (z - \tau_i, y - \gamma)R^* \ \text{ and } \ P_j \ := \ (z - \beta_j)R^*,$$

for $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, s\}$. We apply Theorem 18.10 with $I_i = Q_i$, for $1 \leq i \leq r$, and $I_{r+j} = P_j$, for $1 \leq j \leq s$. Then $\{I_\ell \mid 1 \leq \ell \leq r + s\}$ satisfies the comaximality condition of Theorem 18.10 at the localization at x. As in Notes 18.15, Theorem 17.28 implies each map $R \hookrightarrow (R^*/I_\ell)[1/x]$ is flat and each $A_\ell := K \cap (R^*/I_\ell)$ is excellent. Let $I := I_1 \cap \cdots \cap I_{r+s}$ and $A := K \cap (R^*/I)$. By Theorem 18.10, the map $R \hookrightarrow R^*/I$ is flat and A is Noetherian. Since $I = \bigcap_{1 \leq \ell \leq r+s} I_\ell$ and $\widehat{R}$ is the completion of R^*, we have $I\widehat{R} = \bigcap_{1 \leq \ell \leq r+s}(I_\ell \widehat{R})$, by Remark 2.31.11. Since each R^*/I_ℓ is a regular local ring, the extension $I_\ell \widehat{R}$ is a prime ideal. We have

$$\widehat{A} \ = \ \widehat{A^*} \ = \ \widehat{R^*}/I\widehat{R^*} \ = \ \widehat{R}/I\widehat{R} \ = \ \widehat{R}/(\cap_{1 \leq \ell \leq r+s} I_\ell \widehat{R^*}).$$

Thus the minimal primes of $\widehat{A}$ all have the form $\mathbf{p}_\ell := I_\ell \widehat{A}$.

For J an ideal of R^* containing I, let $\bar{J}$ denote the image of J in R^*/I. Then, for each i with $1 \leq i \leq r$, $\dim(\, (R^*/I)/\bar{Q}_i) = \dim(R^*/Q_i) = 1$ and, for each j with $1 \leq j \leq s$, $\dim(\, (R^*/I)/\bar{P}_j) = 2$. Thus A^* contains r minimal primes of dimension one and s minimal primes of dimension two. Since A^* modulo each of its minimal primes is a regular local ring, the completion $\widehat{A}$ of A also has precisely r minimal primes of dimension one and s minimal primes of dimension two.

We show that the stated properties hold for the integral domain A. From the format of the general Homomorphic Image Construction 17.2 and the details of the construction of this integral domain A, we see that A birationally dominates the $(t + 1)$-dimensional regular local domain R and is birationally dominated by each of the A_i.

By the definition of R and the observations given in Proposition 18.11, the hypotheses of Theorem 18.12 are satisfied. Theorem 18.12 implies that A has geometrically regular formal fibers. Since $\dim(A) = 2$, A is catenary.

We show in Example 18.17 that for every integer $n \geq 2$ there is a Noetherian local domain $(A, \mathbf{m})$ of dimension n that has geometrically regular formal fibers and is catenary but not universally catenary.

EXAMPLE 18.17. Let $R = k[x, y_1, \ldots, y_n]_{(x,y_1,\ldots,y_n)}$ be a localized polynomial ring of dimension $n+1$ where k is a field of characteristic zero. Let $\sigma, \tau_1, \ldots, \tau_n$ be $n+1$ elements of $xk[[x]]$ that are algebraically independent over $k(x)$ and consider the ideals

$$I_1 = (y_1 - \sigma)R^* \quad \text{ and } \quad I_2 = (y_1 - \tau_1, \ldots, y_n - \tau_n)R^*.$$

of the ring $R^* = k[y_1, \ldots, y_n]_{(y_1,\ldots,y_n)}[[x]]$. Then the ring

$$A = k(x, y_1, \ldots, y_n) \cap (R^*/(I_1 \cap I_2))$$

is the desired example. As in Notes 18.15, each ring $k(x, y_1, \ldots, y_n) \cap R^*/I_i$ is excellent. By an argument similar to that of Example 18.16, the completion $\widehat{A}$ of A has two minimal primes, $I_1\widehat{A}$ having dimension n and $I_2\widehat{A}$ having dimension one. Therefore the Henselization A^h has precisely two minimal prime ideals P and Q, which we label so that $P\widehat{A} = I_1\widehat{A}$ and $Q\widehat{A} = I_2\widehat{A}$. Thus $\dim(A^h/P) = n$ and $\dim(A^h/Q) = 1$. By Theorem 18.7, A is catenary but not universally catenary. By Theorem 18.12, A has geometrically regular formal fibers.

In Example 18.18 we construct for each positive integer t and specified nonnegative integers $n_1, \ldots, n_t$ with $n_1 \geq 1$, a t-dimensional Noetherian local domain A that has geometrically regular formal fibers and birationally dominates a $t+1$-dimensional regular local domain such that the completion $\widehat{A}$ of A has, for each r with $1 \leq r \leq t$, exactly n_r minimal primes $\mathbf{p}_{rj}$ of dimension $t+1-r$. Moreover, each $\widehat{A}/\mathbf{p}_{rj}$ is a regular local ring of dimension $t+1-r$. If $n_i > 0$ for some $i \neq 1$, then A is not universally catenary and is not a homomorphic image of a regular local domain. It follows from Remark 18.2 that the derived normal ring $\overline{A}$ of A has exactly n_r maximal ideals of height $t+1-r$ for each r with $1 \leq r \leq t$.

EXAMPLE 18.18. Let t be a positive integer and let n_r be a nonnegative integer for each r with $1 \leq r \leq t$. Assume that $n_1 \geq 1$. We construct a t-dimensional Noetherian local domain A that has geometrically regular formal fibers such that $\widehat{A}$ has exactly n_r minimal primes of dimension $t+1-r$ for each r. Let $x, y_1 \ldots, y_t$ be indeterminates over a field k of characteristic zero.

Let $R = k[x, y_1, \ldots, y_t]_{(x,y_1,\ldots,y_t)}$, let $R^* = k[y_1, \ldots, y_t]_{(y_1,\ldots,y_t)}[[x]]$ denote the (x)-adic completion of R and let K denote the field of fractions of R. For every $r, j, i \in \mathbb{N}$ such that $1 \leq r \leq t$, $1 \leq j \leq n_r$ and $1 \leq i \leq r$, choose elements $\{\tau_{rji}\}$ of $xk[[x]]$ so that the set $\bigcup\{\tau_{rji}\}$ is algebraically independent over $k(x)$.

For each r, j with $1 \leq r \leq t$ and $1 \leq j \leq n_r$, define the prime ideal $P_{rj} := (y_1 - \tau_{rj1}, \ldots, y_r - \tau_{rjr})$ of height r in R^*. Notice that R^*/P_{rj} is a regular local ring of dimension $t+1-r$. Theorems 17.25 and 17.11.4 imply that the extension $R \hookrightarrow (R^*/P_{rj})[1/x]$ is flat, and that the intersection domain $A_{rj} := K \cap (R^*/P_{rj})$ is a regular local ring of dimension $t+1-r$ that has (x)-adic completion R^*/P_{rj}.

Let $I := \bigcap P_{rj}$ be the intersection of all the prime ideals P_{rj}. Since the $\tau_{rji} \in xk[[x]]$ are algebraically independent over $k(x)$, the sum of any two of these ideals P_{rj} and P_{mi}, where we may assume $r \leq m$, has radical $(x, y_1, \ldots, y_m)R^*$, and thus $(P_{rj} + P_{mi})R^*[1/x] = R^*[1/x]$. It follows that the representation of I as the intersection of the P_{rj} is irredundant and $\operatorname{Ass}(R^*/I) = \{P_{rj} \,|\, 1 \leq r \leq t, 1 \leq j \leq n_r\}$. Since each $P_{rj} \cap R = (0)$, we have $R \hookrightarrow R^*/I$, and the intersection domain $A := K \cap (R^*/I)$ is well defined. Moreover the x-adic completion A^* of A is R^*/I by Construction Properties Theorem 17.11.4.

By Theorem 18.10.2, the map $R \hookrightarrow (R^*/I)[1/x]$ is flat, A is Noetherian and A is a localization of a subring of $R[1/x]$. Since $I = \bigcap P_{rj}$ and $\widehat{R}$ is the completion of R^*, we have $I\widehat{R} = \bigcap P_{rj}\widehat{R}$ by Remark 2.31.11. Since R^*/P_{rj} is a regular local ring, the extension $P_{rj}\widehat{R}$ is a prime ideal. We have

$$\widehat{A} \;=\; \widehat{A^*} \;=\; \widehat{R^*}/I\widehat{R^*} \;=\; \widehat{R}/I\widehat{R} \;=\; \widehat{R}/(\cap P_{rj}\widehat{R^*}).$$

Thus the minimal primes of $\widehat{A}$ all have the form $\mathbf{p}_{rj} := P_{rj}\widehat{A}$. Since R^*/P_{rj} is a regular local ring of dimension $t+1-r$, each $\widehat{A}/\mathbf{p}_{rj}$ is a regular local ring of dimension $t+1-r$. The ring A birationally dominates the $(t+1)$-dimensional

regular local domain R. By Theorem 18.12, A has geometrically regular formal fibers.

REMARKS 18.19. (1) Examples 18.16 and 18.17 are special cases of Example 18.18. By Theorem 18.7, the ring A constructed in Example 18.18 is catenary if and only if each minimal prime of $\widehat{A}$ has dimension either one or t. By taking $n_r = 0$ for $r \notin \{1, t\}$ in Example 18.18, we obtain additional examples of catenary Noetherian local domains A of dimension t having geometrically regular formal fibers for which the completion $\widehat{A}$ has precisely n_t minimal primes of dimension one and n_1 minimal primes of dimension t; thus A is not universally catenary.

(2) Let $(A, \mathbf{n})$ be a Noetherian local domain constructed as in Example 18.18, let A^h denote the Henselization of A, and let A^* denote the x-adic completion of A. Since each minimal prime of $\widehat{A}$ is the extension of a minimal prime of A^h and also the extension of a minimal prime of A^*, the minimal primes of A^h and A^* are in a natural one-to-one correspondence. Let P be the minimal prime of A^h corresponding to a minimal prime $\mathbf{p}$ of A^*. Since the minimal primes of A^* extend to pairwise comaximal prime ideals of $A^*[1/x]$, for each prime ideal $Q \supset P$ of A^h with $x \notin Q$, the prime ideal P is the unique minimal prime of A^h contained in Q. Let $\mathbf{q} := Q \cap A$. We have $\operatorname{ht} \mathbf{q} = \operatorname{ht} Q$, and either $\dim(A/\mathbf{q}) > \dim(A^h/Q)$ or else every saturated chain of prime ideals of A containing $\mathbf{q}$ has length less than $\dim A$.

In connection with Remark 18.19.2, we ask:

QUESTION 18.20. Let $(A, \mathbf{n})$ be a Noetherian local domain constructed as in Example 18.18. If A is not catenary, what can be said about the cardinality of the set

$$\Gamma_A \ := \ \{P \in \operatorname{Spec}(A^h) \mid \dim(A^h/P) \ < \ \dim(A/(P \cap A))\}?$$

Is the set Γ_A ever infinite?

18.6. The depth of the constructed rings

We thank Lucho Avramov for suggesting we consider the depth of the rings constructed in Example 18.18; "depth" is defined in Definition 3.25.

REMARK 18.21. The catenary rings that arise from the construction in Example 18.18 all have depth one. However, Example 18.18 can be used to construct, for each integer $t \geq 3$ and integer d with $2 \leq d \leq t-1$, an example of a non-catenary Noetherian local domain A of dimension t and depth d having geometrically regular formal fibers. The (x)-adic completion A^* of A has precisely two minimal primes, one of dimension t and one of dimension d. To establish the existence of such an example, with notation as in Example 18.18, we set $m = t - d + 1$ and take $n_r = 0$ for $r \notin \{1, m\}$ and $n_1 = n_m = 1$. Let

$$P_1 := P_{11} = (y_1 - \tau_{111})R^* \text{ and } P_m := P_{m1} = (y_1 - \tau_{m11}, \ldots, y_m - \tau_{m1m})R^*.$$

Consider $A^* = R^*/(P_1 \cap P_m)$ and the short exact sequence

$$0 \longrightarrow \frac{P_1}{P_1 \cap P_m} \longrightarrow \frac{R^*}{P_1 \cap P_m} \longrightarrow \frac{R^*}{P_1} \longrightarrow 0.$$

Since P_1 is principal and not contained in P_m, we have $P_1 \cap P_m = P_1 P_m$ and $P_1/(P_1 \cap P_m) \cong R^*/P_m$. It follows that $\operatorname{depth} A^* = \operatorname{depth}(R^*/P_m) \ = d$; [**85**, page 103, ex 14] or [**23**, Prop. 1.2.9, page 11]. Since the local ring A and its (x)-adic

completion have the same completion $\widehat{A}$ with respect to their maximal ideals, we have $\operatorname{depth} A = \operatorname{depth} \widehat{A} = \operatorname{depth} A^*$ [**103**, Theorem 17.5]. By Remark 18.2, the derived normal ring $\overline{A}$ of A has precisely two maximal ideals one, of height t and one of height d.

Exercises

(1) Let $(R, \mathbf{m})$ be a three-dimensional Noetherian local domain such that each height-one prime ideal of R is the radical of a principal ideal. Prove that R is catenary.

(2) Let $(R, \mathbf{m})$ be a catenary Noetherian local domain having geometrically normal formal fibers. If R is not universally catenary, prove that R has depth one. **Suggestion**: Use Theorem 18.7 and the following theorem:

THEOREM 18.22. [**103**, Theorem 17.2] *Let $(R, \mathbf{m})$ be a Noetherian local ring and $M \neq (0)$ a finite R-module. Then $\operatorname{depth} M \leq \dim R/\mathbf{p}$, for every prime ideal $\mathbf{p}$ of R associated to M.*

(A prime ideal $\mathbf{p}$ of R is *associated to* M if $\mathbf{p}$ is the *annihilator* ideal in R of an element $x \in M$; that is, $\mathbf{p} = \{a \in R \mid ax = 0\}$.)

(3) Let R be an integral domain with field of fractions K and let R' be a subring of K that contains R. If $P \in \operatorname{Spec} R$ is such that $R' \subseteq R_P$, prove that there exists a unique prime ideal $P' \in \operatorname{Spec} R'$ such that $P' \cap R = P$.

(4) For the rings A and A^* of Example 18.13, prove that A^* is universally catenary.

CHAPTER 19

Multi-ideal-adic completions of Noetherian rings

In this chapter we consider a variation of the usual ideal-adic completion of a Noetherian ring R.[1] Instead of successive powers of a fixed ideal I, we use a *multi-adic* filtration formed from a more general descending sequence $\{I_n\}_{n=0}^{\infty}$ of ideals. We develop the mechanics of a *multi-adic* completion R^* of R. With additional hypotheses on the ideals of the filtration, we show that R^* is Noetherian. In the case where R is local, we prove that R^* is excellent, or Henselian or universally catenary if R has the stated property.

19.1. Ideal filtrations and completions

Let R be a commutative ring with identity. A *filtration* on R is a decreasing sequence $\{I_n\}_{n=0}^{\infty}$ of ideals of R. Associated to a filtration there is a well-defined completion

$$R^* = \varprojlim_n R/I_n,$$

and a canonical homomorphism $\psi : R \to R^*$, [**121**, Chapter 9]. If $\bigcap_{n=0}^{\infty} I_n = (0)$, then ψ is injective and R may be regarded as a subring of R^*, [**121**, page 401]. In the terminology of Northcott, a filtration $\{I_n\}_{n=0}^{\infty}$ is said to be *multiplicative* if $I_0 = R$ and $I_n I_m \subseteq I_{n+m}$, for all $m \geq 0$, $n \geq 0$, [**121**, page 408]. A well-known example of a multiplicative filtration on R is the I-adic filtration $\{I^n\}_{n=0}^{\infty}$, where I is a fixed ideal of R.

In this chapter we consider filtrations of ideals of R that are *not* multiplicative, and examine the completions associated to these filtrations. We assume the ring R is Noetherian. Instead of successive powers of a fixed ideal I, we use a filtration formed from a more general descending sequence $\{I_n\}_{n=0}^{\infty}$ of ideals. We require that, for each $n > 0$, the n^{th} ideal I_n is contained in the n^{th} power of the Jacobson radical of R, and that $I_{nk} \subseteq I_n^k$ for all $k, n \geq 0$. We call the associated completion a *multi-adic* completion, and denote it by R^*. The basics of the multi-adic construction and the relationship between this completion and certain ideal-adic completions are considered in Section 19.2. In Sections 19.3 and 19.4, we prove that the multi-adic completion R^* with respect to such ideals $\{I_n\}$ has the properties stated above

The process of passing to completion gives an analytic flavor to algebra. Often we view completions in terms of power series, or in terms of coherent sequences as in [**11**, pages 103-104]. Sometimes results are established by demonstrating for each n that they hold at the n^{th} stage in the inverse limit.

Multi-adic completions are interesting from another point of view. Many examples in commutative algebra can be considered as subrings of R^*/J, where R^* is

[1] The material in this chapter is adapted from our paper [**78**] dedicated to Melvin Hochster on the occasion of his 65^{th} birthday. Hochster's brilliant work has had a tremendous impact on commutative algebra.

a multi-adic completion of a localized polynomial ring R over a countable ground field and J is an ideal of R^*. In particular, certain counterexamples of Brodmann and Rotthaus, Heitmann, Nishimura, Ogoma, Rotthaus and Weston can be interpreted in this way, see [**20**], [**21**], [**84**], [**118**], [**120**], [**123**], [**124**], [**132**], [**133**], [**156**]. For many of these examples, a particular enumeration, $\{p_1, p_2, \dots\}$, of countably many non-associate prime elements is chosen and the ideals I_n are defined to be $I_n := (p_1 p_2 \dots p_n)^n$. The Noetherian property in these examples is a trivial consequence of the fact that every ideal of R^* that contains a power of one of the ideals I_n is extended from R. An advantage of R^* over the I_n-adic completion $\widehat{R}_n$ is that an ideal of R^* is more likely to be extended from R than is an ideal of $\widehat{R}_n$.

19.2. Basic mechanics for the multi-adic completion

SETTING 19.1. Let R be a Noetherian ring with Jacobson radical $\mathcal{J}$, and let $\mathbb{N}$ denote the set of positive integers. For each $n \in \mathbb{N}$, let Q_n be an ideal of R. Assume that the sequence $\{Q_n\}$ is descending, that is $Q_{n+1} \subseteq Q_n$, and that $Q_n \subseteq \mathcal{J}^n$, for each $n \in \mathbb{N}$. Also assume, for each pair of integers $k, n \in \mathbb{N}$, that $Q_{nk} \subseteq Q_n^k$.

Let $\mathcal{F} = \{Q_k\}_{k \geq 0}$ be a filtration

$$R = Q_0 \supseteq Q_1 \supseteq \cdots \supseteq Q_k \supseteq Q_{k+1} \supseteq \cdots$$

of R satisfying the conditions in the previous paragraph and let

$$R^* := \varprojlim_k R/Q_k \tag{19.1.1}$$

denote the completion of R with respect to $\mathcal{F}$.

Let $\widehat{R} := \varprojlim_k R/\mathcal{J}^k$ denote the completion of R with respect to the powers of the Jacobson radical $\mathcal{J}$ of R, and for each $n \in \mathbb{N}$, let

$$\widehat{R_n} := \varprojlim_k R/Q_n^k \tag{19.1.2}$$

denote the completion of R with respect to the powers of Q_n.

REMARK 19.2. Assume notation as in Setting 19.1. For each fixed $n \in \mathbb{N}$, we have

$$R^* = \varprojlim_k R/Q_k = \varprojlim_k R/Q_{nk},$$

where $k \in \mathbb{N}$ varies. This holds because the limit of a subsequence is the same as the limit of the original sequence.

We establish in Proposition 19.3 canonical inclusion relations among $\widehat{R}$ and the completions defined in (19.1.1) and (19.1.2).

PROPOSITION 19.3. *Let the notation be as in Setting 19.1. For each $n \in \mathbb{N}$, we have canonical inclusions*

$$R \subseteq R^* \subseteq \widehat{R_n} \subseteq \widehat{R_{n-1}} \subseteq \cdots \subseteq \widehat{R_1} \subseteq \widehat{R}.$$

PROOF. The inclusion $R \subseteq R^*$ is clear since the intersection of the ideals Q_k is zero. For the inclusion $R^* \subseteq \widehat{R_n}$, by Remark 19.2, $R^* = \varprojlim_k R/Q_{nk}$. Notice that

$$Q_{nk} \subseteq Q_n^k \subseteq Q_{n-1}^k \subseteq \cdots \subseteq \mathcal{J}^k.$$

□

To complete the proof of Proposition 19.3, we state and prove a general result about completions with respect to ideal filtrations (see also [**121**, Section 9.5]). We define the respective completions using coherent sequences as in [**11**, pages 103-104].

LEMMA 19.4. *Let R be a Noetherian ring with Jacobson radical $\mathcal{J}$ and let $\{H_k\}_{k\in\mathbb{N}}$, $\{I_k\}_{k\in\mathbb{N}}$ and $\{L_k\}_{k\in\mathbb{N}}$ be descending sequences of ideals of R such that, for each $k \in \mathbb{N}$, we have inclusions*

$$L_k \subseteq I_k \subseteq H_k \subseteq \mathcal{J}^k.$$

We denote the families of natural surjections arising from these inclusions as:

$$\delta_k : R/L_k \to R/I_k, \quad \lambda_k : R/I_k \to R/H_k \quad \text{and} \quad \theta_k : R/H_k \to R/\mathcal{J}^k,$$

and the completions with respect to these families as:

$$\widehat{R_L} := \varprojlim_k R/L_k, \quad \widehat{R_I} := \varprojlim_k R/I_k \quad \widehat{R_H} := \varprojlim_k R/H_k \quad \text{and} \quad \widehat{R} := \varprojlim_k R/\mathcal{J}^k.$$

Then

(1) *These families of surjections induce canonical injective maps Δ, Λ and Θ among the completions as shown in the diagram below.*
(2) *For each positive integer k we have a commutative diagram as displayed below, where the vertical maps are the natural surjections.*

$$\begin{array}{ccccccc} R/L_k & \xrightarrow{\delta_k} & R/I_k & \xrightarrow{\lambda_k} & R/H_k & \xrightarrow{\theta_k} & R/\mathcal{J}^k \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \widehat{R_L} & \xrightarrow{\Delta} & \widehat{R_I} & \xrightarrow{\Lambda} & \widehat{R_H} & \xrightarrow{\Theta} & \widehat{R}\,. \end{array}$$

(3) *The composition $\Lambda \cdot \Delta$ is the canonical map induced by the natural surjections $\lambda_k \cdot \delta_k : R/L_k \to R/H_k$. Similarly, the other compositions in the bottom row are the canonical maps induced by the appropriate natural surjections.*

PROOF. In each case there is a unique homomorphism of the completions. For example, the family of homomorphisms $\{\delta_k\}_{k\in\mathbb{N}}$ induces a unique homomorphism

$$\widehat{R_L} \xrightarrow{\Delta} \widehat{R_I}. \tag{19.1}$$

To define Δ, let $x = (x_k)_{k\in\mathbb{N}} \in \widehat{R_L}$, where each $x_k \in R/L_k$. Then $\delta_k(x_k) \in R/I_k$ and we define $\Delta(x) := (\delta_k(x_k))_{k\in\mathbb{N}} \in \widehat{R_I}$.

To show the maps on the completions are injective, consider for example the map Δ. Suppose $x = (x_k)_{k\in\mathbb{N}} \in \varprojlim_k R/L_k$ with $\Delta(x) = 0$. Then $\delta_k(x_k) = 0$ in R/I_k, that is, $x_k \in I_k R/L_k$, for every $k \in \mathbb{N}$. For $v \in \mathbb{N}$, consider the following commutative diagram:

$$\begin{array}{ccc} R/L_k & \xrightarrow{\delta_k} & R/I_k \\ {\scriptstyle \beta_{k,kv}}\uparrow & & \uparrow{\scriptstyle \alpha_{k,kv}} \\ R/L_{kv} & \xrightarrow{\delta_{kv}} & R/I_{kv} \end{array} \tag{19.2}$$

where $\beta_{k,kv}$ and $\alpha_{k,kv}$ are the canonical surjections associated with the inverse limits. We have $x_{kv} \in I_{kv}R/L_{kv}$. Therefore

$$x_k = \beta_{k,kv}(x_{kv}) \in I_{kv}(R/L_k) \subseteq \mathcal{J}^{kv}(R/L_k),$$

for every $v \in \mathbb{N}$. Since $\mathcal{J}(R/L_k)$ is contained in the Jacobson radical of R/L_k and R/L_k is Noetherian, we have

$$\bigcap_{v\in\mathbb{N}} \mathcal{J}^{kv}(R/L_k) = (0).$$

Therefore $x_k = 0$ for each $k \in \mathbb{N}$, and so Δ is injective. The remaining assertions are clear. □

LEMMA 19.5. *With R^* and $\widehat{R_n}$ as in Setting 19.1, we have*

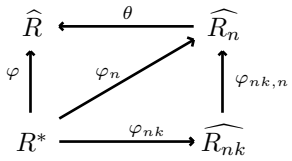
$$R^* = \bigcap_{n\in\mathbb{N}} \widehat{R_n}.$$

PROOF. The inclusion "$\subseteq$" is shown in Proposition 19.3. For the reverse inclusion, fix positive integers n and k, and let $L_\ell = Q_{nk\ell}$, $I_\ell = Q_{nk}^\ell$ and $H_\ell = Q_n^\ell$ for each $\ell \in \mathbb{N}$. Then $L_\ell \subseteq I_\ell \subseteq H_\ell \subseteq \mathcal{J}^\ell$, as in Lemma 19.4 and

$$\widehat{R_L} := \varprojlim_\ell R/Q_{nk\ell} = R^*, \quad \widehat{R_I} := \varprojlim_\ell R/Q_{nk}^\ell = \widehat{R_{nk}}, \quad \widehat{R_H} := \varprojlim_\ell R/Q_n^\ell = \widehat{R_n}.$$

(Also, as before, $\widehat{R} := \varprojlim_\ell R/\mathcal{J}^\ell$.) We define φ_n, φ_{nk}, $\varphi_{nk,n}$, θ and φ to be the canonical injective homomorphisms given by Lemma 19.4 among the rings displayed in the following diagram.

(19.5.1)



By Lemma 19.4, Diagram 19.5.1 is commutative.

Let $\widehat{y} \in \bigcap_{n\in\mathbb{N}} \widehat{R_n}$. We show that there is an element $\xi \in R^*$ such that $\varphi(\xi) = \widehat{y}$. This is sufficient to ensure that $\widehat{y} \in R^*$, since the maps θ_t are injective and Diagram 19.5.1 is commutative.

First, we define ξ: For each $t \in \mathbb{N}$, we have

$$\widehat{y} = (y_{1,t}, y_{2,t}, \dots,) \in \varprojlim_\ell R/Q_t^\ell = \widehat{R_t},$$

where $y_{1,t} \in R/Q_t$, $y_{2,t} \in R/Q_t^2$ and $y_{2,t} + Q_t/Q_t^2 = y_{1,t}$ in R/Q_t, $\cdots$ and so forth, is a coherent sequence as in [**11**, pp. 103-104]. Now take $z_t \in R$ so that $z_t + Q_t = y_{1,t}$. Thus $\widehat{y} - z_t \in Q_t\widehat{R_t}$. For positive integers s and t with $s \geq t$, we have $Q_s \subseteq Q_t$. Therefore $z_t - z_s \in Q_t\widehat{R_t} \cap R = Q_tR$. Thus $\xi := (z_t)_{t\in\mathbb{N}} \in R^*$. We have $\widehat{y} - z_t \in Q_t\widehat{R_t} \subseteq \mathcal{J}^t\widehat{R}$, for all $t \in \mathbb{N}$. Hence $\varphi(\xi) = \widehat{y}$. This completes the proof of Lemma 19.5. □

The following special case of Setting 19.1 is used by Brodmann, Heitmann, Nishimura, Ogoma, Rotthaus, and Weston for the construction of numerous examples.

SETTING 19.6. Let R be a Noetherian ring with Jacobson radical $\mathcal{J}$. For each $i \in \mathbb{N}$, let $p_i \in \mathcal{J}$ be a non-zero-divisor (that is, a regular element) on R.

For each $n \in \mathbb{N}$, let $q_n = (p_1 \cdots p_n)^n$. Let $\mathcal{F}_0 = \{(q_k)\}_{k \geq 0}$ be the filtration

$$R \supseteq (q_1) \supseteq \cdots \supseteq (q_k) \supseteq (q_{k+1}) \supseteq \cdots$$

of R and define $R^* := \varprojlim_k R/(q_k)$ to be the completion of R with respect to $\mathcal{F}_0$.

REMARK 19.7. In Setting 19.6, assume further that $R = K[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}$, a localized polynomial ring over a countable field K, and that $\{p_1, p_2, \ldots\}$ is an enumeration of all the prime elements (up to associates) in R. As in 19.6, let $R^* := \varprojlim_n R/(q_n)$, where each $q_n = (p_1 \cdots p_n)^n$.

The ring R^* is often useful for the construction of Noetherian local rings with a bad locus (regular, Cohen-Macaulay, normal). In particular, Brodmann, Heitmann, Nishimura, Ogoma, Rotthaus, and Weston make use of special subrings of this multi-adic completion R^* for their examples. The first such example was constructed by Rotthaus in [**132**]. In this paper, Rotthaus obtains a regular local Nagata ring A that contains a prime element ω so that the singular locus of the quotient ring $A/(\omega)$ is not closed. This ring A is situated between the localized polynomial ring R and its $*$-completion R^*; thus, in general R^* is bigger than R. In the Rotthaus example, the singular locus of $(A/(\omega))^*$ is defined by a height one prime ideal Q that intersects $A/(\omega)$ in (0). Since all ideals $Q + (p_n)$ are extended from $A/(\omega)$, the singular locus of $A/(\omega)$ is not closed.

REMARK 19.8. For R and R^* as in Remark 19.7, the ring R^* is also the "ideal-completion", or "R-completion of R. This completion is defined and used in the paper of Zelinsky [**166**], the work of Matlis [**99**] and [**100**], and the book of Fuchs and Salce[**43**]. The *ideal-topology*, or *R-topology* on an integral domain R is the linear topology defined by letting the nonzero ideals of R be a subbase for the open neighborhoods of 0. The nonzero principal ideals of R also define a subbase for the open neighborhoods of 0. Recent work on ideal completions has been done by Tchamna in [**153**]. In particular, Tchamna observes in [**153**, Theorem 4.1] that the ideal-completion of a countable Noetherian local domain is also a multi-ideal-adic completion.

19.3. Preserving Noetherian under multi-adic completion

THEOREM 19.9. *Let the notation be as in Setting 19.1. Then the ring R^* defined in (19.1.1) is Noetherian.*

PROOF. It suffices to show each ideal I of R^* is finitely generated. Since $\widehat{R}$ is Noetherian, there exist $f_1, \ldots, f_s \in I$ such that $I\widehat{R} = (f_1, \ldots, f_s)\widehat{R}$. Since $\widehat{R_n} \hookrightarrow \widehat{R}$ is faithfully flat, $I\widehat{R_n} = I\widehat{R} \cap \widehat{R_n} = (f_1, \ldots, f_s)\widehat{R_n}$, for each $n \in \mathbb{N}$.

Let $f \in I \subseteq R^*$. Then $f \in I\widehat{R_1}$, and so

$$f \;=\; \sum_{i=1}^{s} \widehat{b}_{i0} f_i,$$

where $\widehat{b}_{i0} \in \widehat{R_1}$. Consider R as "Q_0", and so $\widehat{b}_{i0} \in Q_0\widehat{R}_1$. Since $\widehat{R_1}/Q_1\widehat{R_1} \cong R/Q_1$, for all i with $1 \leq i \leq s$, we have $\widehat{b}_{i0} = a_{i0} + \widehat{c}_{i1}$, where $a_{i0} \in R = Q_0 R$ and

$\widehat{c}_{i1} \in Q_1\widehat{R_1}$. Then

$$f \;=\; \sum_{i=1}^{s} a_{i0}f_i \;+\; \sum_{i=1}^{s} \widehat{c}_{i1}f_i.$$

Notice that

$$\widehat{d_1} \;:=\; \sum_{i=1}^{s} \widehat{c}_{i1}f_i \in (Q_1I)\widehat{R_1} \cap R^* \subseteq \widehat{R_2}.$$

By the faithful flatness of the extension $\widehat{R_2} \hookrightarrow \widehat{R_1}$, we see $\widehat{d_1} \in (Q_1I)\widehat{R_2}$, and therefore there exist $\widehat{b}_{i1} \in Q_1\widehat{R_2}$ with

$$\widehat{d_1} = \sum_{i=1}^{s} \widehat{b}_{i1}f_i.$$

As before, using that $\widehat{R_2}/Q_2\widehat{R_2} \cong R/Q_2$, we can write $\widehat{b}_{i1} = a_{i1} + \widehat{c}_{i2}$, where $a_{i1} \in R$ and $\widehat{c}_{i2} \in Q_2\widehat{R}_2$. This implies that $a_{i1} \in Q_1\widehat{R_2} \cap R = Q_1$. We have:

$$f \;=\; \sum_{i=1}^{s} (a_{i0} + a_{i1})f_i + \sum_{i=1}^{s} \widehat{c}_{i2}f_i.$$

Now set

$$\widehat{d_2} \;:=\; \sum_{i=1}^{s} \widehat{c}_{i2}f_i.$$

Then $\widehat{d_2} \in (Q_2I)\widehat{R_2} \cap R^* \subseteq \widehat{R_3}$ and, since the extension $\widehat{R_3} \hookrightarrow \widehat{R_2}$ is faithfully flat, we have $\widehat{d_2} \in (Q_2I)\widehat{R_3}$. We repeat the process. By a simple induction argument,

$$f = \sum_{i=1}^{s} (a_{i0} + a_{i1} + a_{i2} + \ldots)f_i,$$

where $a_{ij} \in Q_j$ and $a_{i0} + a_{i1} + a_{i2} + \ldots \in R^*$. Thus $f \in (f_1, \ldots, f_s)R^*$. Hence I is finitely generated and R^* is Noetherian. □

COROLLARY 19.10. *With notation as in Setting 19.1, the maps $R \hookrightarrow R^*$, $R^* \hookrightarrow \widehat{R_n}$ and $R^* \hookrightarrow \widehat{R}$ are faithfully flat.*

We use Proposition 19.11 in the next section on preserving excellence.

PROPOSITION 19.11. *Assume notation as in Setting 19.1, and let the ring R^* be defined as in (19.1.1). If M is a finitely generated R^*-module, then*

$$M \cong \varprojlim_k \; (M/Q_kM),$$

that is, M is $*$-complete.

PROOF. If $F = (R^*)^n$ is a finitely generated free R^*-module, then one can see directly that

$$F \cong \varprojlim_k \; F/Q_kF,$$

and so F is $*$-complete.

Let M be a finitely generated R^*-module. Consider an exact sequence:

$$0 \longrightarrow N \longrightarrow F \longrightarrow M \longrightarrow 0,$$

where F is a finitely generated free R^*-module. This induces an exact sequence:

$$0 \longrightarrow \tilde{N} \longrightarrow F^* \longrightarrow M^* \longrightarrow 0,$$

where $\tilde{N}$ is the completion of N with respect to the induced filtration $\{Q_kF\cap N\}_{k\geq 0}$; see [**11**, (10.3)].

This gives a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \cong\downarrow & & \gamma\downarrow & & \\ 0 & \longrightarrow & \tilde{N} & \longrightarrow & F^* & \longrightarrow & M^* & \longrightarrow & 0 \end{array}$$

where γ is the canonical map $\gamma : M \longrightarrow M^*$. The diagram shows that γ is surjective. We have

$$\bigcap_{k=1}^{\infty}(Q_kM) \subseteq \bigcap_{k=1}^{\infty} J^kM = (0),$$

where the last equality is by [**11**, (10.19)]. Therefore γ is also injective. □

REMARK 19.12. Let the notation be as in Setting 19.1, and let B be a finite R^*-algebra. Let $\widehat{B_n} \cong B \otimes_{R^*} \widehat{R_n}$ denote the Q_n-adic completion of B. By Proposition 19.3, and Corollary 19.10, we have a sequence of inclusions:

$$B \hookrightarrow \cdots \hookrightarrow \widehat{B_{n+1}} \hookrightarrow \widehat{B_n} \hookrightarrow \ldots \hookrightarrow \widehat{B_1} \hookrightarrow \widehat{B},$$

where $\widehat{B}$ denotes the completion of B with respect to $\mathcal{J}B$. Let $\mathcal{J}_0$ denote the Jacobson radical of B. Since every maximal ideal of B lies over a maximal ideal of R^*, we have $\mathcal{J}B \subseteq \mathcal{J}_0$.

THEOREM 19.13. *With the notation of Setting 19.1, let B be a finite R^*-algebra and let $\widehat{B_n} \cong B \otimes_{R^*} \widehat{R_n}$ denote the Q_n-adic completion of B. Let $\widehat{I}$ be an ideal of $\widehat{B}$, let $I := \widehat{I} \cap B$, and let $I_n := \widehat{I} \cap \widehat{B_n}$, for each $n \in \mathbb{N}$. If $\widehat{I} = I_n\widehat{B}$, for all n, then $\widehat{I} = I\widehat{B}$.*

PROOF. By replacing B by B/I, we may assume that $(0) = I = \widehat{I} \cap B$. To prove the theorem, it suffices to show that $\widehat{I} = 0$.

For each $n \in \mathbb{N}$, we define ideals $\mathfrak{c}_n$ of $\widehat{B_n}$ and $\mathfrak{a}_n$ of B:

$$\mathfrak{c}_n := I_n + Q_n\widehat{B_n}, \qquad \mathfrak{a}_n := \mathfrak{c}_n \cap B.$$

Since $B/Q_nB = \widehat{B_n}/Q_n\widehat{B_n}$, the ideals of B containing Q_n are in one-to-one inclusion-preserving correspondence with the ideals of $\widehat{B_n}$ containing $Q_n\widehat{B_n}$, and so

$$\mathfrak{a}_n\widehat{B_n} = \mathfrak{c}_n, \quad \mathfrak{a}_{n+1}\widehat{B_n} = \mathfrak{a}_{n+1}\widehat{B_{n+1}}\widehat{B_n} = \mathfrak{c}_{n+1}\widehat{B_n}. \tag{19.13.1}$$

Since $\widehat{B}$ is faithfully flat over $\widehat{B_n}$ and $\widehat{I}$ is extended,

$$I_{n+1}\widehat{B_n} = (I_{n+1}\widehat{B}) \cap \widehat{B_n} = \widehat{I} \cap \widehat{B_n} = I_n. \tag{19.13.2}$$

Thus, for all $n \in \mathbb{N}$, we have, using (19.13.1), (19.13.2) and $Q_{n+1}\widehat{B_n} \subseteq Q_n\widehat{B_n}$:

$$\mathfrak{a}_n\widehat{B_n} = \mathfrak{c}_n = I_n + Q_n\widehat{B_n} = I_{n+1}\widehat{B_n} + Q_n\widehat{B_n} = \mathfrak{c}_{n+1}\widehat{B_n} + Q_n\widehat{B_n} = \mathfrak{a}_{n+1}\widehat{B_n} + Q_n\widehat{B_n}.$$

Since $\widehat{B_n}$ is faithfully flat over B, the equation above implies that

$$\mathfrak{a}_{n+1} + Q_nB = (\mathfrak{a}_{n+1}\widehat{B_n} + Q_n\widehat{B_n}) \cap B = \mathfrak{a}_n\widehat{B_n} \cap B = \mathfrak{a}_n. \tag{19.13.3}$$

Thus also

$$\mathfrak{a}_n\widehat{B} \subseteq \mathfrak{a}_{n+1}\widehat{B} + Q_n\widehat{B} \subseteq I_{n+1}\widehat{B} + Q_n\widehat{B} = \widehat{I} + Q_n\widehat{B}. \tag{19.13.4}$$

Now $Q_n \subseteq \mathcal{J}^n \widehat{B}$ and $\mathcal{J} \subseteq \mathcal{J}_0$, and so using (19.13.4)

$$\bigcap_{n\in\mathbb{N}} (\mathfrak{a}_n \widehat{B}) \subseteq \bigcap_{n\in\mathbb{N}} (\widehat{I} + Q_n \widehat{B}) \subseteq \bigcap_{n\in\mathbb{N}} (\widehat{I} + \mathcal{J}^n \widehat{B}) = \widehat{I}.$$

Since $\widehat{I} \cap B = (0)$, we have

$$0 = \widehat{I} \cap B \supseteq (\bigcap_{n\in\mathbb{N}} (\mathfrak{a}_n \widehat{B})) \cap B \supseteq \bigcap_{n\in\mathbb{N}} ((\mathfrak{a}_n \widehat{B}) \cap B) = \bigcap_{n\in\mathbb{N}} \mathfrak{a}_n,$$

where the last equality is because $\widehat{B}$ is faithfully flat over B. Thus $\bigcap_{n\in\mathbb{N}} \mathfrak{a}_n = (0)$.

Claim. $\widehat{I} = (0)$.

Proof of Claim. Suppose $\widehat{I} \neq 0$. Then there exists $d \in \mathbb{N}$ so that $\widehat{I} \nsubseteq \mathcal{J}_0^d \widehat{B}$. By hypothesis, $\widehat{I} = I_d \widehat{B}$, and so $I_d \widehat{B} \nsubseteq \mathcal{J}_0^d \widehat{B}$. Since $\widehat{B}$ is faithfully flat over $\widehat{B_d}$, we have $I_d \nsubseteq \mathcal{J}_0^d \widehat{B_d}$. By (19.13.1),

$$\mathfrak{a}_d \widehat{B_d} = \mathfrak{c}_d = I_d + Q_d \widehat{B_d} \nsubseteq \mathcal{J}_0^d \widehat{B_d},$$

and so there exists an element $y_d \in \mathfrak{a}_d$ with $y_d \notin \mathcal{J}_0^d$.

By (19.13.3), $\mathfrak{a}_{d+1} + Q_d B = \mathfrak{a}_d$. Hence there exists $y_{d+1} \in \mathfrak{a}_{d+1}$ and $q_d \in Q_d B$ so that $y_{d+1} + q_d = y_d$. Recursively we construct sequences of elements $y_n \in \mathfrak{a}_n$ and $q_n \in Q_n B$ such that $y_{n+1} + q_n = y_n$, for each $n \geq d$.

The sequence $\xi = (y_n + Q_n B) \in \varprojlim_n B/Q_n B = B$ corresponds to a nonzero element $y \in B$ such that, for every $n \geq d$, we have $y = y_n + g_n$, for some element $g_n \in Q_n B$. This shows that $y \in \mathfrak{a}_n$, for all $n \geq d$, and therefore $\bigcap_{n\in\mathbb{N}} \mathfrak{a}_n \neq 0$, a contradiction. Thus $\widehat{I} = (0)$. □

19.4. Preserving excellence or Henselian under multi-adic completion

The first four results of this section concern preservation of excellence.

THEOREM 19.14. *Assume notation as in Setting 19.1, and let the ring R^* be defined as in (19.1.1). If $(R, \mathbf{m})$ is an excellent local ring, then R^* is excellent.*

The following result is critical to the proof of Theorem 19.14.

LEMMA 19.15. [**103**, Theorem 32.5, page 259] *Let A be a semilocal Noetherian ring. Assume that $(\widehat{B})_Q$ is a regular local ring, for every local domain $(B, \mathbf{n})$ that is a localization of a finite A-algebra and for every prime ideal Q of the $\mathbf{n}$-adic completion $\widehat{B}$ such that $Q \cap B = (0)$. Then A is a G-ring, that is, $A \hookrightarrow \widehat{A_p}$ is regular for every prime ideal p of A; thus all of the formal fibers of all the local rings of A are geometrically regular.*

We use Proposition 19.16 in the proof of Theorem 19.14.

PROPOSITION 19.16. *Let $(R, \mathbf{m})$ be a Noetherian semilocal ring with geometrically regular formal fibers. Then R^* has geometrically regular formal fibers.*

PROOF. Let B be a domain that is a finite R^*-algebra and let $P \in \operatorname{Sing}(\widehat{B})$, that is, $\widehat{B}_P$ is not a regular local ring. To prove that R^* has geometrically regular formal fibers, by Lemma 19.15, it suffices to prove that $P \cap B \neq (0)$.

The Noetherian complete semilocal ring $\widehat{R}$ has the property J-2 in the sense of Matsumura, that is, for every finite $\widehat{R}$-algebra, such as $\widehat{B}$, the subset $\operatorname{Reg}(\operatorname{Spec}(\widehat{B}))$,

of primes where the localization of $\widehat{B}$ is regular, is an open subset in the Zariski topology; see [**101**, pp. 246–249]. Thus there is a reduced ideal $\widehat{I}$ in $\widehat{B}$ so that

$$\mathrm{Sing}(\widehat{B}) \;=\; \mathcal{V}(\widehat{I}).$$

If $\widehat{I} = (0)$, then $\widehat{B}$ is a reduced ring and, for all minimal primes Q of $\widehat{B}$, the localization $\widehat{B}_Q$ is a field, contradicting $Q \in \mathrm{Sing}(\widehat{B})$. Thus $\widehat{I} \neq (0)$. For all $n \in \mathbb{N}$:

$$\widehat{B_n} \cong \widehat{R_n} \otimes_{R^*} B$$

is a finite $\widehat{R}_n$-algebra. Since by [**131**] $\widehat{R_n}$ has geometrically regular formal fibers so has $\widehat{B_n}$. This implies that $\widehat{I}$ is extended from $\widehat{B_n}$ for all $n \in \mathbb{N}$. By Theorem 19.13, $\widehat{I}$ is extended from B, and so $\widehat{I} = I\widehat{B}$, where $0 \neq I := \widehat{I} \cap B$. Since $\widehat{I} \subseteq P$, we have $(0) \neq I \subseteq P \cap B$. □

Proof of Theorem 19.14 It remains to show that R^* is universally catenary. We have injective local homomorphisms $R \hookrightarrow R^* \hookrightarrow \widehat{R}$, and R^* is Noetherian with $\widehat{R^*} = \widehat{R}$. Proposition 19.17 below implies that R^* is universally catenary. □

PROPOSITION 19.17. *Let $(A, \mathbf{m})$ be a Noetherian local universally catenary ring and let $(B, \mathbf{n})$ be a Noetherian local subring of the $\mathbf{m}$-adic completion $\widehat{A}$ of A with $A \subseteq B \subseteq \widehat{A}$ and $\widehat{B} = \widehat{A}$, where $\widehat{B}$ is the $\mathbf{n}$-adic completion of B. Then B is universally catenary.*

PROOF. By [**103**, Theorem 31.7], it suffices to show for $P \in \mathrm{Spec}(B)$ that $\widehat{A}/P\widehat{A}$ is equidimensional. We may assume that $P \cap A = (0)$, and hence that A is a domain.

Let Q and W in $\mathrm{Spec}(\widehat{A})$ be minimal primes over $P\widehat{A}$.

Claim: $\dim(\widehat{A}/Q) = \dim(\widehat{A}/W)$.

Proof of Claim: Since B is Noetherian, the canonical morphisms $B_P \longrightarrow \widehat{A}_Q$ and $B_P \longrightarrow \widehat{A}_W$ are flat. By [**103**, Theorem 15.1],

$$\dim(\widehat{A}_Q) = \dim(B_P) + \dim(\widehat{A}_Q/P\widehat{A}_Q), \quad \dim(\widehat{A}_W) = \dim(B_P) + \dim(\widehat{A}_W/P\widehat{A_W}).$$

Since Q and W are minimal over $P\widehat{A}$, it follows that:

$$\dim(\widehat{A}_Q) = \dim(\widehat{A}_W) = \dim(B_P).$$

Let $q \subseteq Q$ and $w \subseteq W$ be minimal primes of $\widehat{A}$ so that:

$$\dim(\widehat{A}_Q) = \dim(\widehat{A}_Q/q\widehat{A}_Q) \quad \text{and} \quad \dim(\widehat{A}_W) = \dim(\widehat{A}_W/w\widehat{A}_W).$$

Since we have reduced to the case where A is a universally catenary domain, its completion $\widehat{A}$ is equidimensional and therefore:

$$\dim(\widehat{A}/q) = \dim(\widehat{A}/w).$$

Since a complete local ring is catenary [**103**, Theorem 29.4], we have:

$$\dim(\widehat{A}/q) = \dim(\widehat{A}_Q/q\widehat{A}_Q) + \dim(\widehat{A}/Q),$$
$$\dim(\widehat{A}/w) = \dim(\widehat{A}_W/w\widehat{A}_W) + \dim(\widehat{A}/W).$$

Since $\dim(\widehat{A}/q) = \dim(\widehat{A}/w)$ and $\dim(\widehat{A}_Q) = \dim(\widehat{A}_W)$, it follows that

$$\dim(\widehat{A}/Q) = \dim(\widehat{A}/W).$$

This completes the proof of Proposition 19.17. □

REMARK 19.18. Let R be a universally catenary Noetherian local ring. Proposition 19.17 implies that *every* Noetherian local subring B of $\widehat{R}$ with $R \subseteq B$ and $\widehat{B} = \widehat{R}$ is universally catenary. Hence, for each ideal I of R, the I-adic completion of R is universally catenary. Also R^* as in Setting 19.1 is universally catenary. Proposition 19.17 also implies that the Henselization of R is universally catenary. Seydi shows that the I-adic completions of universally catenary rings are universally catenary in [**141**]. Proposition 19.17 establishes this result for a larger class of rings.

PROPOSITION 19.19. *With notation as in Setting 19.1, let $(R, \mathbf{m}, k)$ be a Noetherian local ring. If R is Henselian, then R^* is Henselian.*

PROOF. Assume that R is Henselian. It is well known that every ideal-adic completion of R is Henselian, see [**132**, p.6]. Thus $\widehat{R_n}$ is Henselian for all $n \in \mathbb{N}$. Let $\mathbf{n}$ denote the nilradical of $\widehat{R}$. Then $\mathbf{n} \cap R^*$ is the nilradical of R^*, and to prove R^* is Henselian, it suffices to prove that $R' := R^*/(\mathbf{n} \cap R^*)$ is Henselian [**117**, (43.15)]. To prove R' is Henselian, by [**132**, Prop. 3, page 76], it suffices to show:

If $f \in R'[x]$ is a monic polynomial and its image $\bar{f} \in k[x]$ has a simple root, then f has a root in R'.

Let $f \in R'[x]$ be a monic polynomial such that $\bar{f} \in k[x]$ has a simple root. Since $\widehat{R_n}/(\mathbf{n} \cap \widehat{R_n})$ is Henselian, for each $n \in \mathbb{N}$, there exists $\widehat{\alpha}_n \in \widehat{R_n}/(\mathbf{n} \cap \widehat{R_n})$ with $f(\widehat{\alpha}_n) = 0$. Since f is monic and $\widehat{R}/\mathbf{n}$ is reduced, f has only finitely many roots in $\widehat{R}/\mathbf{n}$. Thus there is an α so that $\alpha = \widehat{\alpha}_n$, for infinitely many $n \in \mathbb{N}$. By Lemma 19.13, $R^* = \bigcap_{n\in\mathbb{N}} \widehat{R_n}$. Hence

$$R' = R^*/(\mathbf{n} \cap R^*) = \bigcap_{n\in\mathbb{N}} \widehat{R_n}/(\mathbf{n} \cap \widehat{R_n}),$$

and so there exists $\alpha \in R'$ such that $f(\alpha) = 0$. □

Exercise

(1) Let R denote the ring and $\{q_n\}$ the family of ideals given in Remark 19.7. Consider the linear topology obtained by letting the ideals q_n be a subbase for the open neighborhoods of 0. Prove the ideals q_n are also a subbase for the ideal-topology on R.

CHAPTER 20

Idealwise algebraic independence I, int

Let $(R, \mathbf{m})$ be an excellent normal local domain with field of fractions K and completion $(\widehat{R}, \widehat{\mathbf{m}})$. We consider three concepts of independence over R for elements $\tau_1, \ldots, \tau_n \in \widehat{\mathbf{m}}$ that are algebraically independent over K. The first of these, *idealwise independence*, is that $K(\tau_1, \ldots, \tau_n) \cap \widehat{R}$ equals the localized polynomial ring $R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)}$. If R is countable with $\dim(R) > 1$, we show the existence of an infinite sequence of elements $\tau_1, \tau_2, \ldots$ of $\widehat{\mathbf{m}}$ such that $\tau_1, \ldots, \tau_n$ are idealwise independent over R for each positive integer n. This implies that the subfield $K(\tau_1, \tau_2, \ldots)$ of $\mathcal{Q}(\widehat{R})$ has the property that the intersection domain $A = K(\tau_1, \tau_2, \ldots) \cap \widehat{R}$ is a localized polynomial ring in infinitely many variables over R. In particular, this intersection domain A is not Noetherian. These topics are continued in Chapter 21.

20.1. Idealwise independence, weakly flat and PDE extensions

We use the following setting throughout this chapter and Chapter 21.

SETTING AND NOTATION 20.1. Let $(R, \mathbf{m})$ be an excellent normal local domain with field of fractions K and completion $(\widehat{R}, \widehat{\mathbf{m}})$. Let $t_1, \ldots, t_n, \ldots$ be indeterminates over R, and assume that $\tau_1, \tau_2, \ldots, \tau_n, \ldots \in \widehat{\mathbf{m}}$ are algebraically independent over K. For each integer $n \geq 0$ and ∞, we consider the following localized polynomial rings:

$S_n := R[t_1, \ldots, t_n]_{(\mathbf{m}, t_1, \ldots, t_n)}$,
$R_n := R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)}$,
$S_\infty := R[t_1, \ldots, t_n, \ldots]_{(\mathbf{m}, t_1, \ldots, t_n, \ldots)}$ and
$R_\infty := R[\tau_1, \ldots, \tau_n, \ldots]_{(\mathbf{m}, \tau_1, \ldots, \tau_n, \ldots)}$.

For $n = 0$, we define $R_0 = R = S_0$. Of course, S_n is R-isomorphic to R_n and S_∞ is R-isomorphic to R_∞ with respect to the R-algebra homomorphism taking $t_i \to \tau_i$ for each i. When working with a particular n or ∞, we sometimes define S to be R_n or R_∞.

The completion $\widehat{S_n}$ of S_n is $\widehat{R}[[t_1, \ldots, t_n]]$, and we have the following commutative diagram:

$$\begin{array}{ccccc} & & S_n = R[t_1, \ldots, t_n]_{(\mathbf{m}, t_1, \ldots, t_n)} & \xrightarrow{\subseteq} & \widehat{S_n} = \widehat{R}[[t_1, \ldots, t_n]] \\ & & \cong \downarrow & & \lambda \downarrow \\ R & \xrightarrow{\subseteq} & S = R_n = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)} & \xrightarrow{\varphi} & \widehat{R}. \end{array}$$

Here the first vertical isomorphism is the R-algebra map taking $t_i \to \tau_i$, the restriction of the R-algebra surjection $\lambda : \widehat{S_n} \to \widehat{R}$ where

$$\mathfrak{p} := \ker(\lambda) = (t_1 - \tau_1, \ldots, t_n - \tau_n)\widehat{S_n}.$$

Note that $\mathfrak{p} \cap S_n = (0)$.

The central definition of this chapter is the following:

DEFINITION 20.2. Let $(R, \mathbf{m})$ and $\tau_1, \ldots, \tau_n \in \widehat{\mathbf{m}}$ be as in Setting 20.1. We say that $\tau_1, \ldots, \tau_n$ are *idealwise independent over* R if

$$\widehat{R} \ \cap \ K(\tau_1, \ldots, \tau_n) \ = \ R_n.$$

Similarly, an infinite sequence $\{\tau_i\}_{i=1}^{\infty}$ of algebraically independent elements of $\widehat{\mathbf{m}}$ is *idealwise independent over* R if $\widehat{R} \cap K(\{\tau_i\}_{i=1}^{\infty}) = R_\infty$.

REMARKS 20.3. Assume Setting and Notation 20.1.

(1) A subset of an idealwise independent set $\{\tau_1, \ldots, \tau_n\}$ over R is also idealwise independent over R. For example, to see that $\tau_1, \ldots, \tau_m$ are idealwise independent over R for $m \leq n$, let K denote the field of fractions of R and observe that

$$\begin{aligned}\widehat{R} \cap K(\tau_1, \ldots, \tau_m) &= \widehat{R} \cap K(\tau_1, \ldots, \tau_n) \cap K(\tau_1, \ldots, \tau_m) \\ &= R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)} \cap K(\tau_1, \ldots, \tau_m) = R[\tau_1, \ldots, \tau_m]_{(\mathbf{m}, \tau_1, \ldots, \tau_m)}.\end{aligned}$$

(2) Idealwise independence is a strong property of the elements $\tau_1, \ldots, \tau_n$ and of the embedding map $\varphi : R_n \hookrightarrow \widehat{R}$. It is often difficult to compute $\widehat{R} \cap L$ for an intermediate field L between the field K and the field of fractions of $\widehat{R}$. In order for $\widehat{R} \cap L$ to be the localized polynomial ring R_n, there can be no new quotients in $\widehat{R}$ other than those in $\varphi(R_n)$; that is, if $f/g \in \widehat{R}$ and $f, g \in R_n$, then $f/g \in R_n$. This does not happen, for example, if one of the τ_i is in the completion of R with respect to a principal ideal; in particular, if $\dim(R) = 1$, then there do not exist idealwise independent elements over R.

The following example, considered in Chapter 4, illustrates Remark 20.3.2. This is Example 4.10; other details are given in Remarks 4.11.

EXAMPLE 20.4. Let $R = \mathbb{Q}[x, y]_{(x,y)}$, the localized ring of polynomials in two variables over the rational numbers. The elements $\tau_1 = e^x - 1, \tau_2 = e^y - 1$, and $e^x - e^y = \tau_1 - \tau_2$ of $\widehat{R} = \mathbb{Q}[[x, y]]$ belong to completions of R with respect to principal ideals (and so are not idealwise independent). If $S = R_2 = \mathbb{Q}[x, y, \tau_1, \tau_2]_{(x,y,\tau_1,\tau_2)}$ and L is the field of fractions of S, then the elements $(e^x - 1)/x, (e^y - 1)/y$, and $(e^x - e^y)/(x - y)$ are certainly in $L \cap \widehat{R}$ but not in S. Theorem 4.8 implies that $L \cap \widehat{R}$ is a two-dimensional regular local ring with completion $\widehat{R}$.

Recall the concepts PDE, weakly flat and height-one preserving from Definitions 2.10 in Chapter 2 and 8.1 in Chapter 8. We state the definitions again here for Krull domains.

DEFINITIONS 20.5. Let $S \hookrightarrow T$ be an extension of Krull domains.

- T is a PDE extension of S if for every height-one prime ideal Q in T, the height of $Q \cap S$ is at most one.
- T is a *height-one preserving* extension of S if for every height-one prime ideal P of S with $PT \neq T$ there exists a height-one prime ideal Q of T with $PT \subseteq Q$.

- T is *weakly flat* over S if every height-one prime ideal P of S with $PT \neq T$ satisfies $PT \cap S = P$.

We summarize the results of this chapter.

SUMMARY 20.6. Let $(R, \mathbf{m})$ be an excellent normal local domain of dimension d with field of fractions K and completion $(\widehat{R}, \widehat{\mathbf{m}})$. In Section 20.1 we consider idealwise independent elements as defined in Definition 20.2. We show in Theorem 20.11 that $\tau_1, \ldots, \tau_n \in \widehat{\mathbf{m}}$ are idealwise independent over R if and only if the extension $R[\tau_1, \ldots, \tau_n] \hookrightarrow \widehat{R}$ is weakly flat in the sense of Definition 20.5. If R has the additional property that every height-one prime of R is the radical of a principal ideal, we show in Section 20.1 that a sufficient condition for $\tau_1, \ldots, \tau_n$ to be idealwise independent over R is that the extension $R[\tau_1, \ldots, \tau_n] \hookrightarrow \widehat{R}$ satisfies PDE ("pas d'éclatement", or in English "no blowing up"), defined in Definitions 20.5. At the end of Section 20.1 we display in a schematic diagram the relationships among these concepts and some others, for extensions of Krull domains.

In Sections 20.2 and 20.3 we present two methods for obtaining idealwise independent elements over a countable ring R. The method in Section 20.2 is to find elements $\tau_1, \ldots, \tau_n \in \widehat{\mathbf{m}}$ so that (1) $\tau_1, \ldots, \tau_n$ are algebraically independent over R, and (2) for every prime ideal P of $S = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)}$ with $\dim(S/P) = n$, the ideal $P\widehat{R}$ is $\widehat{\mathbf{m}}$-primary. In this case, we say that $\tau_1, \ldots, \tau_n$ are *primarily independent* over R. If R is countable and $\dim(R) > 2$, we show in Theorem 20.28 the existence over R of idealwise independent elements that fail to be primarily independent.

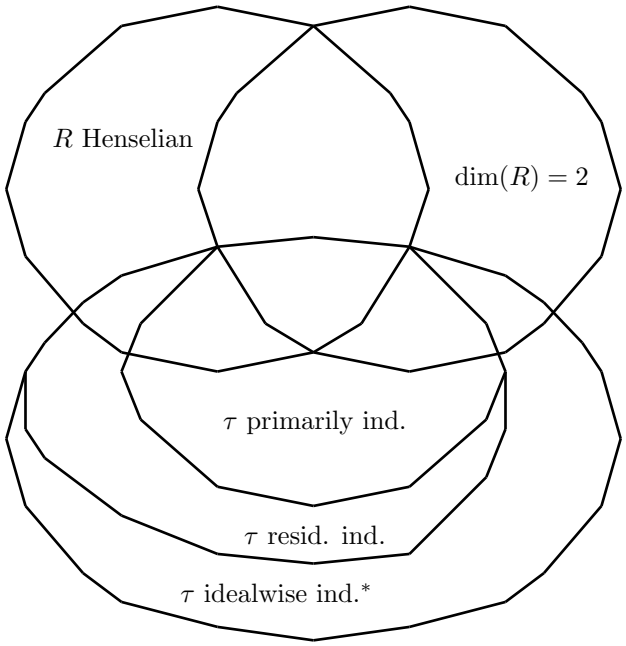
The main theorem of this chapter is Theorem 20.20: For every countable excellent normal local domain R of dimension at least two, there exists an infinite sequence $\tau_1, \tau_2, \ldots$ of elements of $\widehat{\mathbf{m}}$ that are primarily independent over R. It follows that $A = K(\tau_1, \tau_2, \ldots) \cap \widehat{R}$ is an infinite-dimensional non-Noetherian local domain. Thus, for the example $R = k[x, y]_{(x,y)}$ with k a countable field, there exists for every positive integer n and $n = \infty$, an extension $A_n = L_n \cap \widehat{R}$ of R such that $\dim(A_n) = \dim(R) + n$. In particular, the canonical surjection $\widehat{A_n} \to \widehat{R}$ has a nonzero kernel.

In Section 20.3 we define $\tau \in \widehat{\mathbf{m}}$ to be *residually algebraically independent* over R if τ is algebraically independent over R and, for each height-one prime ideal P of $\widehat{R}$ such that $P \cap R \neq 0$, the image of τ in $\widehat{R}/P$ is algebraically independent over $R/(P \cap R)$. We extend the concept of residual algebraic independence to a finite or infinite number of elements $\tau_1, \ldots, \tau_n, \ldots \in \widehat{\mathbf{m}}$ and observe the equivalence of residual algebraic independence to the extension $R[\tau_1, \ldots, \tau_n] \hookrightarrow \widehat{R}$ satisfying PDE.

We show that primary independence implies residual algebraic independence and that primary independence implies idealwise independence. If every height-one prime ideal of R is the radical of a principal ideal, we show that residual algebraic independence implies idealwise independence.

For R of dimension two, we show that primary independence is equivalent to residual algebraic independence. Hence residual algebraic independence implies idealwise independence if $\dim R = 2$. As remarked above, if R has dimension greater than two, then primary independence is stronger than residual algebraic independence. We show in Theorems 20.33 and 20.35 the existence of idealwise independent elements that fail to be residually algebraically independent.

The following diagram summarizes some relationships among the independence concepts for one element τ of $\widehat{\mathbf{m}}$, over a local normal excellent domain $(R, \mathbf{m})$. In the diagram we use "ind." and "resid." to abbreviate "independent" and "residually algebraic".



* In order to conclude that the idealwise independent set contains the residually algebraically independent set for $\dim R > 2$, we assume that every height-one prime of R is the radical of a principal ideal.

In Section 21.4 we include a diagram that displays many more relationships among the independence concepts and other related properties.

In the remainder of this section we discuss some properties of extensions of Krull domains related to idealwise independence. A diagram near the end of this section displays the relationships among these properties.

REMARK 20.7. Let $S \hookrightarrow T$ be an extension of Krull domains. If S is a UFD, or more generally, if every height-one prime ideal of S is the radical of a principal ideal, then T is a height-one preserving extension of S. This is clear from the fact that every minimal prime of a principal ideal in a Krull domain is of height one.

REMARK 20.8. Let $(R, \mathbf{m})$ and $\tau_1, \ldots, \tau_n \in \widehat{\mathbf{m}}$ be as in Setting 20.1. Also assume that each height-one prime of R is the radical of a principal ideal. Since this property is preserved in a polynomial ring extension, Remark 20.7 implies that the embedding

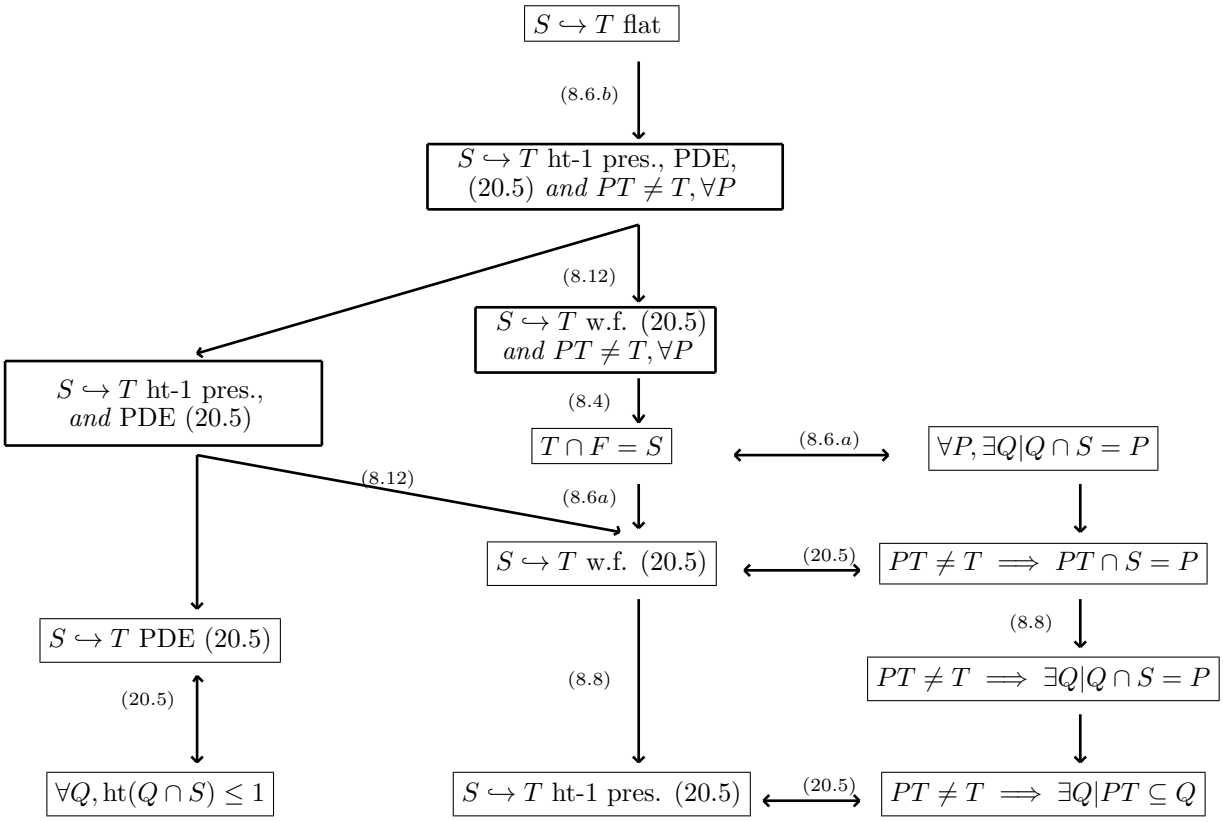
$$\varphi : R_n = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)} \hookrightarrow \widehat{R}$$

is a height-one preserving extension.

Corollary 20.9 is immediate from Remark 20.8 and Proposition 8.12.

COROLLARY 20.9. *Let $(R, \mathbf{m})$ and $\tau_1, \ldots, \tau_n \in \widehat{\mathbf{m}}$ be as in Setting 20.1. Assume that each height-one prime of R is the radical of a principal ideal. Let $S = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)}$. If $S \hookrightarrow \widehat{R}$ satisfies PDE, then $\widehat{R}$ is weakly flat over S.*

Let $S \hookrightarrow T$ be an extension of Krull domains, and let F be the field of fractions of S. Throughout the diagram "Q" denotes a prime ideal $Q \in \operatorname{Spec}(T)$ with $\operatorname{ht}(Q) = 1$, and "P" denotes $P \in \operatorname{Spec}(S)$ with $\operatorname{ht}(P) = 1$. The following diagram illustrates the relationships among the terms in Definitions 20.5 using the results (8.12), (8.4), (8.6), and (8.8):



The relationships among properties of an extension $S \hookrightarrow T$ of Krull domains

REMARK 20.10. Let S be a Krull domain, and let $S \hookrightarrow T$ be an extension of commutative rings such that every nonzero element of S is regular on T. In Corollary 8.4 the condition that $PT \neq T$ for every height-one prime ideal P of S relates weak flatness to $S = \mathcal{Q}(S) \cap T$. This condition holds if S and T are local Krull domains with T dominating S, and so it holds for $R_n \hookrightarrow \widehat{R}$ as in Setting 20.1.

Summarizing from Corollaries 20.9 and 8.4, we have the following implications among the concepts of weakly flat, PDE and idealwise independence in Setting 20.1:

THEOREM 20.11. *Let $(R, \mathbf{m})$ be an excellent normal local domain with $\mathbf{m}$-adic completion $(\widehat{R}, \widehat{\mathbf{m}})$ and let $\tau_1, \ldots, \tau_n \in \widehat{\mathbf{m}}$ be algebraically independent elements over R. Then:*

(1) *$\tau_1, \ldots, \tau_n$ are idealwise independent over $R \iff R[\tau_1, \ldots, \tau_n] \hookrightarrow \widehat{R}$ is weakly flat.*
(2) *If $R[\tau_1, \ldots, \tau_n] \hookrightarrow \widehat{R}$ satisfies PDE and each height-one prime of R is the radical of a principal ideal, then $R[\tau_1, \ldots, \tau_n] \hookrightarrow \widehat{R}$ is weakly flat.*

In view of Remark 8.6.b, these assertions also hold with $R[\tau_1, \ldots, \tau_n]$ replaced by its localization $R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)}$.

In order to demonstrate idealwise independence we develop in the next two sections the concepts of primary independence and residual algebraic independence. Primary independence implies idealwise independence. If we assume that every height-one prime ideal of the base ring R is the radical of a principal ideal, then residual algebraic independence implies idealwise independence.

20.2. Primarily independent elements

In this section we introduce primary independence, a concept we show in Proposition 20.15 implies idealwise independence. We construct in Theorem 20.20 infinitely many primarily independent elements over any countable excellent normal local domain of dimension at least two.

DEFINITION 20.12. Let $(R, \mathbf{m})$ be an excellent normal local domain. We say that elements $\tau_1, \ldots, \tau_n \in \widehat{\mathbf{m}}$ that are algebraically independent over R are *primarily independent over R*, if for every prime ideal P of $S = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)}$ such that $\dim(S/P) \leq n$, the ideal $P\widehat{R}$ is $\widehat{\mathbf{m}}$-primary. A countably infinite sequence $\{\tau_i\}_{i=1}^{\infty}$ of elements of $\widehat{\mathbf{m}}$ is *primarily independent over R* if $\tau_1, \ldots, \tau_n$ are primarily independent over R for each n.

REMARKS 20.13. (1) Referring to the diagram in Setting 20.1, primary independence of $\tau_1, \ldots, \tau_n$ as defined in (20.12) is equivalent to the statement that for every prime ideal P of S with $\dim(S/P) \leq n$, the ideal $\lambda^{-1}(P\widehat{R}) = P\widehat{S_n} + \ker(\lambda)$ is primary for the maximal ideal of $\widehat{S_n}$.

(2) A subset of a primarily independent set is again primarily independent. For example, if $\tau_1, \ldots, \tau_n$ are primarily independent over R, to see that $\tau_1, \ldots, \tau_{n-1}$ are primarily independent, let P be a prime ideal of R_{n-1} with $\dim(R_{n-1}/P) \leq n-1$. Then PR_n is a prime ideal of R_n with $\dim(R_n/PR_n) \leq n$, and so $P\widehat{R}$ is primary for the maximal ideal of $\widehat{R}$.

LEMMA 20.14. *Let $(R, \mathbf{m})$ be an excellent normal local domain of dimension at least 2, let n be a positive integer, and let $S = R_n = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)}$, where $\tau_1, \ldots, \tau_n$ are primarily independent over R. Let P be a prime ideal of S such that $\dim(S/P) \geq n+1$. Then*

(1) *the ideal $P\widehat{R}$ is not $\widehat{\mathbf{m}}$-primary, and*
(2) *$P\widehat{R} \cap S = P$.*

PROOF. For item 1, if $\dim(S/P) \geq n+1$ and if $P\widehat{R}$ is primary for $\widehat{\mathbf{m}}$, then the diagram in Setting 20.1 shows that $\lambda^{-1}(P\widehat{R}) = P\widehat{S}_n + \ker(\lambda)$ is primary for the maximal ideal of $\widehat{S}$. Hence the maximal ideal of $\widehat{S}/P\widehat{S}$ is the radical of an

n-generated ideal. We also have $\widehat{S_n}/P\widehat{S_n} \cong \widehat{(S/P)}$ is the completion of S/P, and $\dim(S/P) \geq n+1$ implies that $\dim(\widehat{S/P}) \geq n+1$. This is a contradiction by Theorem 2.17.

For item 2, if $\dim(S/P) = n+1$, and $P \subsetneq (P\widehat{R} \cap S)$, then $\dim(\frac{S}{(P\widehat{R}\cap S)}) \leq n$. Thus $P\widehat{R} = (P\widehat{R} \cap S)\widehat{R}$ is primary for $\widehat{\mathbf{m}}$, a contradiction to item 1. Therefore $P\widehat{R} \cap S = P$ for each P such that $\dim(S/P) = n+1$.

Assume that $\dim(S/P) > n+1$ and let

$$\mathcal{A} \ := \ \{Q \in \operatorname{Spec} S \mid P \subset Q \text{ and } \dim(S/Q) \ = \ n+1\}.$$

Proposition 3.21 implies that $P = \bigcap_{Q\in\mathcal{A}} Q$. Since for each prime ideal $Q \in \mathcal{A}$, we have $Q\widehat{R} \cap S = Q$, it follows that

$$P \subseteq P\widehat{R} \cap S = (\bigcap_{Q\in\mathcal{A}} Q)\widehat{R} \cap S \subseteq \bigcap_{Q\in\mathcal{A}} Q = P.$$

□

PROPOSITION 20.15. *Let $(R, \mathbf{m})$ be an excellent normal local domain of dimension at least 2.*

1. *Let n be a positive integer, and let $S = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m},\tau_1,\ldots,\tau_n)}$, where $\tau_1, \ldots, \tau_n$ are primarily independent over R. Then $S = L \cap \widehat{R}$, where L is the field of fractions of S. Thus $\tau_1, \ldots, \tau_n$ are idealwise independent elements of $\widehat{R}$ over R.*
2. *If $\{\tau_i\}_{i=1}^{\infty}$ is a countably infinite sequence of primarily independent elements of $\widehat{\mathbf{m}}$ over R, then $\{\tau_i\}_{i=1}^{\infty}$ are idealwise independent over R.*

PROOF. Item 2 is a consequence of item 1. Thus it suffices to prove item 1. Let P be a height-one prime of S. Since S is catenary and $\dim R \geq 2$, $\dim(S/P) \geq n+1$. Lemma 20.14.2 implies that $P\widehat{R} \cap S = P$. Therefore $\widehat{R}$ is weakly flat over S. Hence by Theorem 20.11.1, we have $S = L \cap \widehat{R}$. □

PROPOSITION 20.16. *Let $(R, \mathbf{m})$ and $\tau_1, \ldots, \tau_n \in \widehat{\mathbf{m}}$ be as in Setting 20.1. Let $R_n = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m},\tau_1,\ldots,\tau_n)} \cong S_n = R[t_1, \ldots, t_n]_{(\mathbf{m},t_1,\ldots,t_n)}$, where $t_1, \ldots, t_n$ are indeterminates over R. Then the following are equivalent:*

1. *For each prime ideal P of S_n such that $\dim(S_n/P) \geq n$ and each prime ideal $\widehat{P}$ of $\widehat{S_n}$ minimal over $P\widehat{S_n}$, the images of $t_1 - \tau_1, \ldots, t_n - \tau_n$ in $\widehat{S_n}/\widehat{P}$ generate an ideal of height n in $\widehat{S_n}/\widehat{P}$.*
2. *For each prime ideal P of S_n with $\dim(S_n/P) \geq n$ and each nonnegative integer $i \leq n$, the element $t_i - \tau_i$ is outside every prime ideal $\widehat{Q}$ of $\widehat{S}_n$ minimal over $(P, t_1 - \tau_1, \ldots, t_{i-1} - \tau_{i-1})\widehat{S}_n$.*
3. *For each prime ideal P of S_n such that $\dim(S_n/P) = n$, the images of $t_1 - \tau_1, \ldots, t_n - \tau_n$ in $\widehat{S_n}/P\widehat{S_n}$ generate an ideal primary for the maximal ideal of $\widehat{S}_n/P\widehat{S}_n$.*
4. *The elements $\tau_1, \ldots, \tau_n$ are primarily independent over R*

PROOF. It is clear that item 1 and item 2 are equivalent, that item 1 and item 2 imply item 3 and that item 3 is equivalent to item 4. It remains to observe that item 3 implies item 1. For this, let P be a prime ideal of S_n such that $\dim(S_n/P) = n+h$, where $h \geq 0$. There exist $s_1, \ldots, s_h \in S_n$ so that if $I = (P, s_1, \ldots, s_h)S_n$, then for each minimal prime Q of I we have $\dim(S_n/Q) = n$. Item 3 implies

that the images of $t_1 - \tau_1, \ldots, t_n - \tau_n$ in $\widehat{S_n}/Q\widehat{S_n}$ generate an ideal primary for the maximal ideal of $\widehat{S}_n/Q\widehat{S_n}$. It follows that the images of $t_1-\tau_1, \ldots, t_n-\tau_n$ in $\widehat{S_n}/I\widehat{S_n}$ generate an ideal primary for the maximal ideal of $\widehat{S}_n/I\widehat{S_n}$, and therefore that the images of $s_1, \ldots, s_h, t_1 - \tau_1, \ldots, t_n - \tau_n$ in $\widehat{S_n}/P\widehat{S_n}$ are a system of parameters for the $(n + h)$-dimensional local ring $\widehat{S_n}/P\widehat{S_n}$. Let $\widehat{P}$ be a minimal prime of $P\widehat{S_n}$. Then $\dim(\widehat{S_n}/\widehat{P}) = n + h$, and the images of $s_1, \ldots, s_h, t_1 - \tau_1, \ldots, t_n - \tau_n$ in the complete local domain $\widehat{S_n}/\widehat{P}$ are a system of parameters. It follows that the images of $t_1 - \tau_1, \ldots, t_n - \tau_n$ in $\widehat{S_n}/\widehat{P}$ generate an ideal of height n in $\widehat{S_n}/\widehat{P}$. Therefore item 1 holds. □

COROLLARY 20.17. *With the notation of Setting 20.1 and Proposition 20.16, assume that $\tau_1, \ldots, \tau_n$ are primarily independent over R.*

1. *Let I be an ideal of S_n such that $\dim(S/I) = n$. It follows that the ideal $(I, t_1 - \tau_1, \ldots, t_n - \tau_n)\widehat{S_n}$ is primary to the maximal ideal of S_n.*
2. *Let $P \in \operatorname{Spec}(S_n)$ be a prime ideal with $\dim(S_n/P) > n$. Then the ideal $\widehat{W} = (P, t_1 - \tau_1, \ldots, t_n - \tau_n)\widehat{S}_n$ has $\operatorname{ht}(\widehat{W}) = \operatorname{ht}(P) + n$ and $\widehat{W} \cap S_n = P$.*

PROOF. Part 1 is an immediate corollary of Proposition 20.16.3, and it follows from (20.16.1) that $\operatorname{ht}(\widehat{W}) = \operatorname{ht}(P) + n$. Let λ_n be the restriction to S_n of the canonical homomorphism $\lambda : \widehat{S_n} \to \widehat{R}$ from (20.1) so that $\lambda_n : S_n \xrightarrow{\cong} \to R_n$. Then $\dim(R_n/\lambda_n(P)) > n$, and so by (20.14.2), $\lambda_n(P)\widehat{R} \cap R_n = \lambda_n(P)$. Now

$$\widehat{W} \cap S_n = \lambda^{-1}(\lambda_n(P)\widehat{R}) \cap \lambda_n^{-1}(R_n) = \lambda_n^{-1}(\lambda_n(P)\widehat{R} \cap R_n) = \lambda_n^{-1}(\lambda_n(P)) = P. \quad \square$$

To establish the existence of primarily independent elements, we use the following prime avoidance lemma over a complete local ring. (This is similar to [**24**, Lemma 3],[**159**, Lemma 10], [**145**] and [**91**, Lemma 14.2].) We also use this result in two constructions given in Section 20.3.

LEMMA 20.18. *Let $(T, \mathbf{n})$ be a complete local ring of dimension at least 2, and let $t \in \mathbf{n} - \mathbf{n}^2$. Assume that I is an ideal of T containing t, and that $\mathcal{U}$ is a countable set of prime ideals of T each of which fails to contain I. Then there exists an element $a \in I \cap \mathbf{n}^2$ such that $t - a \notin \bigcup\{Q : Q \in \mathcal{U}\}$.*

PROOF. Let $\{P_i\}_{i=1}^{\infty}$ be an enumeration of the prime ideals of $\mathcal{U}$. We may assume that there are no containment relations among the primes of $\mathcal{U}$. Choose $f_1 \in \mathbf{n}^2 \cap I$ so that $t - f_1 \notin P_1$. Then choose $f_2 \in P_1 \cap \mathbf{n}^3 \cap I$ so that $t - f_1 - f_2 \notin P_2$. Note that $f_2 \in P_1$ implies $t - f_1 - f_2 \notin P_1$. Successively, by induction, choose

$$f_n \in P_1 \cap P_2 \cap \cdots \cap P_{n-1} \cap \mathbf{n}^{n+1} \cap I$$

so that $t - f_1 - \ldots - f_n \notin \bigcup_{i=1}^{n} P_i$ for each positive integer n. Then we have a Cauchy sequence $\{f_1 + \cdots + f_n\}_{n=1}^{\infty}$ in T that converges to an element $a \in \mathbf{n}^2$. Now

$$t - a = (t - f_1 - \ldots - f_n) + (f_{n+1} + \ldots),$$

where $(t - f_1 - \ldots - f_n) \notin P_n, (f_{n+1} + \ldots) \in P_n$. Therefore $t - a \notin P_n$, for all n, and so $t - a \in I$. □

REMARK 20.19. Let $A \hookrightarrow B$ be an extension of Krull domains. If α is a nonzero nonunit of B such that $\alpha \notin Q$ for each height-one prime Q of B such that $Q \cap A \neq (0)$, then $\alpha B \cap A = (0)$. In particular, such an element α is algebraically independent over A.

THEOREM 20.20. *Let $(R, \mathbf{m})$ be a countable excellent normal local domain of dimension at least 2, and let $(\widehat{R}, \widehat{\mathbf{m}})$ be the completion of R. Then:*

1. *There exists $\tau \in \widehat{\mathbf{m}}$ that is primarily independent over R.*
2. *If $\tau_1, \ldots, \tau_{n-1} \in \widehat{\mathbf{m}}$ are primarily independent over R, then there exists $\tau_n \in \widehat{\mathbf{m}}$ such that $\tau_1, \ldots, \tau_{n-1}, \tau_n$ are primarily independent over R.*
3. *There exists an infinite sequence $\tau_1, \ldots, \tau_n, \ldots \in \widehat{\mathbf{m}}$ of elements that are primarily independent over R.*

PROOF. Item 2 implies item 1 and item 3. To prove item 2, let $t_1, \ldots, t_n$ be indeterminates over R, and let the notation be as in Setting 20.1. Thus we have $S_{n-1} \cong R_{n-1}$, under the R-algebra isomorphism taking $t_i \to \tau_i$. Let $\widehat{\mathbf{n}}$ denote the maximal ideal of $\widehat{S_n}$. We show the existence of $a \in \widehat{\mathbf{n}}^2$ such that, if λ denotes the $\widehat{R}$-algebra surjection $\widehat{S_n} \to \widehat{R}$ with kernel $(t_1 - \tau_1, \ldots, t_{n-1} - \tau_{n-1}, t_n - a)\widehat{S_n}$, then $\tau_1, \ldots, \tau_{n-1}$ together with the image τ_n of t_n under the map λ are primarily independent over R.

Since S_n is countable and Noetherian we can enumerate as $\{P_j\}_{j=1}^{\infty}$ the prime ideals of S_n such that $\dim(S_n/P_j) \geq n$. Let $\widehat{I} = (t_1 - \tau_1, \ldots, t_{n-1} - \tau_{n-1})\widehat{S_{n-1}}$, and let $\mathcal{U}$ be the set of all prime ideals of $\widehat{S_n} = \widehat{R}[[t_1, \ldots, t_n]]$ minimal over ideals of the form $(P_j, \widehat{I})\widehat{S_n}$ for some P_j; then $\mathcal{U}$ is countable and $\widehat{\mathbf{n}} \notin \mathcal{U}$ since $(P_j, \widehat{I})\widehat{S_n}$ is generated by $n-1$ elements over $P_j\widehat{S_n}$ and $\dim(\widehat{S_n}/P_j\widehat{S_n}) \geq n$. By Lemma 20.18 with the ideal I of that lemma taken to be $\widehat{\mathbf{n}}$, there exists an element $a \in \widehat{\mathbf{n}}^2$ so that $t_n - a$ is not in $\widehat{Q}$, for every prime ideal $\widehat{Q} \in \mathcal{U}$. Let $\tau_n \in \widehat{R}$ denote the image of t_n under the $\widehat{R}$-algebra surjection $\lambda : \widehat{S_n} \to \widehat{R}$ with kernel $(\widehat{I}, t_n - a)\widehat{S_n}$. The kernel of λ is also generated by $(\widehat{I}, t_n - \tau_n)\widehat{S}_n$. Therefore the setting will be as in the diagram of Setting 20.1 after we establish Claim 20.21.

CLAIM 20.21. $(\widehat{I}, t_n - \tau_n)\widehat{S_n} \cap S_n = (0)$.

PROOF. (of Claim 20.21) Since $\tau_1, \ldots, \tau_{n-1}$ are algebraically independent over R, we have $\widehat{I} \cap S_{n-1} = (0)$. Let $R'_n = R_{n-1}[t_n]_{(\max(R_{n-1}), t_n))}$. Consider the diagram:

$$\begin{array}{ccc} S_n = S_{n-1}[t_n]_{(\max(S_{n-1}), t_n)} & \xrightarrow{\subset} & \widehat{S_n} = \widehat{S_{n-1}}[[t_n]] \\ \cong \downarrow & & \lambda_1 \downarrow \\ R'_n = R_{n-1}[t_n]_{(\max(R_{n-1}), t_n)} & \xrightarrow{\subset} & \widehat{R}[[t_n]] \cong (\widehat{S_{n-1}}/\widehat{I})[[t_n]], \end{array}$$

where $\lambda_1 : \widehat{S_n} \to \widehat{S_n}/(\widehat{I}\widehat{S_n}) \cong \widehat{R}$ is the canonical projection.

For $\widehat{Q}$ a prime ideal of $\widehat{S_n}$, we have $\widehat{Q} \in \mathcal{U} \iff \lambda_1(\widehat{Q}) = \widehat{P}$, where $\widehat{P}$ is a prime ideal of $\widehat{R}[[t_n]] \cong (\widehat{S_{n-1}}/\widehat{I})[[t_n]]$ minimal over $\lambda_1(P_j)\widehat{R}[[t_n]]$ for some prime ideal P_j of S_n such that $\dim(S_n/P_j) \leq n$. Since $t_n - a$ is outside every $\widehat{Q} \in \mathcal{U}$, $t_n - \lambda_1(a) = \lambda_1(t_n - a)$ is outside every prime ideal $\widehat{P}$ of $\widehat{R}[[t_n]]$, such that $\widehat{P}$ is minimal over $\lambda_1(P_j)\widehat{R}[[t_n]]$. Since S_n is catenary and $\dim(S_n) = n + \dim(R)$, a prime ideal P_j of S_n is such that $\dim(S_n/P_j) \geq n \iff \mathrm{ht}(P_j) \leq \dim(R)$. Suppose $\widehat{P}$ is a height-one prime ideal of $\widehat{R}[[t_n]]$ such that $\widehat{P} \cap R'_n = P \neq (0)$. Then $\widehat{P}$ is a minimal prime ideal of $P\widehat{R}[[t_n]]$. But also $P = \lambda_1(Q)$, where Q is a height-one prime of S_n and $\dim(S_n/Q) = n + \dim(R) - 1 \geq n$. Therefore $Q \in \{P_j\}_{j=1}^{\infty}$. Hence by choice of a, we have $t_n - \lambda_1(a) \notin \widehat{P}$. By Remark 20.19, $(t_n - \lambda_1(a))\widehat{R}[[t_n]]) \cap R'_n = (0)$. Hence $(\widehat{I}, t_n - \tau_n)\widehat{S_n} \cap S_n = (0)$. □

CLAIM 20.22. *Let P be a prime ideal of S_n such that $\dim(S_n/P) = n$. Then the ideal $(P, \widehat{I}, t_n - \tau_n)\widehat{S_n}$ is $\widehat{\mathbf{n}}$-primary.*

PROOF. (of Claim 20.22) Let $Q = P \cap S_{n-1}$. Either $QS_n = P$, or $QS_n \subsetneq P$. If $QS_n = P$, then $\dim(S_{n-1}/Q) = n-1$ and the primary independence of $\tau_1, \ldots, \tau_{n-1}$ implies that $(Q, \widehat{I})\widehat{S_{n-1}}$ is primary for the maximal ideal of $\widehat{S_{n-1}}$. Therefore $(Q, \widehat{I}, t_n - \tau_n)\widehat{S_n} = (P, \widehat{I}, t_n - \tau_n)\widehat{S_n}$ is $\widehat{\mathbf{n}}$-primary in this case. On the other hand, if $QS_n \subsetneq P$, then $\dim(S_{n-1}/Q) = n$. Let $\widehat{Q'}$ be a minimal prime of $(Q, \widehat{I})\widehat{S_{n-1}}$. By Proposition 20.16, $\dim(\widehat{S_{n-1}}/\widehat{Q'}) = 1$, and hence $\dim(\widehat{S_n}/\widehat{Q'}\widehat{S_n}) = 2$. The primary independence of $\tau_1, \ldots \tau_{n-1}$ implies that $\widehat{Q'} \cap S_{n-1} = Q$. Therefore $\widehat{Q'}\widehat{S_{n-1}}[[t_n]] \cap S_n = QS_n \subsetneq P$, so P is not contained in $\widehat{Q'}\widehat{S_n}$. Therefore $\dim(\widehat{S_n}/(P, \widehat{I})\widehat{S_n})) = 1$ and our choice of a implies that $(P, \widehat{I}, t_n - \tau_n)\widehat{S_n}$ is $\widehat{\mathbf{n}}$-primary. □

This completes the proof of Theorem 20.20. □

COROLLARY 20.23. *Let $(R, \mathbf{m})$ be a countable excellent normal local domain of dimension at least 2, and let K denote the field of fractions of R. Then there exist $\tau_1, \ldots, \tau_n, \ldots \in \widehat{\mathbf{m}}$ such that $A = K(\tau_1, \tau_2, \ldots) \cap \widehat{R}$ is an infinite-dimensional non-Noetherian local domain. In particular, for k a countable field, the localized polynomial ring $R = k[x, y]_{(x,y)}$ has such extensions inside $\widehat{R} = k[[x, y]]$.*

PROOF. By Theorem 20.20.3, there exist $\tau_1, \ldots, \tau_n, \ldots \in \widehat{\mathbf{m}}$ that are primarily independent over R. It follows that $A = K(\tau_1, \tau_2, \ldots) \cap \widehat{R}$ is an infinite-dimensional local domain. In particular, A is not Noetherian. □

20.3. Residually algebraically independent elements

We introduce in this section a third concept, that of residual algebraic independence. Residual algebraic independence is weaker than primary independence. In Theorem 20.28 we show that over every countable normal excellent local domain $(R, \mathbf{m})$ of dimension at least three there exists an element residually algebraically independent over R that is not primarily independent over R. In Theorems 20.33 and 20.35 we show the existence of idealwise independent elements that fail to be residually algebraically independent.

DEFINITION 20.24. Let $(\widehat{R}, \widehat{\mathbf{m}})$ be a complete normal Noetherian local domain and let A be a Krull subdomain of $\widehat{R}$ such that $A \hookrightarrow \widehat{R}$ satisfies PDE.

(1) An element $\tau \in \widehat{\mathbf{m}}$ is *residually algebraically independent with respect to* $\widehat{R}$ over A if τ is algebraically independent over A and for each height-one prime $\widehat{P}$ of $\widehat{R}$ such that $\widehat{P} \cap A \neq (0)$, the image of τ in $\widehat{R}/\widehat{P}$ is algebraically independent over the integral domain $A/(\widehat{P} \cap A)$.
(2) Elements $\tau_1, \ldots \tau_n \in \widehat{\mathbf{m}}$ are said to be *residually algebraically independent* over A if for each $0 \leq i < n$, τ_{i+1} is residually algebraically independent over $A[\tau_1, \ldots, \tau_i]$.
(3) An infinite sequence $\{\tau_i\}_{i=1}^{\infty}$ of elements of $\widehat{\mathbf{m}}$ is *residually algebraically independent* over A, if $\tau_1, \ldots \tau_n$ are residually algebraically independent over A for each positive integer n.

Proposition 20.25 relates residual algebraic independence for τ over A to the PDE property of Definition 20.5 for $A[\tau] \hookrightarrow \widehat{R}$. By Proposition 8.11, for an extension of Krull domains, the PDE property is equivalent to the LF_1 property of Definition 8.1.3.

PROPOSITION 20.25. *Let $(R, \mathbf{m})$ and $\tau \in \widehat{\mathbf{m}}$ be as in Setting 20.1. Let A be a Krull subdomain of $\widehat{R}$ such that $A \hookrightarrow \widehat{R}$ satisfies PDE. Then τ is residually algebraically independent with respect to $\widehat{R}$ over $A \iff A[\tau] \hookrightarrow \widehat{R}$ satisfies PDE.*

PROOF. Assume $A[\tau] \hookrightarrow \widehat{R}$ does not satisfy PDE. Then there exists a prime ideal $\widehat{P}$ of $\widehat{R}$ of height one such that $\mathrm{ht}(\widehat{P} \cap A[\tau]) \geq 2$. Now $\mathrm{ht}(\widehat{P} \cap A) = 1$, since PDE holds for $A \hookrightarrow \widehat{R}$. Thus, with $\mathbf{p} = \widehat{P} \cap A$, we have $\mathbf{p}A[\tau] \subsetneq \widehat{P} \cap A[\tau]$; that is, there exists $f(\tau) \in (\widehat{P} \cap A[\tau]) \setminus \mathbf{p}A[\tau]$, or equivalently there is a nonzero polynomial $\bar{f}(x) \in (A/\mathbf{p})[x]$ so that $\bar{f}(\bar{\tau}) = \bar{0}$ in $A[\tau]/(\widehat{P} \cap A[\tau])$, where $\bar{\tau}$ denotes the image of τ in $\widehat{R}/\widehat{P}$. This means that $\bar{\tau}$ is algebraic over $A/(\widehat{P} \cap A)$. Hence τ is not residually algebraically independent with respect to $\widehat{R}$ over A.

For the converse, assume that $A[\tau] \hookrightarrow \widehat{R}$ satisfies PDE and let $\widehat{P}$ be a height-one prime of $\widehat{R}$ such that $\widehat{P} \cap A = \mathbf{p} \neq 0$. Since $A[\tau] \hookrightarrow \widehat{R}$ satisfies PDE, $\widehat{P} \cap A[\tau] = \mathbf{p}A[\tau]$ and $A[\tau]/(\mathbf{p}A[\tau])$ canonically embeds in $\widehat{R}/\widehat{P}$. Hence the image of τ in $A[\tau]/\mathbf{p}A[\tau]$ is algebraically independent over $A/\mathbf{p}$. It follows that τ is residually algebraically independent with respect to $\widehat{R}$ over A. □

THEOREM 20.26. *Let $(R, \mathbf{m})$ be an excellent normal local domain with completion $(\widehat{R}, \widehat{\mathbf{m}})$ and let $\tau_1, \ldots, \tau_n \in \widehat{\mathbf{m}}$ be algebraically independent over R. The following statements are equivalent:*

1. *The elements $\tau_1, \ldots, \tau_n$ are residually algebraically independent with respect to $\widehat{R}$ over R.*
2. *For each integer i with $1 \leq i \leq n$, if $\widehat{P}$ is a height-one prime ideal of $\widehat{R}$ such that $\widehat{P} \cap R[\tau_1, \ldots, \tau_{i-1}] \neq 0$, then $\mathrm{ht}(\widehat{P} \cap R[\tau_1, \ldots, \tau_i]) = 1$.*
3. *$R[\tau_1, \ldots, \tau_n] \hookrightarrow \widehat{R}$ satisfies PDE.*

If each height-one prime of R is the radical of a principal ideal, then these equivalent conditions imply the map $R[\tau_1, \ldots, \tau_n] \hookrightarrow \widehat{R}$ is weakly flat.

PROOF. The equivalence of the three items follows from Proposition 20.25. The last sentence follows from Theorem 20.11) □

THEOREM 20.27. *Let $(R, \mathbf{m})$ and $\{\tau_i\}_{i=1}^m \subseteq \widehat{\mathbf{m}}$ be as in Setting 20.1, where $\dim(R) \geq 2$ and m is either a positive integer or $m = \infty$.*

1. *If $\{\tau_i\}_{i=1}^m$ is primarily independent over R, then $\{\tau_i\}_{i=1}^m$ is residually algebraically independent over R.*
2. *If $\dim(R) = 2$, then $\{\tau_i\}_{i=1}^m$ is primarily independent over R if and only if it is residually algebraically independent over R.*
3. *If each height-one prime of R is the radical of a principal ideal and $\{\tau_i\}_{i=1}^m$ is residually algebraically independent over R, then $\{\tau_i\}_{i=1}^m$ is idealwise independent over R.*

PROOF. To prove item 1, it suffices by Theorem 20.26 to show that for each positive integer $n \leq m$, if $\tau_1, \ldots, \tau_n$ are primarily independent over R, then the

extension $R[\tau_1,\dots,\tau_n] \hookrightarrow \widehat{R}$ satisfies PDE. Let $S = R[\tau_1,\dots,\tau_n]_{(\mathbf{m},\tau_1,\dots,\tau_n)}$ and let the notation be as in the diagram of Setting 20.1.

Let $\widehat{P}$ be a height-one prime ideal of $\widehat{R}$ with $\mathbf{p} := \widehat{P} \cap R \neq (0)$. Consider the ideal $\widehat{W} := (\mathbf{p}, t_1 - \tau_1, \dots, t_n - \tau_n)\widehat{S}_n$. Using the diagram of Setting 20.1, we see that $\lambda(\widehat{W}) = \mathbf{p}\widehat{R} \subseteq \widehat{P}$. By Corollary 20.17.2, $\operatorname{ht}(\widehat{W}) = \operatorname{ht}(\mathbf{p}) + n$. However, $\widehat{W} \subseteq (\widehat{P}, t_1 - \tau_1, \dots, t_n - \tau_n) = \lambda^{-1}(\widehat{P})$ and thus

$$1 + n \leq \operatorname{ht}(\mathbf{p}) + n = \operatorname{ht}(\widehat{W}) \leq \operatorname{ht}(\lambda^{-1}(\widehat{P})) \leq \operatorname{ht}(\widehat{P}) + n = 1 + n.$$

Therefore $\operatorname{ht}(\mathbf{p}) = 1$.

In view of item 1, to prove item 2, we assume that $\dim R = 2$ and $n \leq m$ is a positive integer such that $\tau_1,\dots,\tau_n$ are residually algebraically independent over R. Let $S = R[\tau_1,\dots,\tau_n]_{(\mathbf{m},\tau_1,\dots,\tau_n)}$. By Theorem 20.26, $S \hookrightarrow \widehat{R}$ satisfies PDE. Let P be a prime ideal of S such that $\dim(S/P) \leq n$. Since $\dim(S) = n + 2$ and S is catenary, it follows that $\operatorname{ht}(P) \geq 2$. To show $\tau_1,\dots,\tau_n$ are primarily independent over R, it suffices to show that $P\widehat{R}$ is primary for the maximal ideal of $\widehat{R}$. Since $\dim(\widehat{R}) = 2$, this is equivalent to showing P is not contained in a height-one prime of $\widehat{R}$, and this last statement holds since $S \hookrightarrow \widehat{R}$ satisfies PDE.

The proof of item 3 follows from Theorems 20.26 and 20.11. □

THEOREM 20.28. *Let $(R,\mathbf{m})$ be a countable excellent normal local domain of dimension d and let $(\widehat{R},\widehat{\mathbf{m}})$ be the completion of R. If $d \geq 3$, then there exists an element $\tau \in \widehat{\mathbf{m}}$ that is residually algebraically independent over R, but not primarily independent over R.*

PROOF. We use techniques similar to those in the proof of Theorem 20.20. Let t be an indeterminate over R and set $S_1 = R[t]_{(\mathbf{m},t)}$. Thus $\widehat{S_1} = \widehat{R}[[t]]$. Let $\widehat{\mathbf{n}_1}$ denote the maximal ideal of $\widehat{S_1}$. We have the top line

$$S_1 \;=\; R[t]_{(\mathbf{m},t)} \;\hookrightarrow\; \widehat{S_1} = \widehat{R}[[t]]$$

of a diagram as in Setting 20.1 for $n = 1$. We seek an appropriate mapping

$$\lambda : \widehat{S_1} \;\longrightarrow\; \widehat{R}.$$

Then we use the element $\tau := \lambda(t) \in \widehat{\mathbf{m}}$ to complete the diagram of Setting 20.1. Let $\widehat{Q_0}$ be a prime ideal of $\widehat{S_1}$ of height d that contains t and is such that $Q_0 := \widehat{Q_0} \cap S_1$ also has height d. Let

$$\mathcal{U} \;=\; \{\widehat{Q} \;\in\; \operatorname{Spec}\widehat{S_1} \mid \operatorname{ht}\widehat{Q} \;\leq\; d,\ \operatorname{ht}(\widehat{Q} \cap S_1) = \operatorname{ht}\widehat{Q} \text{ and } \widehat{Q} \neq \widehat{Q_0}\ \}.$$

Since S_1 is countable, the set $\mathcal{U}$ is countable. We apply Lemma 20.18 with $T = \widehat{S_1}, \mathbf{n} = \widehat{\mathbf{n}_1}$ and $I = \widehat{Q_0}$, to obtain an element $a \in \widehat{Q_0} \;\cap\; \widehat{\mathbf{n}_1}^2$ so that $t - a \in \widehat{Q}_0$ but $t - a$ is not in any prime ideal in $\mathcal{U}$. Since $a \in \widehat{\mathbf{n}_1}^2$, we have $\widehat{R}[[t]] = \widehat{R}[[t - a]]$. Define λ to be the natural surjection

$$\lambda : \widehat{S_1} \;\longrightarrow\; \widehat{S_1}/(t-a)\widehat{S_1} \;=\; \widehat{R}.$$

We have $\tau = \lambda(t) = \lambda(a)$.

Since λ restricted to S_1 is an isomorphism from S_1 onto $S := R[\tau]_{(\mathbf{m},\tau)}$, the prime ideal $\lambda(Q_0)$ in $S = R[\tau]_{(\mathbf{m},\tau)}$ is of height d. Thus $\dim(S/\lambda(Q_0)) = 1$. Since the diagram of Setting 20.1 is commutative, we have $\lambda(Q_0)\widehat{R} \subseteq \lambda(\widehat{Q_0})$. Since $(t - \tau)\widehat{S_1} = (t - a)\widehat{S_1} \subseteq \widehat{Q_0}$, the prime ideal $\lambda(\widehat{Q_0})$ is of height $d - 1$. Therefore $\lambda(Q_0)\widehat{R}$ is not $\widehat{\mathbf{m}}$-primary. Hence τ is not primarily independent.

To prove that τ is residually algebraically independent over R, by Theorem 20.26, it suffices to show the extension

$$S \;=\; R[\tau]_{(\mathbf{m},\tau)} \;\hookrightarrow\; \widehat{R}$$

satisfies PDE.

If $\widehat{P}$ is a height-one prime ideal of $\widehat{R}$ with $\widehat{P} \cap R \neq 0$, then the height of $\widehat{P} \cap R$ is 1, and so the height of $\widehat{P} \cap S$ is at most 2. Let $\widehat{Q_2} := \lambda^{-1}(\widehat{P})$ in $\widehat{S_1}$. Then $\operatorname{ht}(\widehat{Q_2}) = 2$—since it is generated by the inverse images of the generators of $\widehat{P}$ and $\ker(\lambda) = (t-a)\widehat{S_1}$.

Suppose that the height of $\widehat{P} \cap S = 2$. Then under the R-isomorphism of S_1 to S taking t to τ, $\widehat{P} \cap S$ corresponds to a height-two prime P of S_1. Since $\widehat{S_1}$ is flat over S_1, the height of $\widehat{Q_2} \cap S_1$ is at most two. We have $P \subseteq \widehat{Q_2} \cap S_1$. Hence $P = \widehat{Q_2} \cap S_1$. The following diagram illustrates this situation:

$$\begin{array}{ccccc} P = \widehat{Q}_2 \cap S_1 (\text{ht } 2) & \xrightarrow{\subseteq} & \widehat{Q}_2 = \lambda^{-1}(\widehat{P}) & \xleftarrow{\supseteq} & (\widehat{P}, (t-a))\widehat{S_1} \ (\text{ht } 2) \\ \cong\big\downarrow & & \lambda\big\downarrow & & \\ \widehat{P} \cap S \ (\text{ht } 2) & \xrightarrow{\subseteq} & \widehat{P} & & (\text{ht } 1 \text{ in } \widehat{R}). \end{array}$$

Since $\operatorname{ht} \widehat{Q_2} = 2 < d = \operatorname{ht} \widehat{Q_0}$, we have $\widehat{Q_2} \in \mathcal{U}$. However, $t - a \in \widehat{Q_2}$, a contradiction.

We conclude that $\operatorname{ht}(\widehat{P} \cap S) = 1$. Thus τ is residually algebraically independent over R. □

Example 20.29. The following construction, similar to that in Theorem 20.28, shows that condition 2 in Definition 20.24 is stronger than

(2′) For each height-one prime ideal $\widehat{P}$ of $\widehat{R}$ with $\widehat{P} \cap R \neq 0$, the images of $\tau_1, \ldots, \tau_n$ in $\widehat{R}/\widehat{P}$ are algebraically independent over $R/(\widehat{P} \cap R)$.

Construction 20.30. Let $(R, \mathbf{m})$ be a countable excellent local unique factorization domain (UFD) of dimension two and let $(\widehat{R}, \widehat{\mathbf{m}})$ be the completion of R, for example $R = \mathbb{Q}[x,y]_{(x,y)}$ and $\widehat{R} = \mathbb{Q}[[x,y]]$. As in Theorem 20.20, construct $\tau_1 \in \widehat{\mathbf{m}}$ primarily independent over R (or equivalently residually algebraically independent in this context). Let t_1, t_2 be variables over R and let $S_2 := R[t_1,t_2]_{(\mathbf{m},t_1,t_2)}$. Consider the ideal $I := (t_1, t_2, t_1 - \tau_1)\widehat{S_2}$ and define

$$\mathcal{U} = \{\widehat{Q} \in \operatorname{Spec}(\widehat{S_2}) \mid I \nsubseteq \widehat{Q} \ ,\ \operatorname{ht}(\widehat{Q}) = \operatorname{ht}(\widehat{Q} \cap S_2) \text{ and } \widehat{Q} \cap S_2 = (P, t_1 - \tau_1), \\ \text{for some } P \in \operatorname{Spec} S_2 \text{ with } \operatorname{ht}(P) \leq 2\}.$$

Thus $P \neq (t_1, t_2)S_2$. Let $\widehat{\mathbf{n}}$ denote the maximal ideal of $\widehat{S_2}$. By Lemma 20.18, there exists $a \in \widehat{\mathbf{n}}^2 \cap I$ so that $t_2 - a \notin \bigcup\{\widehat{Q} : \widehat{Q} \in \mathcal{U}\}$. We have $\widehat{S_2} = \widehat{R}[[t_1,t_2]] = \widehat{R}[[t_1 - \tau_1, t_2 - a]]$. Let λ denote the canonical R-algebra surjection

$$\lambda : \ \widehat{S_2} \ \longrightarrow \ \widehat{S_2}/(t_1 - \tau_1, t_2 - a)\widehat{S_2} \ = \widehat{R},$$

and $\tau_2 = \lambda(t_2)$. Notice that $\ker(\lambda)$ has height two.

Claim 20.31. *The element τ_2 is not residually algebraically independent over $R[\tau_1]$; thus τ_1, τ_2 do not satisfy item 2 of Definition 20.24.*

Proof. (of Claim 20.31) Let $\widehat{W}$ be a prime ideal of $\widehat{S_2}$ that is minimal over I. Then $\operatorname{ht} \widehat{W} \leq 3$. Also we have $t_2 \in I$ and $a \in I$, and so $t_2 - a \in I \subseteq \widehat{W}$.

Thus $\ker(\lambda) \subseteq \widehat{W}$. Let $\widehat{P} = \lambda(\widehat{W}) \subset \widehat{R}$. Thus $\operatorname{ht}\widehat{P} \leq 1$. In fact $\operatorname{ht}\widehat{P} = 1$, since $0 \neq \tau_1 = \lambda(t_1) \in \widehat{P}$. Since τ_1 is residually algebraically independent over R, the extension $R[\tau_1] \hookrightarrow \widehat{R}$ satisfies PDE by Proposition 20.25. Therefore $\operatorname{ht}(\widehat{P} \cap R[\tau_1]) \leq 1$. But $\tau_1 \in \widehat{P} \cap R[\tau_1]$, and so $\operatorname{ht}(\widehat{P} \cap R[\tau_1]) = 1$ and $\widehat{P} \cap R = (0)$. Also $\tau_2 = \lambda(t_2) \in \widehat{P}$; thus $\tau_1, \tau_2 \in \widehat{P} \cap R[\tau_1, \tau_2]$, and so $\operatorname{ht}(\widehat{P} \cap R[\tau_1, \tau_2]) \geq 2$. Thus the extension $R[\tau_1, \tau_2] \hookrightarrow \widehat{R}$ does not satisfy PDE. By Proposition 20.25, τ_2 is not residually algebraically independent over $R[\tau_1]$. □

Claim 20.32. *For each height-one prime ideal $\widehat{P}$ of $\widehat{R}$ with $\widehat{P} \cap R \neq 0$, the images of τ_1 and τ_2 in $\widehat{R}/\widehat{P}$ are algebraically independent over $R/(\widehat{P} \cap R)$. That is, τ_1, τ_2 satisfy item $2'$ above.*

Proof. (of Claim 20.32) Suppose $\widehat{P}$ is a height-one prime ideal of $\widehat{R}$ with $\widehat{P} \cap R \neq (0)$ and let $\widehat{Q} = \lambda^{-1}(\widehat{P})$. Then $\operatorname{ht}(\widehat{Q}) = 3$ and $\operatorname{ht}(\widehat{P} \cap R) = 1$. Set $R_1 := R[\tau_1]_{(\mathbf{m}, \tau_1)}$ and $R_2 := R[\tau_1, \tau_2]_{(\mathbf{m}, \tau_1, \tau_2)}$. By Proposition 20.25 and the residual algebraic independence of τ_1 over R, we have $\operatorname{ht}(\widehat{P} \cap R_1) = 1$, and so $\operatorname{ht}(\widehat{P} \cap R_2) \leq 2$. If $\operatorname{ht}(\widehat{P} \cap R_2) = 1$, we are done by Proposition 20.25. Suppose $\operatorname{ht}(\widehat{P} \cap R_2) = 2$. The following diagram illustrates this situation:

$$\begin{array}{ccccccc}
 & & \widehat{Q} \cap S_1 & \xrightarrow{\subseteq} & \widehat{Q} \cap S_2 & \xrightarrow{\subseteq} & \widehat{Q} = \lambda^{-1}(\widehat{P}) \xrightarrow{\subseteq} \widehat{S_2} \\
 & & \cong\downarrow & & \cong\downarrow & & \lambda\downarrow \qquad\qquad \lambda\downarrow \\
\widehat{P} \cap R & \xrightarrow{\subseteq} & \widehat{P} \cap R_1 & \xrightarrow{\subseteq} & \widehat{P} \cap R_2 & \xrightarrow{\subseteq} & \widehat{P} \xrightarrow{\subseteq} \widehat{R}.
\end{array}$$

Thus $\widehat{Q} \cap S_2 = P$ is a prime ideal of height 2, and $\operatorname{ht}(\widehat{Q} \cap S_1) = 1$. Also $P \neq (t_1, t_2)S_2$ because $(t_1, t_2)S_2 \cap R = (0)$. But this means that $\widehat{Q} \in \mathcal{U}$ since $\widehat{Q}$ is minimal over $(P, t_1 - \tau_1)\widehat{S_2}$ where P is a prime of S_2 with $\dim(S_2/P) = 2$ and $P \neq (t_1, t_2)S_2$. This contradicts the choice of a and establishes that item $2'$ holds. □

We present in Theorem 20.33 a method to obtain an idealwise independent element that fails to be residually algebraically independent.

Theorem 20.33. *Let $(R, \mathbf{m})$ be a countable excellent local UFD of dimension at least two. Assume there exists a height-one prime P of R such that P is contained in at least two distinct height-one primes $\widehat{P}$ and $\widehat{Q}$ of $\widehat{R}$. Also assume that $\widehat{P}$ is not the radical of a principal ideal in $\widehat{R}$. Then there exists $\tau \in \mathbf{m}\widehat{R}$ that is idealwise independent but not residually algebraically independent over R.*

Proof. Let t be an indeterminate over R and set $S_1 = R[t]_{(\mathbf{m}, t)}$ so that $\widehat{S_1} = \widehat{R}[[t]]$. Let $\widehat{\mathbf{n}_1}$ denote the maximal ideal of $\widehat{S_1}$. By Lemma 20.18 with $I = (\widehat{P}, t)\widehat{S_1}$ and

$$\mathcal{U} = \{\mathbf{p} \in \operatorname{Spec}(\widehat{S_1}) \mid \mathbf{p} \neq I,\ \operatorname{ht}(\mathbf{p}) \leq 2,\ \text{and } \mathbf{p} \text{ minimal over } \mathbf{p} \cap S_1\},$$

there exists $a \in (\widehat{P}, t)\widehat{S_1} \cap \widehat{\mathbf{n}_1}^2$, such that $t - a \notin \bigcup\{\mathbf{p} \mid \mathbf{p} \in \mathcal{U}\}$, but $t - a \in (\widehat{P}, t)\widehat{S_1}$. That is, if $t - a \in \mathbf{p}$, for some prime ideal $\mathbf{p} \neq (\widehat{P}, t)\widehat{S_1}$ of $\widehat{S_1}$ with $\operatorname{ht}(\mathbf{p}) \leq 2$, then $\operatorname{ht}(\mathbf{p}) > \operatorname{ht}(\mathbf{p} \cap S_1)$. Let λ be the surjection $\widehat{S_1} \to \widehat{R}$ with kernel $(t - a)\widehat{S_1}$. By construction, $(t - a)\widehat{S_1} \cap S_1 = (0)$. Therefore the restriction of λ to S_1 maps S_1 isomorphically onto $S = R[\tau]_{(\mathbf{m}, \tau)}$, where $\lambda(t) = \tau \in \mathbf{m}\widehat{R}$ is algebraically independent over R.

That τ is not residually algebraically independent over R follows because the prime ideal $\lambda((P,t)S_1) = (P,\tau)S$ has height two and is the contraction to S of the prime ideal $\lambda((\widehat{P},t)\widehat{S_1}) = \widehat{P}$ of $\widehat{R}$. Since $(t-\tau)\widehat{S_1} = (t-a)\widehat{S_1} \subseteq (\widehat{P},t)\widehat{S_1}$, $\lambda((\widehat{P},t)\widehat{S_1})$ has height one and equals $\widehat{P}$. Therefore τ is not residually algebraically independent over R.

Our choice of $t-a$ insures that each height-one prime $\widehat{\mathbf{q}}$ other than $\widehat{P}$ of $\widehat{R}$ has the property that $\mathrm{ht}(\widehat{\mathbf{q}} \cap S) \leq 1$. We show that τ is idealwise independent over R by showing each height-one prime of S is the contraction of a height-one prime of $\widehat{R}$. Let $\varphi : S_1 \to S$ denote the restriction of λ. For $\mathbf{q}$ a height-one prime of S, let $\mathbf{q}_1 := \varphi^{-1}(\mathbf{q})$ denote the corresponding height-one prime of S_1. Then $(\mathbf{q}_1, t-a)\widehat{S_1}$ is an ideal of height two. Let $\mathbf{w}_1$ be a height-two prime of $\widehat{S_1}$ containing $(\mathbf{q}_1, t-a)$. If $\mathbf{q}_1$ is not contained in $(\widehat{P},t)\widehat{S_1}$, then by the choice of $t-a$, $\mathbf{w}_1 \cap S_1$ has height at most one. Therefore $\mathbf{w}_1 \cap S_1 = \mathbf{q}_1$. Let $\mathbf{w} = \lambda(\mathbf{w}_1)$. Then $\mathbf{w}$ is a height-one prime of $\widehat{R}$ and $\mathbf{w} \cap S = \mathbf{q}$.

Therefore each height-one prime $\mathbf{q}$ of S such that $\mathbf{q}_1 := \varphi^{-1}(\mathbf{q})$ is not contained in $(\widehat{P},t)\widehat{S_1}$ is the contraction of a height-one prime of $\widehat{R}$. Since $\lambda((\widehat{P},t)\widehat{S_1}) \cap S = (P,\tau)S$, it remains to consider height-one primes $\mathbf{q}$ of S such that $\mathbf{q} \subseteq (P,\tau)S$. By hypothesis we have $PS = \widehat{Q} \cap S$. Let $\mathbf{q}$ be a height-one prime of S such that $\mathbf{q} \neq PS$ and $\mathbf{q} \subseteq (P,\tau)S$. Since R is a UFD, S is a UFD and $\mathbf{q} = fS$ for an element $f \in \mathbf{q}$. Since $\widehat{P}$ is not the radical of a principal ideal, there exists a height-one prime $\widehat{\mathbf{q}} \neq \widehat{P}$ of $\widehat{R}$ such that $f \in \widehat{\mathbf{q}}$. Since $\mathrm{ht}(\widehat{\mathbf{q}} \cap S) \leq 1$, we have $\widehat{\mathbf{q}} \cap S = fS = \mathbf{q}$. Therefore τ is idealwise independent over R. □

EXAMPLE 20.34. An example of a countable excellent local UFD having a height-one prime P satisfying the conditions in Theorem 20.33 is $R = k[x,y,z]_{(x,y,z)}$, where k is the algebraic closure of the field $\mathbb{Q}$ and $z^2 = x^3 + y^7$. That R is a UFD is shown in [**138**, page 32]. Since $z - xy$ is an irreducible element of R, the ideal $P = (z-xy)R$ is a height-one prime of R. It is observed in [**56**, pages 300-301] that in the completion $\widehat{R}$ of R there exist distinct height-one primes $\widehat{P}$ and $\widehat{Q}$ lying over P. Moreover, the blowup of $\widehat{P}$ has a unique exceptional prime divisor and this exceptional prime divisor is not the unique exceptional prime divisor on the blowup of an $\widehat{\mathbf{m}}$-primary ideal. Therefore $\widehat{P}$ is not the radical of a principal ideal of $\widehat{R}$.

In Theorem 20.35 we present an alternative method to obtain idealwise independent elements that are not residually algebraically independent.

THEOREM 20.35. *Let $(R,\mathbf{m})$ be a countable excellent local UFD of dimension at least two. Assume there exists a height-one prime P_0 of R such that P_0 is contained in at least two distinct height-one primes $\widehat{P}$ and $\widehat{Q}$ of $\widehat{R}$. Also assume that the Henselization $(R^h, \mathbf{m}^h)$ of R is a UFD. Then there exists $\tau \in \mathbf{m}\widehat{R}$ that is idealwise independent but not residually algebraically independent over R.*

PROOF. Since R is excellent, $P := \widehat{P} \cap R^h$ and $Q := \widehat{Q} \cap R^h$ are distinct height-one primes of R^h with $P\widehat{R} = \widehat{P}$, and $Q\widehat{R} = \widehat{Q}$. Let $x \in R^h$ be such that $xR^h = P$. Theorem 20.20 implies there exists $y \in \mathbf{m}\widehat{R}$ that is primarily independent and hence residually algebraically independent over R^h.

We show that $\tau = xy$ is idealwise independent but not residually algebraically independent over R. Since x is nonzero and algebraic over R, xy is algebraically

independent over R. Let $S = R[xy]_{(\mathbf{m},xy)}$. Then S is a UFD and $\widehat{P} \cap S = x\widehat{R} \cap S \supseteq (P_0, xy)S$ has height at least two in S. Therefore by Theorem 20.26, xy is not residually algebraically independent over R.

Since y is idealwise independent over R^h, every height-one prime of the polynomial ring $R^h[y]$ contained in the maximal ideal $\mathbf{n} = (\mathbf{m}^h, y)R^h[y]$ is the contraction of a height-one prime of $\widehat{R}$. To show xy is idealwise independent over R, it suffices to show every prime element $w \in (\mathbf{m}, xy)R[xy]$ is such that $wR[xy]$ is the contraction of a height-one prime of $R^h[y]$ contained in $\mathbf{n}$. If $w \notin (P, xy)R^h[xy]$, then the constant term of w as a polynomial in $R^h[xy]$ is in $\mathbf{m}^h \setminus P$. Thus $w \in \mathbf{n}$ and $w \notin xR^h[y]$. Since $R^h[xy][1/x] = R^h[y][1/x]$ and $xR^h[y] \cap R^h[xy] = (x, xy)R^h[xy]$, it follows that there is a prime factor u of w in $R^h[xy]$ such that $u \in \mathbf{n} \setminus xR^h[y]$. Then $uR^h[y]$ is a height-one prime of $R^h[y]$ and $uR^h[x] \cap R^h[xy] = uR^h[xy]$. Since $R^h[xy]$ is faithfully flat over $R[xy]$, it follows that $uR^h[y] \cap R[xy] = wR[xy]$.

We have $QR^h[xy] = QR^h[y] \cap R^h[xy]$ and $QR^h[xy] \cap R[xy] = P_0R[xy]$. Thus it remains to show, for a prime element $w \in (\mathbf{m}, xy)R[xy]$ such that $w \in (P, xy)R^h[xy]$ and $wR[xy] \neq P_0R[xy]$, that $wR[xy]$ is the contraction of a height-one prime of R^h contained in $\mathbf{n}$. Since $(P, xy)R^h[xy] \cap R[xy] = (P_0, xy)R[xy]$, it follows that w is a nonconstant polynomial in $R[xy]$ and the constant term w_0 of w is in P_0. In the polynomial ring $R^h[y]$ we have $w = x^n v$, where $v \notin xR^h[y]$. If v_0 denotes the constant term of v as a polynomial in $R^h[y]$, then $x^n v_0 = w_0 \in P_0 \subseteq R$ implies $x^n v_0 \in Q \subseteq R^h$. Since $x \in R^h \setminus Q$, we must have $v_0 \in Q$ and hence $v \in \mathbf{n}$. Also $v \notin xR^h[y]$ implies there is a height-one prime ideal $\mathbf{v}$ of $R^h[y]$ with $v \in \mathbf{v}$ and $x \notin \mathbf{v}$. Then, since $R^h[y]_{\mathbf{v}}$ is a localization of $R^h[xy]$, $\mathbf{v} \cap R^h[xy]$ is a height-one prime of $R^h[xy]$ that is contained in $(\mathbf{m}^h, xy)R^h[xy]$. It follows that $\mathbf{v} \cap R^h[xy] = wR^h[xy]$, which completes the proof of Theorem 20.35. □

Example 4.12 is a specific example with the hypothesis of Theorem 20.35. In more generality, we have:

EXAMPLE 20.36. Let $R = k[s,t]_{(s,t)}$ be a localized polynomial ring in two variables s and t over a countable field k where k has characteristic not equal to 2. Let $P_0 = (s^2 - t^2 - t^3)R$. Then P_0 is a height-one prime of R and $P_0\widehat{R} = (s^2 - t^2 - t^3)k[[s,t]]$ is the product of two distinct height-one primes of $\widehat{R}$.

REMARK 20.37. Let $(R, \mathbf{m})$ be excellent normal local domain and let $(\widehat{R}, \widehat{\mathbf{m}})$ be its completion. Assume that $\tau \in \widehat{\mathbf{m}}$ is algebraically independent over R. By Theorem 20.11, the extension $R[\tau] \hookrightarrow \widehat{R}$ is weakly flat if and only if τ is idealwise independent over R. By Theorem 20.26, this extension satisfies PDE (or equivalently LF_1) if and only if τ is residually independent over R. Thus Examples 20.34 and 20.36 give extensions of Krull domains $R[\tau] \hookrightarrow \widehat{R}$, that are weakly flat, but do not satisfy PDE. In fact, in these examples the ring $R[\tau]$ is a 3-dimensional excellent UFD.

Exercises

(1) As in Remark 20.19, let $A \hookrightarrow B$ be an extension of Krull domains, and let α be a nonzero nonunit of B such that $\alpha \notin Q$ for each height-one prime Q of B such that $Q \cap A \neq (0)$.
   (a) Prove that $\alpha B \cap A = (0)$ as asserted in Remark 20.19.
   (b) Prove that α is algebraically independent over A.

(2) Let $R = k[s,t]_{(s,t)}$ and the field k be as in Example 20.36.
   (a) Prove as asserted in Example 20.36 that $(s^2 - t^2 - t^3)R$ is a prime ideal.
   (b) Prove that $s^2 - t^2 - t^3$ factors in the power series ring $k[[s,t]]$ as the product of two nonassociate prime elements.

CHAPTER 21

Idealwise algebraic independence II

We relate the three concepts of independence from Chapter 20 to flatness conditions of extensions of Krull domains, establish implications among them, and draw some conclusions concerning their equivalence in special situations. We also investigate their stability under change of base ring.

We use Setting 20.1, from Chapter 20, for this chapter. Thus $(R, \mathbf{m})$ is an excellent normal local domain with field of fractions K and completion $(\widehat{R}, \widehat{\mathbf{m}})$, and $t_1, \ldots, t_n$ are indeterminates over R. The elements $\tau_1, \ldots, \tau_n \in \widehat{\mathbf{m}}$ are algebraically independent over R, and we have embeddings:

$$R \hookrightarrow S = R_n = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)} \overset{\varphi}{\hookrightarrow} \widehat{R}$$

Using this setting and other terminology of Chapter 20, we summarize the results of this chapter.

Summary 21.1. In Section 21.1 we describe the three concepts of idealwise independence, residual algebraic independence, and primary independence defined in Definitions 20.2,20.24 and 20.12 of Chapter 20 in terms of certain flatness conditions on the embedding

$$\varphi : R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)} \hookrightarrow \widehat{R},$$

where $(R, \mathbf{m})$ is an excellent normal local ring and $(\widehat{R}, \widehat{\mathbf{m}})$ is the $\mathbf{m}$-adic completion of R. In Section 21.2 we investigate the stability of these independence concepts under base change, composition and polynomial extension. We prove in Corollary 21.19 the existence of uncountable excellent normal local domains R such that $\widehat{R}$ contains infinite sets of primarily independent elements.

We show in Section 21.3 that both residual algebraic independence and primary independence hold for elements over the original ring R exactly when they hold over the Henselization R^h of R (21.21). Also idealwise independence descends from the Henselization to the ring R.

A large diagram in Section 21.4 displays the relationships among the independence concepts and many other related properties.

21.1. Primary independence and flatness

In this section we describe the concept of primary independence in terms of flatness of certain localizations of the canonical embedding of Setting 20.1

$$\varphi : R_n = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)} \hookrightarrow \widehat{R}.$$

We establish in Chapter 20 flatness conditions for φ that are equivalent to idealwise independence and residual algebraic independence. We summarize these conditions in Remark 21.2.

REMARK 21.2. Let $(R, \mathbf{m})$ and $\tau_1, \dots, \tau_n \in \widehat{\mathbf{m}}$ be as in Setting 20.1, and let $\varphi : R_n = R[\tau_1, \dots, \tau_n]_{(\mathbf{m}, \tau_1, \dots, \tau_n)} \hookrightarrow \widehat{R}$ denote the canonical embedding. Then:

(1) $\tau_1, \dots, \tau_n$ are idealwise independent over R if and only if the map $R_n \hookrightarrow \widehat{R}$ is weakly flat; see Definitions 20.2 and 20.5 and Theorem 20.11..
(2) The elements $\tau_1, \dots, \tau_n$ are residually algebraically independent over R $\iff$ $\varphi : R_n = R[\tau_1, \dots, \tau_n]_{(\mathbf{m}, \tau_1, \dots, \tau_n)} \longrightarrow \widehat{R}$ satisfies LF_1; see Definition 8.1.3, Proposition 8.11 and Theorem 20.26.
(3) If each height-one prime of R is the radical of a principal ideal and the elements $\tau_1, \dots, \tau_n$ are residually algebraically independent over R, then the elements $\tau_1, \dots, \tau_n$ are idealwise independent over R. See Theorem 20.27.3.

These items follow from Theorem 20.11, Proposition 8.11 and Theorem 20.26.

To describe primary independence in terms of flatness of certain localizations of the embedding $\varphi : R_n \longrightarrow \widehat{R}$, we use the following lemma.

LEMMA 21.3. *Let $d \in \mathbb{N}$ and $n \in \mathbb{N}_0$, and let $(S, \mathbf{m}) \hookrightarrow (T, \mathbf{n})$ be a local embedding of catenary Noetherian local domains with $\dim T = d$ and $\dim S = d+n$. Assume the extension $S \hookrightarrow T$ satisfies the following property for every $P \in \operatorname{Spec} S$:*

$$\operatorname{ht} P \ \geq \ d \quad \Longrightarrow \quad PT \quad \text{is} \quad \mathbf{n}\text{-primary.} \tag{21.3.0}$$

Then, for every $Q \in \operatorname{Spec} T$ with $\operatorname{ht} Q \leq d-1$, we have $\operatorname{ht}(Q \cap S) \leq \operatorname{ht} Q$.

PROOF. If $Q \in \operatorname{Spec} T$ is such that $\operatorname{ht}(Q \cap S) \geq d$, then, by Property 21.3.0, $(Q \cap S)T$ is $\mathbf{n}$-primary, and so $Q = \mathbf{n}$ and $\operatorname{ht} Q = d$. Thus, for every $Q \in \operatorname{Spec} T$ with $\operatorname{ht} Q \leq d-1$, we have $\operatorname{ht}(Q \cap S) \leq d-1$. In particular, if $\operatorname{ht} Q = d-1$, then $\operatorname{ht}(Q \cap S) \leq \operatorname{ht} Q$.

We proceed by induction on $s \geq 1$: Assume $s \geq 2$ and $\operatorname{ht}(P \cap S) \leq \operatorname{ht} P$, for every $P \in \operatorname{Spec} T$ with $d > \operatorname{ht} P \geq d - s + 1$. Let $Q \in \operatorname{Spec} T$ with $\operatorname{ht} Q = d - s$. Suppose $\operatorname{ht}(Q \cap S) \geq d - s + 1$; choose $b \in \mathbf{m} \setminus Q$ and let $Q_1 \in \operatorname{Spec} T$ be minimal over $(b, Q)T$. Since T is catenary and Noetherian, we have $\operatorname{ht} Q_1 = d - s + 1$. By the inductive hypothesis, $\operatorname{ht}(Q_1 \cap S) \leq d - s + 1$. Since $b \in Q_1 \cap S$, the ideal $Q_1 \cap S$ properly contains $Q \cap S$. But this implies

$$d - s + 1 \ \geq \operatorname{ht}(Q_1 \cap S) \ > \ \operatorname{ht}(Q \cap S) \ \geq d - s + 1,$$

a contradiction. Thus $\operatorname{ht}(Q \cap S) \leq \operatorname{ht} Q$, for every $Q \in \operatorname{Spec} T$ with $\operatorname{ht} Q \leq d-1$. □

We use the LF_d notation of Definition 8.1.3 and Remark 8.2 in the following theorem.

THEOREM 21.4. *Let $(R, \mathbf{m})$ be an excellent normal local domain, and let the elements $\tau_1, \dots, \tau_n \in \widehat{\mathbf{m}}$ be as in Setting 20.1. Assume that $\dim R = d$. Then the elements $\tau_1, \dots, \tau_n$ are primarily independent over R if and only if*

$$\varphi : R_n = R[\tau_1, \dots, \tau_n]_{(\mathbf{m}, \tau_1, \dots, \tau_n)} \longrightarrow \widehat{R}$$

satisfies LF_{d-1}.

PROOF. To prove the $\Longrightarrow$ direction: Since R_n is a localized polynomial ring over R, the map $R \hookrightarrow R_n$ has regular fibers. Since R is excellent, the map $R \hookrightarrow \widehat{R}$ has regular, hence Cohen-Macaulay, fibers. Consider the sequence

$$R \ \longrightarrow \ R_n \ \xrightarrow{\varphi} \ \widehat{R}.$$

To show φ satisfies LF_{d-1}, we show that $\varphi_{\widehat{Q}}$ is flat for every $\widehat{Q} \in \operatorname{Spec}\widehat{R}$ with $\operatorname{ht}\widehat{Q} \leq d-1$. For this, by (2) $\implies$ (1) of Theorem 7.3, it suffices to show $\operatorname{ht}(\widehat{Q} \cap R_n) \leq \operatorname{ht}\widehat{Q}$ for every $\widehat{Q} \in \operatorname{Spec}\widehat{R}$ with $\operatorname{ht}\widehat{Q} \leq d-1$. This holds by Lemma 21.3, since primary independence implies Property 21.3.0.

For $\Longleftarrow$, let $P \in \operatorname{Spec} R_n$ be a prime ideal with $\dim(R_n/P) \leq n$. Suppose that $P\widehat{R}$ is not $\widehat{\mathbf{m}}$-primary and let $\widehat{Q} \supseteq P\widehat{R}$ be a minimal prime of $P\widehat{R}$. Then $\operatorname{ht}(\widehat{Q}) \leq d-1$. Set $Q = \widehat{Q} \cap R_n$, then LF_{d-1} implies that the map

$$\varphi_{\widehat{Q}} : (R_n)_Q \longrightarrow \widehat{R}_{\widehat{Q}}$$

is faithfully flat. Hence by going-down (Remark 2.31.10), $\operatorname{ht} Q \leq d-1$. But $P \subseteq Q$ and R_n is catenary, so $d-1 \geq \operatorname{ht} Q \geq \operatorname{ht} P \geq d$, a contradiction. We conclude that $\tau_1, \ldots, \tau_n$ are primarily independent. □

Remark 21.5. Theorem 21.4 yields a different proof of statements (1) and (3) of Theorem 20.27, that primarily independent elements are residually algebraically independent and that in dimension two, the two concepts are equivalent. Considering again our basic setting from (20.1) , with $d = \dim(R)$, Theorem 21.4 equates the LF_{d-1} condition on the extension $R_n = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)} \longrightarrow \widehat{R}$, to the primary independence of the τ_i. Also Proposition 8.11 and Theorem 20.26 yield that residual algebraic independence of the τ_i is equivalent to the extension $R_n = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)} \longrightarrow \widehat{R}$ satisfying LF_1. Clearly $\mathrm{LF}_i \implies \mathrm{LF}_{i-1}$, for $i > 1$, and if $d = \dim(R) = 2$, then $\mathrm{LF}_{d-1} = \mathrm{LF}_1$.

Remark 21.6. In Setting 20.1, if $\tau_1, \ldots, \tau_n$ are primarily independent over R and $\dim(R) = d$, then $\varphi : R_n \longrightarrow \widehat{R}$ satisfies LF_{d-1}, but *not* LF_d, that is, φ fails to be faithfully flat; for faithful flatness would imply going-down and hence that $\dim(R_n) \leq d = \dim(\widehat{R})$.

Example 21.7. By a modification of Example 8.13, it is possible to obtain, for each integer $d \geq 2$, an injective local map $\varphi : (A, \mathbf{m}) \longrightarrow (B, \mathbf{n})$ of normal Noetherian local domains with B essentially of finite type over A, $\varphi(\mathbf{m})B = \mathbf{n}$, and $\dim(B) = d$ such that φ satisfies LF_{d-1}, but fails to be faithfully flat over A. Let k be a field and let $x_1, \ldots, x_d, y$ be indeterminates over k. Let A be the localization of $k[x_1, \ldots, x_d, x_1y, \ldots, x_dy]$ at the maximal ideal generated by $(x_1, \ldots, x_d, x_1y, \ldots, x_dy)$, and let B be the localization of $A[y]$ at the prime ideal $(x_1, \ldots, x_d)A[y]$. Then A is an $d+1$-dimensional normal Noetherian local domain and B is an d-dimensional regular local domain birationally dominating A. For any nonmaximal prime Q of B we have $B_Q = A_{Q \cap A}$. Hence $\varphi : A \longrightarrow B$ satisfies LF_{d-1}, but φ is not faithfully flat since $\dim(B) < \dim(A)$.

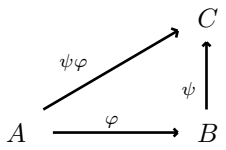
The local injective map $\varphi : (A, \mathbf{m}) \longrightarrow (B, \mathbf{n})$ of Example 21.7 is not height-one preserving. Remark 20.8 shows that if each height-one prime ideal of R is the radical of a principal ideal then the maps studied in this chapter are height-one preserving. We have the following question:

Question 21.8. Let $\varphi : (A, \mathbf{m}) \longrightarrow (B, \mathbf{n})$ be a local injective map of normal Noetherian local integral domains. Assume that B is essentially of finite type over A with $\dim B = d \geq 2$. If φ is both LF_{d-1} and height-one preserving, does it follow that φ is faithfully flat?

21.2. Composition, base change and polynomial extensions

In this section we investigate idealwise independence, residual algebraic independence, and primary independence under polynomial ring extensions and localizations of these polynomial extensions.

We start with a more general situation. Let



be a commutative diagram of commutative rings and injective maps. Proposition 21.9 implies that many of the properties of injective maps that we consider are stable under composition:

PROPOSITION 21.9. *Let $\varphi : A \longrightarrow B$ and $\psi : B \longrightarrow C$ be injective maps of commutative rings, and let $s \in \mathbb{N}$.*

1. *If φ and ψ satisfy LF_s, then $\psi\varphi$ satisfies LF_s.*
2. *If C is Noetherian, ψ is faithfully flat and the composite map $\psi\varphi$ satisfies LF_s, then φ satisfies LF_s.*
3. *Assume that A, B and C are Krull domains, and that $QC \neq C$, for each height-one prime Q of B. If φ and ψ are height-one preserving (respectively weakly flat), then $\psi\varphi$ is height-one preserving (respectively weakly flat).*

PROOF. The first item follows because a flat map satisfies going-down, see Remark 2.31.10. For item , since C is Noetherian and ψ is faithfully flat, B is Noetherian; see Remark 2.31.8. Let $Q \in \operatorname{Spec}(B)$ with $\operatorname{ht}(Q) = d \leq k$. We show $\varphi_Q : A_{Q\cap A} \longrightarrow B_Q$ is faithfully flat. By localization of B and C at $B \setminus Q$, we may assume that B is local with maximal ideal Q. Since C is faithfully flat over B, $QC \neq C$. Let $Q' \in \operatorname{Spec}(C)$ be a minimal prime of QC. Since C is Noetherian and B is local with maximal ideal Q, we have $\operatorname{ht}(Q') \leq d$ and $Q' \cap B = Q$. Since the composite map $\psi\varphi$ satisfies LF_k, the composite map

$$A_{Q'\cap A} = A_{Q\cap A} \xrightarrow{\varphi_Q} B_Q = B_{Q'\cap B} \xrightarrow{\psi_{Q'}} C_{Q'}$$

is faithfully flat. This and the faithful flatness of $\psi_{Q'} : B_{Q'\cap B} \longrightarrow C_{Q'}$ implies that φ_Q is faithfully flat [**101**, (4.B) page 27].

For item 3, let P be a height-one prime of A such that $PC \neq C$. Then $PB \neq B$ so if φ and ψ are height-one preserving then there exists a height-one prime Q of B such that $PB \subseteq Q$. By assumption, $QC \neq C$ (and ψ is height-one preserving), so there exists a height-one prime Q' of C such that $QC \subseteq Q'$. Hence $PC \subseteq Q'$.

If φ and ψ are weakly flat, then by Proposition 8.8 there exists a height-one prime Q of B such that $Q \cap A = P$. Again by assumption, $QC \neq C$, thus weakly flatness of ψ implies $QC \cap B = Q$. Now

$$P \subseteq PC \cap A \subseteq QC \cap A = QC \cap B \cap A = Q \cap A = P.$$

□

REMARKS 21.10. If in Proposition 21.9.3 the Krull domains B and C are local, but not necessarily Noetherian and ψ is a local map, then clearly $QC \neq C$ for each height-one prime Q of B.

If a map λ of Krull domains is faithfully flat, then λ is height-one preserving, weakly flat and satisfies condition LF_k for every integer $k \in \mathbb{N}$. Thus if $\varphi : A \longrightarrow B$ and $\psi : B \longrightarrow C$ are injective maps of Krull domains, such that one of φ or ψ is faithfully flat and the other is weakly flat (respectively height-one preserving or satisfies LF_k), then the composition $\psi\varphi$ is again weakly flat (respectively height-one preserving or satisfies LF_k). Moreover, if the map ψ is faithfully flat, we also obtain the following converse to Proposition 21.9.3:

PROPOSITION 21.11. *Let $\varphi : A \longrightarrow B$ and $\psi : B \longrightarrow C$ be injective maps of Krull domains. Assume that ψ is faithfully flat. If $\psi\varphi$ is height-one preserving (respectively weakly flat), then φ is height-one preserving (respectively weakly flat).*

PROOF. Let P be a height-one prime ideal of A such that $PB \neq B$. Since ψ is faithfully flat, $PC \neq C$; so if $\psi\varphi$ is height-one preserving, then there exists a height-one prime ideal Q' of C containing PC. Now $Q = Q' \cap B$ has height one by going-down for flat extensions, and $PB \subseteq Q' \cap B = Q$, so φ is height-one preserving. The proof of the weakly flat statement is similar, using Proposition 8.8. □

Next we consider a commutative square of commutative rings and injective maps:

$$\begin{array}{ccc} A' & \xrightarrow{\varphi'} & B' \\ {\scriptstyle\mu}\uparrow & & \uparrow{\scriptstyle\nu} \\ A & \xrightarrow{\varphi} & B \end{array}$$

PROPOSITION 21.12. *In the diagram above, assume that μ and ν are faithfully flat, and let $k \in \mathbb{N}$. Then:*

1. *(Ascent) Assume that $B' = B \otimes_A A'$, or that B' is a localization of $B \otimes_A A'$. Let ν denote the canonical map associated with this tensor product. If $\varphi : A \longrightarrow B$ satisfies LF_k, then $\varphi' : A' \longrightarrow B'$ satisfies LF_k.*
2. *(Descent) If B' is Noetherian and $\varphi' : A' \longrightarrow B'$ satisfies LF_k, then $\varphi : A \longrightarrow B$ satisfies LF_k.*
3. *(Descent) Assume that the rings A, A', B and B' are Krull domains. If $\varphi' : A' \longrightarrow B'$ is height-one preserving (respectively weakly flat), then $\varphi : A \longrightarrow B$ is height-one preserving (respectively weakly flat).*

PROOF. For (1), assume that φ satisfies LF_k and let $Q' \in \operatorname{Spec}(B')$ with $\operatorname{ht}(Q') \leq k$. Put $Q = (\nu)^{-1}(Q')$, $P' = (\varphi')^{-1}(Q')$, and $P = \mu^{-1}(P') = \varphi^{-1}(Q)$ and consider the commutative diagrams:

$$\begin{array}{ccc} A' & \xrightarrow{\varphi'} & B' \\ {\scriptstyle\mu}\uparrow & & \uparrow{\scriptstyle\nu} \\ A & \xrightarrow{\varphi} & B \end{array} \qquad\qquad \begin{array}{ccc} A'_{P'} & \xrightarrow{\varphi'_{Q'}} & B'_{Q'} \\ {\scriptstyle\mu_{P'}}\uparrow & & \uparrow{\scriptstyle\nu_{Q'}} \\ A_P & \xrightarrow{\varphi_Q} & B_Q \end{array}$$

The flatness of ν implies that $\operatorname{ht}(Q) \leq k$ and so by assumption, φ_Q is faithfully flat. The ring $B'_{Q'}$ is a localization of $B_Q \otimes_{A_P} A'_{P'}$ and B_Q is faithfully flat over A_P implies $B'_{Q'}$ is faithfully flat over $A'_{P'}$.

For (2), by Proposition 21.9.1, $\varphi'\mu = \nu\varphi$ satisfies LF_k. Now by Proposition 21.9 .2, φ satisfies LF_k.

Item 3 follows immediately from the assumption that μ and ν are faithfully flat maps and hence going-down holds [**101**, Theorem 4, page 33]. □

Next we examine the situation for polynomial extensions.

PROPOSITION 21.13. *Let $(R, \mathbf{m})$ and $\{\tau_i\}_{i=1}^m \subseteq \widehat{\mathbf{m}}$ be as in Setting 20.1, where m is either an integer or $m = \infty$, and the dimension of R is at least 2. Let z be an indeterminate over $\widehat{R}$. Then:*

(1) *$\{\tau_i\}_{i=1}^m$ is residually algebraically independent over $R \iff \{\tau_i\}_{i=1}^m$ is residually algebraically independent over $R[z]_{(\mathbf{m},z)}$.*
(2) *If $\{\tau_i\}_{i=1}^m$ is idealwise independent over $R[z]_{(\mathbf{m},z)}$, then $\{\tau_i\}_{i=1}^m$ is idealwise independent over R.*

PROOF. Let $n \in \mathbb{N}$ be an integer with $n \leq m$. Set $R_n = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m},\tau_1,\ldots,\tau_n)}$. Let $\varphi : R_n \longrightarrow \widehat{R}$ and $\mu : R_n \longrightarrow R_n[z]$ be the inclusion maps. We have the following commutative diagram:

$$\begin{array}{ccccc} R_n[z]_{(\max(R_n),z)} & \xrightarrow{\varphi'} & R' = \widehat{R}[z]_{(\widehat{\mathbf{m}},z)} & \xrightarrow{\psi} & \widehat{R'} = \widehat{R}[[z]] \\ \mu\uparrow & & \mu'\uparrow & & \\ R_n & \xrightarrow{\varphi} & \widehat{R} & & \end{array}$$

The ring R' is a localization of the tensor product $\widehat{R} \otimes_{R_n} R_n[z]$ and Proposition 21.12 applies. Thus, for (1), φ satisfies LF_1 if and only if φ' satisfies LF_1. Since the inclusion map ψ of $R' = \widehat{R}[z]_{(\widehat{\mathbf{m}},z)}$ to its completion $\widehat{R}[[z]]$ is faithfully flat, we obtain equivalences:

φ satisfies $LF_1 \iff \varphi'$ satisfies $LF_1 \iff \psi\varphi'$ satisfies LF_1.

(2) If the τ_i are idealwise independence over $R[z]_{(\mathbf{m},z)}$, the map $\psi\varphi'$ is weakly flat. Thus φ' is weakly flat and the statement follows by Proposition 21.9 . □

We also obtain:

PROPOSITION 21.14. *Let $A \hookrightarrow B$ be an extension of Krull domains such that for each height-one prime $P \in \operatorname{Spec}(A)$ we have $PB \neq B$, and let Z be a (possibly uncountable) set of indeterminates over A. Then $A \hookrightarrow B$ is weakly flat if and only if $A[Z] \hookrightarrow B[Z]$ is weakly flat.*

PROOF. Let F denote the field of fractions of A. By Corollary 8.4, the extension $A \hookrightarrow B$ is weakly flat if and only if $F \cap B = A$. Thus the assertion follows from $F \cap B = A \iff F(Z) \cap B[Z] = A[Z]$. □

It would be interesting to know whether the converse of Proposition 21.13.2 is true. In this connection we have:

REMARKS 21.15. Let $\varphi : A \longrightarrow B$ be a weakly flat map of Krull domains, and let P be a height-one prime in A.

(1) Let Q be a minimal prime of the extended ideal PB. If the map $\varphi_Q : A \longrightarrow B_Q$ is weakly flat, then $\operatorname{ht} Q = 1$. To see this, observe that QB_Q is the unique minimal prime of PB_Q, so QB_Q is the radical of PB_Q. If φ_Q is weakly flat, then $PB_Q \cap A = P$ and hence $QB_Q \cap A = P$. It follows

that $A_P \hookrightarrow B_Q$. Since A_P is a DVR and its maximal ideal PA_P extends to an ideal primary for the maximal ideal QB_Q of the Krull domain B_Q, we must have that B_Q is a DVR and hence $\operatorname{ht} Q = 1$.

(2) Thus if there exists a weakly flat map of Krull domains $\varphi : A \longrightarrow B$ and a minimal prime Q of PB such that $\operatorname{ht} Q > 1$, then the map $\varphi_Q : A \longrightarrow B_Q$ fails to be weakly flat,

(3) If P is the radical of a principal ideal, then each minimal prime of PB has height one.

QUESTION 21.16. Let $\varphi : A \longrightarrow B$ be a weakly flat map of Krull domains, and let P be a height-one prime in A, as in Remarks 21.15. Is it possible that the extended ideal PB has a minimal prime Q with $\operatorname{ht} Q > 1$?

REMARK 21.17. Primary independence never lifts to polynomial rings. To see that $\tau \in \widehat{\mathbf{m}}$ fails to be primarily independent over $R[z]_{(\mathbf{m},z)}$, observe that $\mathbf{m}R[z]_{(\mathbf{m},z)}$ is a dimension-one prime ideal that extends to $\widehat{\mathbf{m}}[[z]]$, which also has dimension one and is *not* $(\mathbf{m}, z)$-primary in $\widehat{R}[[z]]$. Alternatively, in the language of locally flat maps, if the elements $\{\tau_i\}_{i=1}^m \subseteq \widehat{\mathbf{m}}$ are primarily independent over R, then Proposition 21.9 implies that the map

$$\varphi' : R_n[z]_{(\max(R_n),z)} \longrightarrow \widehat{R}[[z]]$$

satisfies condition LF_{d-1}, where $d = \dim(R)$. For $\{\tau_i\}_{i=1}^m$ to be primarily independent over $R[z]_{(\mathbf{m},z)}$, however, the map φ' has to satisfy LF_d, since $\dim R[z]_{(\mathbf{m},z)} = d+1$. Using Proposition 21.9 again this forces $\varphi : R_n \longrightarrow \widehat{R}$ to satisfy condition LF_d and thus φ is flat, which can happen only if $n = 0$. This is an interesting phenomenon; the construction of primarily independent elements involves all parameters of the ring R.

In the remainder of this section we consider localizations of polynomial extensions so that the dimension does not increase. Theorem 21.18 gives a method to obtain residually algebraically independent and primarily independent elements over an uncountable excellent local domain. In Theorem 21.18 we make use of the fact that if A is a Noetherian ring and Z is a set of indeterminates over A, then the ring $A(Z)$ obtained by localizing the polynomial ring $A[Z]$ at the multiplicative system of polynomials whose coefficients generate the unit ideal of A is again a Noetherian ring [**47**, Theorem 6].

THEOREM 21.18. *Let $(R, \mathbf{m})$ and $\{\tau_i\}_{i=1}^m \subset \widehat{\mathbf{m}}$ be as in Setting 20.1, where m is either an integer or $m = \infty$, and $\dim(R) = d \geq 2$. Let Z be a set (possibly uncountable) of indeterminates over $\widehat{R}$ and let $R(Z) = R[Z]_{(\mathbf{m}R[Z])}$. Then:*

(1) *$\{\tau_i\}_{i=1}^m$ is primarily independent over $R \iff \{\tau_i\}_{i=1}^m$ is primarily independent over $R(Z)$.*

(2) *$\{\tau_i\}_{i=1}^m$ is residually algebraically independent over $R \iff \{\tau_i\}_{i=1}^m$ is residually algebraically independent over $R(Z)$.*

(3) *If $\{\tau_i\}_{i=1}^m$ is idealwise independent over $R(Z)$, then $\{\tau_i\}_{i=1}^m$ is idealwise independent over R.*

PROOF. Let $n \in \mathbb{N}$ be an integer with $n \leq m$, put $R_n = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m},\tau_1,\ldots,\tau_n)}$ and let $\mathbf{n}$ denote the maximal ideal of R_n. Let $\varphi : R_n \longrightarrow \widehat{R}$ and $\mu : R_n \longrightarrow R_n(Z) = R_n[Z]_{\mathbf{n}R_n[Z]}$ be the inclusion maps. We have the following commutative

diagram:

$$\begin{array}{ccc} R_n(Z) & \xrightarrow{\varphi'} & \widehat{R}(Z) \\ {\scriptstyle\mu}\uparrow & & {\scriptstyle\mu'}\uparrow \\ R_n & \xrightarrow{\varphi} & \widehat{R} \end{array}$$

The ring $\widehat{R}(Z)$ is a localization of the tensor product $\widehat{R} \otimes_{R_n} R_n[Z]$ and Proposition 21.12 applies. Thus, for item 1, φ satisfies LF_{d-1} if and only if φ' satisfies LF_{d-1}. Similarly, for item 2, φ satisfies LF_1 if and only if φ' satisfies LF_1.

Since the inclusion map ψ taking $\widehat{R}(Z)$ to its completion is faithfully flat, we obtain equivalences:

$$\varphi \text{ satisfies } LF_k \iff \varphi' \text{ satisfies } LF_k \iff \psi\varphi' \text{ satisfies } LF_k.$$

Since primary independence is equivalent to LF_{d-1} by Theorem 21.4 and residual algebraic independence is equivalent to LF_1 by Proposition 8.11, statements 1 and 2 follow.

For item 3, if the τ_i are idealwise independence over $R(Z)$, the morphism $\psi\varphi'$ is weakly flat. Thus φ' is weakly flat. The statement follows by Proposition 21.9. □

COROLLARY 21.19. *Let k be a countable field, let Z be an uncountable set of indeterminates over k and let x, y be additional indeterminates. Then $R := k(Z)[x,y]_{(x,y)}$ is an uncountable excellent normal local domain of dimension two, and, for m a positive integer or $m = \infty$, there exist m primarily independent elements (and hence also residually algebraically and idealwise independent elements) over R.*

PROOF. Apply Proposition 20.15 and Theorems 20.20, 20.27 and 21.18. □

21.3. Passing to the Henselization

In this section we investigate idealwise independence, residual algebraic independence, and primary independence as we pass from R to the Henselization R^h of R. In particular, we show in Proposition 21.24 that for a single element $\tau \in \mathbf{m}\widehat{R}$ the notions of idealwise independence and residual algebraic independence coincide if $R = R^h$. This implies that for every excellent normal local Henselian domain of dimension 2 all three concepts coincide for an element $\tau \in \widehat{\mathbf{m}}$; that is, τ is idealwise independent $\iff$ τ is residually algebraically independent $\iff$ τ is primarily independent.

We use the commutative square of Propostion 21.12 and obtain the following result for Henselizations:

PROPOSITION 21.20. *Let $\varphi : (A, \mathbf{m}) \hookrightarrow (B, \mathbf{n})$ be an injective local map of normal Noetherian local domains, and let $\varphi^h : A^h \longrightarrow B^h$ denote the induced map of the Henselizations. Then:*

1. *For each k with $1 \le k \le \dim(B)$, φ satisfies LF_k $\iff$ φ^h satisfies LF_k. Thus φ satisfies PDE $\iff$ φ^h satisfies PDE.*
2. *(Descent) If φ^h is height-one preserving (respectively weakly flat), then φ is height-one preserving (respectively weakly flat).*

Using shorthand and diagrams, we show Proposition 21.20 schematically:

$$\boxed{\varphi \text{ is } LF_k} \iff \boxed{\varphi^h \text{ is } LF_k} \quad ; \quad \boxed{\varphi \text{ is PDE}} \iff \boxed{\varphi^h \text{ is PDE}}$$

$$\boxed{\varphi \text{ ht-1 pres}} \impliedby \boxed{\varphi^h \text{ ht-1 pres}} \quad ; \quad \boxed{\varphi \text{ w.f.}} \impliedby \boxed{\varphi^h \text{ w.f.}}.$$

PROOF. (of Proposition 21.20) Consider the commutative diagram:

$$\begin{array}{ccc} A^h & \xrightarrow{\varphi^h} & B^h \\ {\scriptstyle\mu}\uparrow & & \uparrow{\scriptstyle\nu} \\ A & \xrightarrow{\varphi} & B \end{array}$$

where μ and ν are the faithfully flat canonical injections [**117**, (43.8), page 182]. Since φ is injective and A is normal, φ^h is injective by [**117**, (43.5)]. By Proposition 20.25, Proposition 8.11 and Proposition 20.26), we need only show "⇒" in (1).

Let $Q' \in \operatorname{Spec}(B^h)$ with $\operatorname{ht}(Q') \leq k$. Put $Q = Q' \cap B$, $P' = Q' \cap A^h$, and $P = P' \cap A$. We consider the localized diagram:

$$\begin{array}{ccc} A^h_{P'} & \xrightarrow{\varphi^h_{Q'}} & B^h_{Q'} \\ {\scriptstyle\mu_{P'}}\uparrow & & \uparrow{\scriptstyle\nu_{Q'}} \\ A_P & \xrightarrow{\varphi_Q} & B_Q \end{array}$$

The faithful flatness of ν implies $\operatorname{ht}(Q) \leq k$.

In order to show that $\varphi^h_{Q'} : A^h_{P'} \longrightarrow B^h_{Q'}$ is faithfully flat, we apply Remark 7.2.2 with $M = B^h_{Q'}$ and $I = PB^h_{Q'}$.

First note that P' is a minimal prime divisor of PA^h and that $(A^h/PA^h)_{P'} = (A^h/P')_{P'}$ is a field [**117**, (43.20)]. Thus

$$\overline{\varphi^h_{Q'}} : (A^h/PA^h)_{P'} \longrightarrow (B^h/PB^h)_{Q'}$$

is faithfully flat and it remains to show that

$$PA^h_{P'} \otimes_{A^h_{P'}} B^h_{Q'} \cong PB^h_{Q'}.$$

This can be seen as follows:

$$\begin{aligned} PA^h_{P'} \otimes_{A^h_{P'}} B^h_{Q'} &\cong (P \otimes_{A_P} A^h_{P'}) \otimes_{A^h_{P'}} B^h_{Q'} && \text{by flatness of } \mu \\ &\cong P \otimes_{A_P} B^h_{Q'} \\ &\cong (P \otimes_{A_P} B_Q) \otimes_{B_Q} B^h_{Q'} \\ &\cong PB_Q \otimes_{B_Q} B^h_{Q'} && \text{by flatness of } \varphi_Q \\ &\cong PB^h_{Q'} && \text{by flatness of } \nu. \end{aligned}$$

□

COROLLARY 21.21. *Let $(R, \mathbf{m})$ and $\{\tau_i\}_{i=1}^m$ be as in Setting 20.1, where m is either a positive integer or $m = \infty$ and $\dim(R) = d \geq 2$. Then:*

(1) *$\{\tau_i\}_{i=1}^m$ is primarily independent over R $\iff$ $\{\tau_i\}_{i=1}^m$ is primarily independent over R^h.*

(2) $\{\tau_i\}_{i=1}^m$ *is residually algebraically independent over* $R \iff \{\tau_i\}_{i=1}^m$ *is residually algebraically independent over* R^h.
(3) *(Descent) If* $\{\tau_i\}_{i=1}^m$ *is idealwise independent over* R^h *then* $\{\tau_i\}_{i=1}^m$ *is idealwise independent over* R.

PROOF. For (1) and (2) it suffices to show the equivalence for every positive integer $n \leq m$. By [**117**, (43.5)]. the local rings $R_n = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)}$ and $\tilde{R}_n = R^h[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)}$ have the same Henselization R_n^h. Also $R_n \subseteq \tilde{R}_n$. By Theorem 21.4 and Proposition 21.20 we have:

$$\begin{aligned} &\tau_1, \ldots, \tau_n \text{ are primarily (respectively residually algebraically)} \\ &\qquad \text{independent over } R \qquad \Longleftrightarrow \\ &R_n \longrightarrow \widehat{R} \text{ satisfies } LF_{d-1} \text{ (respectively } LF_1) \iff \\ &R_n^h \longrightarrow \widehat{R} = \widehat{R}^h \text{ satisfies } LF_{d-1} \text{ (respectively } LF_1) \iff \\ &\tilde{R}_n \longrightarrow \widehat{R} \text{ satisfies } LF_{d-1} \text{ (respectively } LF_1). \end{aligned}$$

The third statement on idealwise independence follows from Theorem 21.12.3 by considering

$$\begin{array}{ccc} \tilde{R}_n & \xrightarrow{\varphi'} & \widehat{R} \\ {\scriptstyle\mu}\uparrow & & \| \\ R_n & \xrightarrow{\varphi} & \widehat{R}. \end{array}$$

□

REMARK 21.22. The examples given in Theorems 20.33 and 20.35 show the converse to part (3) of (21.21) fails: weak flatness need not lift to the Henselization. With the notation of Proposition 21.20, if φ is weakly flat, then for every $P \in \operatorname{Spec}(A)$ of height one with $PB \neq B$ there exists by Proposition 8.8, $Q \in \operatorname{Spec}(B)$ of height one such that $P = Q \cap A$. In the Henselization A^h of A, the ideal PA^h is a finite intersection of height-one prime ideals P_i' of A^h [**117**, (43.20)]. Only one of the P_i' is contained in Q. Thus as in Theorems 20.33 and 20.35, one of the minimal prime divisors P_i' may fail the condition for weak flatness.

Let R be an excellent normal local domain with Henselization R^h and let K, and K^h denote the fields of fractions of R and R^h respectively. Let L be an intermediate field with $K \subseteq L \subseteq K^h$. It is shown in [**135**] that the intersection ring $T = L \cap \widehat{R}$ is an excellent normal local domain with Henselization $T^h = R^h$. Henselian excellent normal local domains are algebraically closed in their completion; see Remark **??.??**. Thus we have:

COROLLARY 21.23. *Let* $(R, \mathbf{m})$ *and* $\{\tau_i\}_{i=1}^m$ *be as in Setting 20.1, where* m *denotes a positive integer or* $m = \infty$. *Let* T *be a Noetherian local domain dominating* R *and algebraic over* R *and dominated by* $\widehat{R}$ *with* $\widehat{R} = \widehat{T}$. *Then:*

(1) $\{\tau_i\}_{i=1}^m$ *is primarily independent over* $R \iff \{\tau_i\}_{i=1}^m$ *is primarily independent over* T.
(2) $\{\tau_i\}_{i=1}^m$ *is residually algebraically independent over* $R \iff \{\tau_i\}_{i=1}^m$ *is residually algebraically independent over* T.
(3) *If* $\{\tau_i\}_{i=1}^m$ *is idealwise independent over* T, *then* $\{\tau_i\}_{i=1}^m$ *is idealwise independent over* R.

PROOF. By [**135**], R and T have a common Henselization, and the statements follow from Corollary 21.21. □

We have seen in Theorem 20.27 that, if R has the property that every height-one prime ideal is the radical of a principal ideal and $\tau \in \widehat{\mathbf{m}}$ is residually algebraically independent over R, then τ is idealwise independent over R. In Proposition 21.24 we describe a situation in which idealwise independence implies residual algebraic independence.

PROPOSITION 21.24. *Let $(R, \mathbf{m})$ and $\tau \in \widehat{m}$ be as in Setting 20.1. Suppose R has the property that, for each $P \in \operatorname{Spec}(R)$ with $\operatorname{ht}(P) = 1$, the ideal $P\widehat{R}$ is prime.*

1. *If τ is idealwise independent over R, then τ is residually algebraically independent over R.*
2. *If R has the additional property that every height-one prime ideal is the radical of a principal ideal, then τ is idealwise independent over $R \iff \tau$ is residually algebraically independent over R.*

PROOF. For item 1, let $\widehat{P} \in \operatorname{Spec}(\widehat{R})$ be such that $\operatorname{ht}(\widehat{P}) = 1$ and $\widehat{P} \cap R \neq 0$. Then $\operatorname{ht}(\widehat{P} \cap R) = 1$ and $(\widehat{P} \cap R)R[\tau]$ is a prime ideal of $R[\tau]$ of height 1. Idealwise independence of τ implies that $(\widehat{P} \cap R)R[\tau] = (\widehat{P} \cap R)\widehat{R} \cap R[\tau]$. Since $(\widehat{P} \cap R)\widehat{R}$ is nonzero and prime, we have $\widehat{P} = (\widehat{P} \cap R)\widehat{R}$ and $\widehat{P} \cap R[\tau] = (\widehat{P} \cap R)R[\tau]$. Therefore $\operatorname{ht}(\widehat{P} \cap R[\tau]) = 1$ and Theorem 20.26 implies that τ is residually algebraically independent over R.

Item 1 implies item 2 by Theorem 20.27.3. □

REMARK 21.25. If R is Henselian, or if R/P is Henselian for each height-one prime P of R, then R has the property that, for each $P \in \operatorname{Spec}(R)$ with $\operatorname{ht}(P) = 1$, the ideal $P\widehat{R}$ is prime, as in the hypothesis of Proposition 21.24. To see this, let P be a height-one prime of R such that R/P is Henselian. Then the integral closure of the domain R/P in its field of fractions is again local, in fact an excellent normal local domain and so analytically normal. This implies that the extended ideal $P\widehat{R}$ is prime, because of the behavior of completions of finite integral extensions [**117**, (17.7), (17.8)]. There is an example in [**6**] of a normal Noetherian local domain R that is not Henselian but, for each prime ideal P of R of height-one, the domain R/P is Henselian.

It is unclear whether Proposition 21.24 extends to more than one algebraically independent element $\tau \in \widehat{\mathbf{m}}$, because even if R is Henselian, the localized polynomial ring $R[\tau]_{(\mathbf{m},\tau)}$ fails to be Henselian.

COROLLARY 21.26. *Let R be an excellent Henselian normal local domain of dimension 2, and assume the notation of Setting 20.1. Then:*

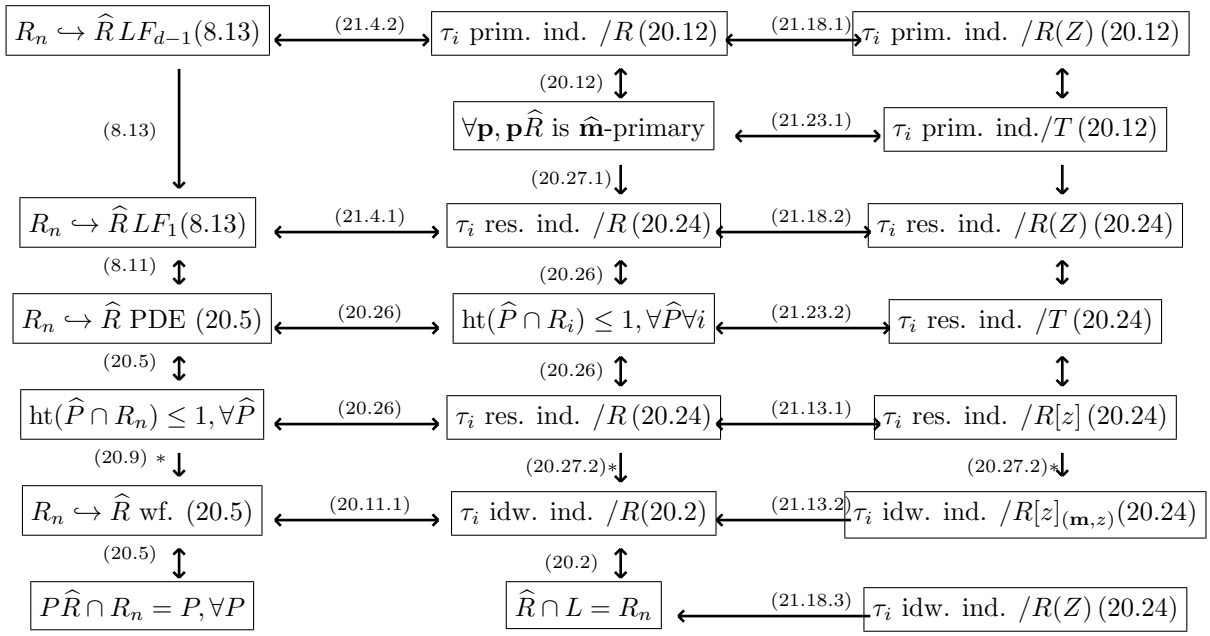
1. *τ is residually algebraically independent over $R \iff \tau$ is primarily independent over R.*
2. *Either of these equivalent conditions implies τ is idealwise independent over R.*
3. *If R has the additional property that every height-one prime ideal is the radical of a principal ideal, then the three conditions are equivalent.*

PROOF. This follows from Theorem 20.27, Proposition 20.15.1 and Proposition 21.24. □

21.4. Summary diagram for the independence concepts

With the notation of Setting 20.1 for $R, \mathbf{m}, R_n, \tau_1 \ldots, \tau_n$, let $d = \dim(R)$, L the field of fractions of R_n, $\mathbf{p} \in \operatorname{Spec}(R_n)$ such that $\dim(R_n/\mathbf{p}) \leq d-1$, $P \in \operatorname{Spec}(R_n)$ with $\operatorname{ht}(P) = 1$, $\widehat{P} \in \operatorname{Spec}(\widehat{R})$ with $\operatorname{ht}(\widehat{P}) = 1$, R^h = the Henselization of R in $\widehat{R}$, T a local Noetherian domain dominating and algebraic over R and dominated by $\widehat{R}$ with $\widehat{R} = \widehat{T}$, z an indeterminate over the field of fractions of $\widehat{R}$ and Z a possibly uncountable set of set of indeterminates over the field of fractions of $\widehat{R}$. Then we have the implications shown below. We use the abbreviations "prim. ind.", "res. ind." and "idw. ind" for "primarily independent", "residually independent" and "idealwise independent".

NOTE 21.27. $R_n \hookrightarrow \widehat{R}$ is always height-one preserving by Proposition 20.8.



* We assume that every height-one prime ideal of R is a principal ideal in order to have these arrows.

CHAPTER 22

Rings between excellent normal local domains and their completions I

In this chapter we continue the investigation of Chapters 4 through 10. We adjust the focus to include as base rings certain Krull domains that are not necessarily Noetherian. We take our working setting here to be Krull domains because in the setting of Krull domains it is possible to iterate the construction. The intersection of a normal Noetherian domain with a subfield of its field of fractions is always a Krull domain, but may fail to be Noetherian. As in Chapters 4 to 10 we consider completions with respect to a principal ideal.

For an excellent normal local domain $(R, \mathbf{m})$, the construction in Chapters 20 and 21 uses the entire $\mathbf{m}$-adic completion rather than a completion with respect to a principal ideal. With $(S, \mathbf{n})$ a localized polynomial ring in several variables over R, Chapters 20 and 21 contain non-trivial examples of ideals $\mathbf{a}$ of the $\mathbf{n}$-adic completion $\widehat{S}$ of S such that the constructed ring $D := \mathcal{Q}(S) \cap (\widehat{S}/\mathbf{a})$ of Homomorphic Image Construction 17.5 results in the ring $D = \mathcal{Q}(S) \cap \widehat{S}/\mathbf{a} = S$. Chapters 20 and 21 also contain examples of subfields L of the field of fractions of $\widehat{R}$ such that the ring $D := L \cap \widehat{R}$ of Inclusion Construction 5.3 is a localized polynomial ring over R in finitely many or infinitely many variables. In particular, this gives examples where the intersection ring is a non-Noetherian Krull domain.

We use the completion with respect to a principal ideal again in this chapter because in most examples of new Noetherian domains produced using Homomorphic Image Construction 17.5, that is, $D = \mathcal{Q}(S) \cap (\widehat{S}/\mathbf{a})$, the ideal $\mathbf{a}$ of $\widehat{S}$ is extended from a completion of R with respect to a principal ideal. Furthermore the completion with respect to a larger ideal can be obtained by appropriately iterating the procedure with the completion with respect to a principal ideal.

Let z be a nonzero nonunit of a Noetherian integral domain R and let R^* denote the (z)-adic completion of R. Let $\tau_1, \ldots, \tau_s$ be elements of zR^* that are algebraically independent over R. Assume that every nonzero element of the polynomial ring $R[\tau_1, \ldots, \tau_s]$ is a regular element of R^*. In Definition 5.10, we define $\tau_1, \ldots, \tau_s$ to be *limit-intersecting* over R if the intersection domain A is equal to the approximation domain B as defined in Section 5.2.

We also investigate here two stronger forms of the limit-intersecting condition, given in Definitions 22.9; these are useful for constructing examples and for determining if A is Noetherian or excellent. We give criteria for $\tau_1, \tau_2, \ldots, \tau_s$ to have these properties. These properties are analogs to types of "idealwise independence" over R defined in Chapter 20. These modified independence conditions enable us to produce concrete examples illustrating the concepts.

Many concepts from our earlier chapters are useful in this study, including several flatness conditions for extensions of Krull domains. For convenience, we recall some definitions that are relevant for this chapter:

DEFINITIONS 22.1. Let $S \hookrightarrow T$ be an extension of Krull domains.

- T is a *PDE* extension of S if for every height-one prime ideal Q in T, the height of $Q \cap S$ is at most one.
- T is a *height-one preserving* extension of S if for every height-one prime ideal P of S with $PT \neq T$ there exists a height-one prime ideal Q of T with $PT \subseteq Q$.
- T is *weakly flat* over S if every height-one prime ideal P of S with $PT \neq T$ satisfies $PT \cap S = P$.
- Let $r \in \mathbb{N}$ be an integer with $1 \leq r \leq d = \dim(T)$ where d is an integer or $d = \infty$. Then φ is called *locally flat in height* r, abbreviated LF_r, if, for every prime ideal Q of T with $\mathrm{ht}(Q) \leq r$, the induced map on the localizations $\varphi_Q : S_{Q\cap S} \longrightarrow T_Q$ is faithfully flat.

We recall the following proposition, a restatement of Corollary 8.4 of Chapter 8.

PROPOSITION 22.2. *Let $\varphi : S \hookrightarrow T$ be an extension of Krull domains and let F denote the field of fractions of S.*

(1) *Assume that $PT \neq T$ for every height-one prime ideal P of S. Then $S \hookrightarrow T$ is weakly flat $\iff$ $S = F \cap T$.*
(2) *If $S \hookrightarrow T$ is weakly flat, then φ is height-one preserving and, moreover, for every height-one prime ideal P of S with $PT \neq T$, there is a height-one prime ideal Q of T with $Q \cap S = P$.*

REMARK 22.3. By Proposition 22.2 weakly flat extensions are height-one preserving. Example 8.9 of Chapter 8 shows that the height-one preserving condition does not imply weakly flat.

22.1. Intersections and directed unions

In general the intersection of a normal Noetherian domain with a subfield of its field of fractions is a Krull domain, but need not be Noetherian. The Krull domain B in the motivating example of Section 12.1 in Chapter 12 (in the case where $B \neq A$) illustrates that a directed union of normal Noetherian domains may be a non-Noetherian Krull domain. Thus, in order to apply an iterative procedure in Section 22.2, we consider a local Krull domain $(T, \mathbf{n})$ that is not assumed to be Noetherian, but is assumed to have a Noetherian completion. To distinguish from the earlier Noetherian hypothesis on R, we let T denote the base domain.

SETTING AND NOTATION 22.4. Let $(T, \mathbf{n})$ be a local Krull domain with field of fractions F. Assume there exists a nonzero element $y \in \mathbf{n}$ such that the y-adic completion $\widehat{(T,(y))} := (T^*, \mathbf{n}^*)$ of T is an analytically normal Noetherian local domain. It then follows that the $\mathbf{n}$-adic completion $\widehat{T}$ of T is also a normal Noetherian local domain, since the $\mathbf{n}$-adic completion of T is the same as the $\mathbf{n}^*$-adic completion of T^*. Let F^* denote the field of fractions of T^*. Since T^* is Noetherian, $\widehat{T}$ is faithfully flat over T^* and we have $T^* = \widehat{T} \cap F^*$. Therefore $F \cap T^* = F \cap \widehat{T}$.

Let d denote the dimension of the Noetherian domain T^*. It follows that d is also the dimension of $\widehat{T}$. [1]

(1) Assume that $T = F \cap T^* = F \cap \widehat{T}$, or equivalently by (22.2.1), that T^* and $\widehat{T}$ are weakly flat over T.

(2) Let $\widehat{T}[1/y]$ denote the localization of $\widehat{T}$ at the powers of y, and similarly, let $T^*[1/y]$ denote the localization of T^* at the powers of y. The domains $\widehat{T}[1/y]$ and $T^*[1/y]$ have dimension $d-1$.

(3) Let $\tau_1, \ldots, \tau_s \in \mathbf{n}^*$ be algebraically independent over F.

(4) For each i with $1 \leq i \leq s$, we have an expansion $\tau_i := \Sigma_{j=1}^{\infty} c_{ij} y^j$ where $c_{ij} \in T$.

(5) For each $n \in \mathbb{N}$ and each i with $1 \leq i \leq s$, we define the n^{th}-endpiece of τ_i with respect to y as in Notation 5.4, so that

$$\tau_{in} := \Sigma_{j=n+1}^{\infty} c_{ij} y^{j-n}, \qquad \tau_{in} = y\tau_{i,n+1} + c_{i,n+1} y.$$

(6) For each $n \in \mathbb{N}$, we define $B_n := T[\tau_{1n}, \ldots, \tau_{sn}]_{(\mathbf{n}, \tau_{1n}, \ldots, \tau_{sn})}$. In view of (5), we have $B_n \subseteq B_{n+1}$ and B_{n+1} dominates B_n for each n. We define

$$B := \varinjlim_{n \in \mathbb{N}} B_n = \bigcup_{n=1}^{\infty} B_n, \quad \text{and} \quad A := F(\tau_1, \ldots, \tau_s) \cap \widehat{T}.$$

Thus B and A are local domains. We show that B and A are local Krull domains and that A birationally dominates B in Theorem 22.8. We are especially interested in conditions which imply that $B = A$.

(7) Let A^* denote the y-adic completion of A and let B^* denote the y-adic completion of B.

REMARK 22.5. The iterative example of Section 12.1 in Chapter 12 as given in Theorem 12.3, with $T := B \neq A$, from the notation of (12.2) and Example 12.6, shows that $T \longrightarrow T^*[1/y]$ can satisfy the other conditions of (22.4) but not satisfy the assumption (22.4.1); that is, such an extension is not in general weakly flat.

We show that the definitions of B and B_n are independent of the representations for $\tau_1, \ldots, \tau_s$ as power series in y with coefficients in T; the proof is analogous to that of Proposition 5.9.

PROPOSITION 22.6. *The definitions of B and B_n are independent of representations for $\tau_1, \ldots, \tau_s$ as power series in y with coefficients in T.*

PROOF. For $1 \leq i \leq s$, assume that τ_i and $\omega_i = \tau_i$ have representations

$$\tau_i = \sum_{j=1}^{\infty} a_{ij} y^j \qquad \text{and} \qquad \omega_i = \sum_{j=1}^{\infty} b_{ij} y^j,$$

where each $a_{ij}, b_{ij} \in T$. We define the n^{th}-endpieces τ_{in} and ω_{in} as in Section 5.4:

$$\tau_{in} = \sum_{j=n+1}^{\infty} a_{ij} y^{j-n} \qquad \text{and} \qquad \omega_{in} = \sum_{j=n+1}^{\infty} b_{ij} y^{j-n}.$$

[1] If T is Noetherian, then d is also the dimension of T. However, if T is not Noetherian, then the dimension of T may be greater than d. This is illustrated by taking T to be the ring B of Example 12.6.

Then we have

$$\tau_i = \Sigma_{j=1}^{\infty} a_{ij}y^j = \Sigma_{j=1}^{n} a_{ij}y^j + y^n\tau_{in} = \Sigma_{j=1}^{\infty} b_{ij}y^j = \Sigma_{j=1}^{n} b_{ij}y^j + y^n\omega_{in} = \omega_i.$$

Therefore, for $1 \le i \le s$ and each positive integer n,

$$y^n\tau_{in} - y^n\omega_{in} = \Sigma_{j=1}^{n} b_{ij}y^j - \Sigma_{j=1}^{n} a_{ij}y^j, \quad \text{and so} \quad \tau_{in} - \omega_{in} = \frac{\Sigma_{j=1}^{n}(b_{ij} - a_{ij})y^j}{y^n}.$$

Since $\Sigma_{j=1}^{n}(b_{ij} - a_{ij})y^j \in T$ is divisible by y^n in T^* and $T = F \cap T^*$, it follows that y^n divides $\Sigma_{j=1}^{n}(b_{ij} - a_{ij})y^j$ in T. Therefore $\tau_{in} - \omega_{in} \in T$. It follows that B_n and $B = \cup_{n=1}^{\infty} B_n$ are independent of the representation of the τ_i. □

Theorems 22.7 and 22.8 are adaptations of Construction Properties Theorem 5.14, Theorem 5.17 and Noetherian Flatness Theorem 6.3 of Chapters 5 and 6 that hold in Setting 22.4 even though the base ring T might not be Noetherian.

THEOREM 22.7. *Assume the setting and notation of (22.4). Then the intermediate rings B_n, B and A have the following properties:*

1. *$yA = yT^* \cap A$ and $yB = yA \cap B = yT^* \cap B$. More generally, for every $t \in \mathbb{N}$, we have $y^tA = y^tT^* \cap A$ and $y^tB = y^tA \cap B = y^tT^* \cap B$.*
2. *$T/y^tT = B/y^tB = A/y^tA = T^*/y^tT^*$, for each positive integer t.*
3. *Every ideal of T, B or A that contains y is finitely generated by elements of T. In particular, the maximal ideal $\mathbf{n}$ of T is finitely generated, and the maximal ideals of B and A are $\mathbf{n}B$ and $\mathbf{n}A$.*
4. *For every $n \in \mathbb{N}$: $yB \cap B_n = (y, \tau_{1n}, \dots, \tau_{sn})B_n$, an ideal of B_n of height $s+1$.*
5. *Let $P \in \operatorname{Spec}(A)$ be minimal over yA, and let $Q = P \cap B$ and $W = P \cap T$. Then $T_W \subseteq B_Q = A_P$, and all three localizations are DVRs.*
6. *For every $n \in \mathbb{N}$, $B[1/y]$ is a localization of B_n, i.e., for each $n \in \mathbb{N}$, there exists a multiplicatively closed subset S_n of B_n such that $B[1/y] = S_n^{-1}B_n$.*
7. *$B = B[1/y] \cap B_{\mathbf{q}_1} \cap \cdots \cap B_{\mathbf{q}_r}$, where $\mathbf{q}_1, \dots, \mathbf{q}_r$ are the prime ideals of B minimal over yB.*

PROOF. Let $K := F(\tau_1, \dots \tau_s)$, the field of fractions of A and B. Then $A = T^* \cap K$ implies $yA \subseteq yT^* \cap A$. Let $g \in yT^* \cap A \subseteq yT^* \cap K$. Then $g/y \in T^* \cap K = A \implies g \in yA$. Since $B = \bigcup_{n=1}^{\infty} B_n$, we have $yB = \bigcup_{n=1}^{\infty} yB_n$. It is clear that $yB \subseteq yA \cap B \subseteq yT^* \cap B$. We next show $yT^* \cap B = yB$. Let $g \in yT^* \cap B$. Then there is an $n \in \mathbf{N}$ with $g \in B_n$ and, multiplying g by a unit of B_n if necessary, we may assume that $g \in T[\tau_{1n}, \dots, \tau_{sn}]$. Write $g = r_0 + g_0$ where $g_0 \in (\tau_{1n}, \dots, \tau_{sn})T[\tau_{1n}, \dots, \tau_{sn}]$ and $r_0 \in T$. Substituting $\tau_{jn} = y\tau_{jn+1} + c_{jn}y \in yT^*$ from (22.4.5) yields that $g_0 \in yT^*$ and so $r_0 \in yT^* \cap T = yT$. Since by (22.4.5), $(\tau_{1n}, \dots, \tau_{sn})B_n \subseteq yB_{n+1}$, it follows that $g \in yB$. Now $yB = yT^* \cap B$ implies $y^2B = y(yT^* \cap B) = y^2T^* \cap yB = y^2T^* \cap B$. Similarly $y^tB = y^tT^* \cap B$ for every $t \in \mathbb{N}$.

Since $y^tT^* \cap T = y^tT$, $T/y^tT = T^*/y^tT^*$, and $T/(y^tT) \hookrightarrow B/(y^tB) \hookrightarrow A/(y^tA) \hookrightarrow T^*/y^tT^*$, the assertions in item 2 follow.

Since T^* is Noetherian, item 3 follows from item 2.

For item 4, let $f \in yB \cap B_n$. After multiplication by a unit of B_n, we may assume that $f \in T[\tau_{1n}, \dots, \tau_{sn}]$, and hence f is of the form

$$f = \sum_{(i) \in \mathbf{N}^s} a_{(i)}\tau_{1n}^{i_1} \dots \tau_{sn}^{i_s}$$

with $a_{(i)} \in T$. Since $\tau_{jn} \in yB$, we see that $a_{(0)} \in yB \cap T \subseteq yT^* \cap T$, and we can write $a_{(0)} = y\widehat{b}$ for some element $\widehat{b} \in T^*$. This implies that $\widehat{b} \in F \cap T^* = T$; the last equality uses (22.4.1). Therefore $a_{(0)} \in yT$ and $f \in (y, \tau_{1n}, \dots, \tau_{sn})B_n$. Furthermore if $g \in (y, \tau_{1n}, \dots, \tau_{sn})B_n$, then $\tau_{in} \subseteq yB \cap B_n$, so $g \in yB \cap B_n$.

For item 5, since T^* and hence A is Krull, P has height one and A_P is a DVR. Also A_P has the same fraction field as B_Q. By (2), W is a minimal prime of yT. Since T is a Krull domain, T_W is a DVR and the maximal ideal of T_W is generated by $u \in T$. Thus by item 2 the maximal ideal of B_Q is generated by u and so B_Q is a DVR dominated by A_P. Therefore they must be the same DVR.

Item 6 follows from (22.4.5).

For item 7, suppose $\beta \in B[1/y] \cap B_{\mathbf{q}_1} \cap \cdots \cap B_{\mathbf{q}_r}$. Now $B_{\mathbf{q}_1} \cap \cdots \cap B_{\mathbf{q}_r} = (B \setminus (\cup\ \mathbf{q}_i))^{-1}B$. There exist $t \in \mathbb{N}, a, b, c \in B$ with $c \notin \mathbf{q}_1 \cup \cdots \cup \mathbf{q}_r$ such that $\beta = a/y^t = b/c$. We may assume that either $t = 0$ (and we are done) or that $t > 0$ and $a \notin yB$. Since $yB = yA \cap B$, it follows that $\mathbf{q}_1, \dots, \mathbf{q}_r$ are the contractions to B of the minimal primes $\mathbf{p}_1, \dots, \mathbf{p}_r$ of yA in A. Since A is a Krull domain, $A = A[1/y] \cap A_{\mathbf{p}_1} \cap \cdots \cap A_{\mathbf{p}_n}$. Thus $\beta \in A$, and $a = y^t\beta \in yA \cap B = yB$, a contradiction. Thus $t = 0$ and $\beta = a \in B$. □

THEOREM 22.8. *With the setting and notation of (22.4), the intermediate rings A and B have the following properties:*

(1) *A and B are local Krull domains.*
(2) *$B \subseteq A$, with A dominating B.*
(3) *$A^* = B^* = T^*$.*
(4) *If B is Noetherian, then $B = A$.*

Moreover, if T is a unique factorization domain (UFD) and y is a prime element of T, then B is a UFD.

PROOF. As noted in the proof of Theorem 22.7, A is a Krull domain. By (22.7.6), $B[1/y]$ is a localization of B_0. Since B_0 is a Krull domain, it follows that $B[1/y]$ is a Krull domain. By (22.7.7), B is the intersection of $B[1/y]$ and the DVR's $B_{\mathbf{q}_1}, \dots, B_{\mathbf{q}_r}$. Therefore B is a Krull domain. Items 2 and 3 are immediate from Theorem 22.7. If B is Noetherian, then B^* is faithfully flat over B, and hence $B = F(\tau_1, \dots, \tau_s) \cap B^* = A$. For the last statement, if T is a UFD, so is the localized polynomial ring B_0. By (22.7.6), $B[1/y] = S_0^{-1}B_0[1/y]$, which implies that $B[1/y]$ is also a UFD. By (22.7.2), y is a prime element of B; hence it follows from Theorem 2.21 that B is a UFD. □

22.2. Limit-intersecting elements

Let $(R, \mathbf{m})$ be an excellent normal local domain and let $\widehat{R}$ be the $\mathbf{m}$-adic completion of R. We are interested in the structure of $L \cap \widehat{R}$, for intermediate fields L between the fields of fractions of R and $\widehat{R}$. This structure is difficult to determine in general. We show in Theorem 22.14 that each of the *limit-intersecting* properties of Definitions 22.9 implies $L \cap \widehat{R}$ is a directed union of localized polynomial ring extensions of R. These limit-intersecting properties are related to the idealwise independence concepts defined in Chapter 20 and to the LF_d properties defined in Definitions 22.1.

DEFINITIONS 22.9. Let $(T, \mathbf{n})$ be a local Krull domain, let $0 \neq y \in \mathbf{n}$ be such that the y-adic completion $\widehat{(T,(y))} := (T^*, \mathbf{n}^*)$ of T is an analytically normal Noetherian local domain of dimension d. Assume that T^* and $\widehat{T}$ are weakly flat over T. Let $\tau_1, \ldots, \tau_s \in \mathbf{n}^*$ be algebraically independent over T as in Setting 22.4.

(1) The elements $\tau_1, \ldots, \tau_s$ are said to be *limit-intersecting* in y over T provided the approximation domain B and the intersection domain A defined in Notation 22.4.6 are equal.

(2) The elements $\tau_1 \ldots, \tau_s$ are said to be *residually limit-intersecting* in y over T provided the inclusion map

$$B_0 := T[\tau_1, \ldots, \tau_s]_{(\mathbf{n}, \tau_1, \ldots, \tau_s)} \longrightarrow T^*[1/y] \quad \text{is } LF_1 \qquad (22.9.2).$$

(3) The elements $\tau_1 \ldots, \tau_s$ are said to be *primarily limit-intersecting* in y over T provided the inclusion map

$$B_0 := T[\tau_1, \ldots, \tau_s]_{(\mathbf{n}, \tau_1, \ldots, \tau_s)} \longrightarrow T^*[1/y] \quad \text{is flat.} \qquad (22.9.3).$$

Since $T^*[1/y]$ and $\widehat{T}[1/y]$ have dimension $d-1$, the condition LF_{d-1} is equivalent to primarily limit-intersecting, that is, to the flatness of the map $B_0 \longrightarrow T^*[1/y]$.

REMARKS 22.10. We show in Theorem 22.14 that the elements $\tau_1, \cdots, \tau_s$ are limit-intersecting in the sense of Definition 22.9.1 if and only if the inclusion map

$$B_0 := T[\tau_1, \ldots, \tau_s]_{(\mathbf{n}, \tau_1, \ldots, \tau_s)} \longrightarrow T^*[1/y] \quad \text{is weakly flat} \qquad (22.10.0).$$

Here are some other remarks concerning Definitions 22.9.

(1) The terms "residually" and "primarily" come from Chapter 20. We justify this terminology in Proposition 22.16) and Theorem 23.3. It is clear that primarily limit-intersecting implies residually limt-intersecting. By Theorem 20.8 if T is an excellent normal local domain, then the extension $B_0 \hookrightarrow T^*$ is height-one preserving. By Proposition 8.12 an extension of Krull domains that is height-one preserving and satisfies PDE is weakly flat.

(2) Since $\widehat{T}[1/y]$ is faithfully flat over $T^*[1/y]$, the statements obtained by replacing $T^*[1/y]$ by $\widehat{T}[1/y]$ give equivalent definitions to those of Definitions 22.9; see Propositions 21.9 and 21.11 of Chapter 21).

(3) We remark that

$$B \longrightarrow T^*[1/y] \text{ is weakly flat } \iff B \longrightarrow T^* \text{ is weakly flat .}$$

To see this, observe that by Theorem 22.7.2, every height-one prime of B containing y is the contraction of a height-one prime of T^*. If $\mathbf{p}$ is a height-one prime of B with $y \notin \mathbf{p}$, then $\mathbf{p}T^* \cap B = \mathbf{p}$ if and only if $\mathbf{p}T^*[1/y] \cap B = \mathbf{p}$.

(4) The ring $B[1/y]$ is a localization $S_0^{-1}B_0$ of B by Theorem 22.7.6. Since S_0 consists of units of $T^*[1/y]$, Remark 8.6.b implies the extension $B_0 \hookrightarrow T^*[1/y]$ is weakly flat if and only if the canonical map

$$S_0^{-1}B_0 = B[1/y] \longrightarrow T^*[1/y]$$

is weakly flat. In view of Proposition 22.11 below, we have $\tau_1, \ldots, \tau_s$ are residually (resp. primarily) limit-intersecting in y over T if and only if the canonical map

$$S_0^{-1}B_0 = B[1/y] \longrightarrow T^*[1/y]$$

is LF_1 (resp. LF_{d-1} or equivalently flat).

(5) If $d = 2$, then obviously $LF_1 = LF_{d-1}$. Hence in this case primarily limit-intersecting is equivalent to residually limit-intersecting.

(6) Since $T \longrightarrow B_n$ is faithfully flat for every n, it follows [**17**, Chap.1, Sec.2.3, Prop.2, p.14] that $T \longrightarrow B$ is always faithfully flat. Thus if residually limit-intersecting elements exist over T, then $T \longrightarrow T^*[1/y]$ must be LF_1. If primarily limit-intersecting elements exist over T, then $T \longrightarrow T^*[1/y]$ must be flat.

(7) The examples of Remarks 8.9 and 22.5 show that in some situations T^* contains no limit-intersecting elements. Indeed, if T is complete with respect to some nonzero ideal I, and y is outside every minimal prime over I, then every element $\tau = \sum a_i y^i$ of T^* that is transcendental over T fails to be limit-intersecting in y. To see this, choose an element $x \in I$, x outside every minimal prime ideal of yT; define $\sigma := \sum a_i x^i \in T$. Then $\tau - \sigma \in (x-y)T^* \cap T[\tau]$. Thus a minimal prime over $x - y$ in T^* intersects $T[\tau]$ in an ideal of height greater than one, because it contains $x - y$ and $\tau - \sigma$.

PROPOSITION 22.11. *Assume the notation and setting of (22.4) and let k be a positive integer with $1 \leq k \leq d-1$. Then the following are equivalent:*

(1) *The canonical injection $\varphi : B_0 := T[\tau_1, \ldots, \tau_s]_{(\mathbf{m}, \tau_1, \ldots, \tau_s)} \longrightarrow T^*[1/y]$ is LF_k.*
(1′) *The canonical injection $\varphi_1 : B_0 := T[\tau_1, \ldots, \tau_s]_{(\mathbf{m}, \tau_1, \ldots, \tau_s)} \longrightarrow \widehat{T}[1/y]$ is LF_k.*
(2) *The canonical injection $\varphi' : U_0 := T[\tau_1, \ldots, \tau_s] \longrightarrow T^*[1/y]$ is LF_k.*
(2′) *The canonical injection $\varphi'_1 : U_0 := T[\tau_1, \ldots, \tau_s] \longrightarrow \widehat{T}[1/y]$ is LF_k.*
(3) *The canonical injection $\theta : B_n := R[\tau_{1n}, \ldots, \tau_{sn}]_{(\mathbf{m}, \tau_{1n}, \ldots, \tau_{sn})} \longrightarrow T^*[1/y]$ is LF_k.*
(3′) *The canonical injection $\theta_1 : B_n := R[\tau_{1n}, \ldots, \tau_{sn}]_{(\mathbf{m}, \tau_{1n}, \ldots, \tau_{sn})} \longrightarrow \widehat{T}[1/y]$ is LF_k.*
(4) *The canonical injection $\psi : B \longrightarrow T^*[1/y]$ is LF_k.*
(4′) *The canonical injection $\psi : B \longrightarrow \widehat{T}[1/y]$ is LF_k.*

Moreover, these statements are also all equivalent to LF_k of the corresponding canonical injections obtained by replacing B_0, U_0 and B by $B_0[1/y]$, $U_0[1/y]$ and $B[1/y]$.

PROOF. We have:

$$U_0 \xrightarrow{\text{loc.}} B_0 \xrightarrow{\varphi} T^*[1/y] \xrightarrow{\text{f.f.}} \widehat{T}[1/y].$$

The injection $\varphi'_1 : U_0 \longrightarrow \widehat{T}[1/y]$ factors as $\varphi' : U_0 \longrightarrow T^*[1/y]$ followed by the faithfully flat injection $T^*[1/y] \longrightarrow \widehat{T}[1/y]$. Therefore φ' is LF_k if and only if φ'_1 is LF_k. The injection φ' factors through the localization $U_0 \longrightarrow B_0$ and so φ is LF_k if and only if φ' is LF_k.

Now set $U_n := T[\tau_{1n}, \ldots, \tau_{sn}]$ for each $n > 1$ and $U := \bigcup_{n=0}^{\infty} U_n$. For each positive integer i, $\tau_i = y^n \tau_{in} + \sum_{i=0}^{n} a_i y^i$. Thus $U_n \subseteq U_0[1/y]$, and $U_0[1/y] = \bigcup U_n[1/y] = U[1/y]$. Moreover, for each n, B_n is a localization of U_n, and hence B is a localization of U.

We have:

$$\begin{aligned} B[1/y] \longrightarrow T^*[1/y] \text{ is } LF_k &\iff U[1/y] \longrightarrow T^*[1/y] \text{ is } LF_k \\ \iff U_0[1/y] \longrightarrow T^*[1/y] \text{ is } LF_k &\iff B_n[1/y] \longrightarrow T^*[1/y] \text{ is } LF_k \\ &\iff B_0[1/y] \longrightarrow T^*[1/y] \text{ is } LF_k\,. \end{aligned}$$

Thus

$$\begin{aligned} \psi : B \longrightarrow T^*[1/y] \text{ is } LF_k &\iff U \longrightarrow T^*[1/y] \text{ is } LF_k \\ \iff \varphi' : U_0 \longrightarrow T^*[1/y] \text{ is } LF_k &\iff \theta : B_n \longrightarrow T^*[1/y] \text{ is } LF_k \\ &\iff \varphi : B_0 \longrightarrow T^*[1/y] \text{ is } LF_k \,. \end{aligned}$$

□

REMARKS 22.12. (1) If $(T, \mathbf{n})$ is a one-dimensional local Krull domain, then T is a rank-one discrete valuation domain (DVR). Hence T^* is also a DVR and $T^*[1/y]$ is flat over $U_0 = T[\tau_1, \ldots, \tau_s]$. Therefore, in this situation, $\tau_1, \ldots, \tau_s$ are primarily limit-intersecting in y over T if and only if $\tau_1, \ldots, \tau_s$ are algebraically independent over T.

(2) Let $\tau_1, \ldots, \tau_s \in k[[y]]$ be transcendental over $k(y)$, where k is a field. Then $\tau_1, \ldots, \tau_s$ are primarily limit-intersecting in y over $k[y]_{(y)}$ by (1) above. If $x_1, \ldots, x_m$ are additional indeterminates over $k(y)$, then by Prototype Theorem 9.2 and Noetherian Flatness Theorem 6.3, the elements $\tau_1, \ldots, \tau_s$ are primarily limit-intersecting in y over $k[x_1, \ldots, x_m, y]_{(x_1,\ldots,x_m,y)}$.

(3) Assume the notation of Setting 22.4, and also assume that B is Noetherian. We show that $\tau_1, \ldots, \tau_s$ are primarily limit-intersecting in y over T. Since T^* is the (y)-adic completion of B and B is Noetherian, it follows that T^* is flat over B. Hence $T^*[1/y]$ is also flat over B, and it follows from Proposition 22.11 that $\tau_1, \ldots, \tau_s$ are primarily limit-intersecting in y over T.

(4) By the equivalence of (1) and (2) of Proposition 22.11, we see that $\tau_1, \ldots, \tau_s$ are primarily limit-intersecting in y over T if and only if the endpiece power series $\tau_{1n}, \ldots, \tau_{sn}$ are primarily limit-intersecting in y over T.

THEOREM 22.13. *Assume the notation of Setting 22.4. Thus $(T, \mathbf{n})$ is a local Krull domain with field of fractions F, and $y \in \mathbf{n}$ is such that the (y)-adic completion $(T^*, \mathbf{n}^*)$ of T is an analytically normal Noetherian local domain and $T = T^* \cap F$. For elements $\tau_1, \ldots, \tau_s \in \mathbf{n}^*$ that are algebraically independent over T, the following are equivalent:*

1. *The extension $T[\tau_1, \ldots, \tau_s] \hookrightarrow T^*[1/y]$ is flat.*
2. *The elements $\tau_1, \ldots, \tau_s$ are primarily limit-intersecting in y over T.*
3. *The intermediate rings A and B are equal and are Noetherian.*
4. *The constructed ring B is Noetherian.*

Moreover, if these equivalent conditions hold, then the Krull domain T is Noetherian.

PROOF. By Theorem 22.7, we have $T/y^tT = B/y^tB = A/y^tA = T^*/y^tT^*$, for each positive integer t. By Definition 22.9.3, the elements $\tau_1 \ldots, \tau_s$ are primarily limit-intersecting in y over T if and only if the inclusion map

$$B_0 := T[\tau_1, \ldots, \tau_s]_{(\mathbf{n}, \tau_1, \ldots, \tau_s)} \longrightarrow T^*[1/y]$$

is flat. Thus item 1 is equivalent to item 2. By Theorem 22.7.6, $B[1/y]$ is a localization of B_0. Hence flatness of the map $B_0 \hookrightarrow T^*[1/y]$ implies flatness of the map $B \hookrightarrow T^*[1/y]$. Applying Lemma 6.2 to the extension $B \hookrightarrow T^*$, we conclude that flatness of the map $B \hookrightarrow T^*[1/y]$ implies that T^* is flat over B and B is Noetherian. Therefore item 1 implies item 4. On the other hand, if B is Noetherian, then T^* is faithfully flat over B since T^* is the (y)-adic completion of B. Therefore $B = A$ and $B \hookrightarrow T^*[1/y]$ is flat. Thus item 4 is equivalent to item 3 and implies

item 1. If these equivalent conditions hold, then $T \hookrightarrow T^*[1/y]$ is flat, and Lemma 6.2 implies that $T \hookrightarrow T^*$ is flat and T is Noetherian. □

THEOREM 22.14. *Assume the notation of Setting 22.4. Thus $(T, \mathbf{n})$ is a local Krull domain with field of fractions F, and $y \in \mathbf{n}$ is such that the (y)-adic completion $(T^*, \mathbf{n}^*)$ of T is an analytically normal Noetherian local domain and $T = T^* \cap F$. For elements $\tau_1, \ldots, \tau_s \in \mathbf{n}^*$ that are algebraically independent over T, the following are equivalent:*

(1) *The elements $\tau_1, \ldots, \tau_s$ are limit-intersecting in y over T, that is, the intermediate rings A and B are equal.*
(2) *$B_0 \longrightarrow T^*[1/y]$ is weakly flat.*
(3) *$B \longrightarrow T^*[1/y]$ is weakly flat.*
(4) *$B \longrightarrow T^*$ is weakly flat.*

PROOF. (2)⇒(1): Since A and B are Krull domains with the same field of fractions and $B \subseteq A$ it is enough to show that every height-one prime ideal $\mathbf{p}$ of B is the contraction of a (height-one) prime ideal of A. By Theorem 22.7.3, each height-one prime of B containing yB is the contraction of a height-one prime of A.

Let $\mathbf{p}$ be a height-one prime of B which does not contain yB. Consider the prime ideal $\mathbf{q} = T[\tau_1, \ldots, \tau_s] \cap \mathbf{p}$. Since $B[1/y]$ is a localization of the ring $T[\tau_1, \ldots, \tau_s]$, we see that $B_{\mathbf{p}} = T[\tau_1, \ldots, \tau_s]_{\mathbf{q}}$ and so $\mathbf{q}$ has height one in $T[\tau_1, \ldots, \tau_s]$. The weakly flat hypothesis implies $\mathbf{q}T^* \cap T[\tau_1, \ldots, \tau_s] = \mathbf{q}$, and there is a height-one prime ideal $\mathbf{w}$ of T^* with $\mathbf{w} \cap T[\tau_1, \ldots, \tau_s] = \mathbf{q}$. This implies that $\mathbf{w} \cap B = \mathbf{p}$ and thus also $(\mathbf{w} \cap A) \cap B = \mathbf{p}$. Hence every height-one prime ideal of B is the contraction of a prime ideal of A. Since A is birational over B, this prime ideal of A can be chosen to have height one.

(3) ⟺ (4): This is shown in Remark 22.10.3.

(1)⇒(4): If $B = A = F \cap T^*$, then by Proposition 22.2 every height-one prime ideal of B is the contraction of a height-one prime ideal of T^*.

(4)⇒(2): If $B \hookrightarrow T^*$ is weakly flat so is the localization $B[1/y] \hookrightarrow T^*[1/y]$. Since $B[1/y] = S_0^{-1}B_{0y}$ for a suitable multiplicative subset $S_0 \subseteq B_{0y}$ the embedding $B_{0y} \hookrightarrow T^*[1/y]$ is weakly flat. Now (2) holds by Remark 22.10.4. □

REMARKS 22.15. (1) If an injective map of Krull domains is weakly flat, then it is height-one preserving by Proposition 22.2.2. Thus any of the equivalent conditions of Theorem 22.14 imply that $B \longrightarrow T^*$ is height-one preserving.

(2) In Theorem 22.14 if B is Noetherian, then by Theorem 22.8.4, $A = B$ and so all the conclusions of Theorem 22.14 hold.

(3) Example 10.9 yields the existence of a three-dimensional regular local domain $R = k[x, y, z]_{(x,y,z)}$, over an arbitrary field k, and an element $f = y\tau_1 + z\tau_2$ in the (x)-adic completion of R such that f is residually limit-intersecting in x over R, but fails to be primarily limit-intersecting in x over R. In particular, the rings A and B constructed using f are equal, yet A and B are not Noetherian. Here the elements τ_1 and τ_2 are chosen to be elements of $xk[[x]]$ that are algebraically independent over $k(x)$.

Proposition 22.16 gives criteria for elements to be residually limit-intersecting similar to those in Chapter 20 for elements to be residually algebraically independent. The corresponding result for primarily limit-intersecting is given in Theorem 23.3.

PROPOSITION 22.16. *With the setting and notation of (22.4) and $s = 1$, the following are equivalent:*

(1) *The element $\tau = \tau_1$ is residually limit-intersecting in y over T.*
(2) *If $\widehat{P}$ is a height-one prime ideal of $\widehat{T}$ such that $y \notin \widehat{P}$ and $\widehat{P} \cap T \neq 0$, then $\mathrm{ht}(\widehat{P} \cap T[\tau]_{(\mathbf{n},\tau)}) = 1$.*
(3) *For every height-one prime ideal P of T such that $y \notin P$ and for every minimal prime divisor $\widehat{P}$ of $P\widehat{T}$ in $\widehat{T}$, the image $\bar{\tau}$ of τ in $\widehat{T}/\widehat{P}$ is algebraically independent over T/P.*
(4) *$B \longrightarrow T^*[1/y]$ is LF_1.*

PROOF. For (1)$\Rightarrow$ (2), suppose (2) fails; that is, there exists a prime ideal $\widehat{P}$ of $\widehat{T}$ of height one such that $y \notin \widehat{P}$, $\widehat{P} \cap T \neq 0$, but $\mathrm{ht}(\widehat{P} \cap T[\tau]) \geq 2$. Let $\widehat{Q} := \widehat{P}\widehat{T}[1/y]$. Then $Q := \widehat{Q} \cap T[\tau]_{(\mathbf{n},\tau)}$ has height greater than or equal to 2. But by the definition of residually limit-intersecting in (22.9.2), the injective morphism $T[\tau]_{(\mathbf{n},\tau)} \longrightarrow \widehat{T}[1/y]$ is LF_1 and so by Definition 22.1), $(T[\tau]_{(\mathbf{n},\tau)})_Q \longrightarrow \widehat{(T[1/y])}_{\widehat{Q}}$ is faithfully flat, a contradiction to $\mathrm{ht}(Q) > \mathrm{ht}(\widehat{P}) = \mathrm{ht}(\widehat{Q})$.

For (2)$\Rightarrow$ (1), the argument of (1) $\Rightarrow$ (2) can be reversed since $(T[\tau]_{(\mathbf{n},\tau)})_Q \longrightarrow (\widehat{T}[1/y])_{\widehat{Q}}$ is faithfully flat.

For (3)$\Rightarrow$ (2), again suppose (2) fails; that is, there exists a prime ideal $\widehat{P}$ of $\widehat{T}$ of height one such that $y \notin \widehat{P}$, $\widehat{P} \cap T \neq 0$, but $\mathrm{ht}(\widehat{P} \cap T[\tau]) \geq 2$. Now $\mathrm{ht}(\widehat{P} \cap T) = 1$, since LF_1 holds for $T \hookrightarrow \widehat{T}$. Thus, with $P = \widehat{P} \cap T$, we have $PT[\tau] < \widehat{P} \cap T[\tau]$; that is, there exists $f(\tau) \in (\widehat{P} \cap T[\tau]) - PT[\tau]$, or equivalently there is a nonzero polynomial $\bar{f}(x) \in (T/(\widehat{P} \cap T))[x]$ so that $\bar{f}(\bar{\tau}) = \bar{0}$ in $T[\tau]/(\widehat{P} \cap T[\tau])$, where $\bar{\tau}$ denotes the image of τ in $\widehat{T}/\widehat{P}$. This means that $\bar{\tau}$ is algebraic over the field of fractions of $T/(\widehat{P} \cap T)$, a contradiction to (3).

For (2)$\Rightarrow$ (3), let $\widehat{P}$ be a height-one prime of $\widehat{R}$ such that $\widehat{P} \cap T = P \neq 0$. Since $\mathrm{ht}(\widehat{P} \cap T[\tau]) = 1$, $\widehat{P} \cap T[\tau] = PT[\tau]$ and $T[\tau]/(PT[\tau])$ canonically embeds in $\widehat{T}/\widehat{P}$. Thus the image of τ in $T[\tau]/PT[\tau]$ is algebraically independent over T/P.

For (1) $\Longleftrightarrow$ (4), we see by (22.11) that (1) is equivalent to the embedding $\psi : B \longrightarrow T^*[1/y]$ being LF_1. □

REMARK 22.17. Assume the notation of Setting 22.4. If T has the property that every height-one prime of T is the radical of a principal ideal, and τ is residually limit-intersecting in y over T, then the extension $B \hookrightarrow T^*[1/y]$ is height-one preserving by Remark 8.6,c, and hence weakly flat by Propositions 22.16, 8.11 and 8.12. Thus with these assumptions, if τ is residually limit-intersecting, then τ is limit-intersecting.

We have the following transitive property of limit-intersecting elements.

PROPOSITION 22.18. *Assume the setting and notation of (22.4). Also assume that $s > 1$ and for all $j \in \{1, \ldots s\}$, set $A(j) := F(\tau_1, \ldots, \tau_j) \cap \widehat{T}$. Then the following statements are equivalent:*

(1) *$\tau_1, \ldots, \tau_s$ are limit-intersecting, respectively, residually limit-intersecting, respectively, primarily limit-intersecting in y over T*
(2) *$\forall j \in \{1, \ldots, s\}$, the elements $\tau_1, \ldots, \tau_j$ are limit-intersecting, respectively, residually limit-intersecting, respectively, primarily limit-intersecting in y over T and the elements $\tau_{j+1}, \ldots, \tau_s$ are limit-intersecting, respectively,*

residually limit-intersecting, respectively, primarily limit-intersecting in y over $A(j)$.

(3) *There exists a $j \in \{1, \dots, s\}$, such that the elements $\tau_1, \dots, \tau_j$ are limit-intersecting, respectively, residually limit-intersecting, respectively, primarily limit-intersecting in y over T and the elements $\tau_{j+1}, \dots, \tau_s$ are limit-intersecting, respectively, residually limit-intersecting, respectively, primarily limit-intersecting in y over $A(j)$.*

PROOF. Set $B(j) := \bigcup_{n=1}^{\infty} T[\tau_{1n}, \dots, \tau_{jn}]_{(\mathbf{n}, \tau_{1n}, \dots, \tau_{jn})}$. That (2) implies (3) is clear.

For (3) $\Longrightarrow$ (1), items (22.14) and (22.10.1) imply that $A(j) = B(j)$ under each of the conditions on $\tau_1, \dots, \tau_j$. The definitions of $\tau_{j+1}, \dots, \tau_s$ being limit-intersecting, respectively, residually limit-intersecting, respectively, primarily limit-intersecting in y over $A(j)$ together with (22.10.4) imply the equivalence of the stated flatness properties for each of the maps

$$\begin{aligned}
\varphi_1 &: A(j)[\tau_{j+1}, \dots, \tau_s]_{(-)} && \longrightarrow A(j)^*[1/y] = T^*[1/y] \\
\varphi_2 &: (A(j)[\tau_{j+1}, \dots, \tau_s]_{(-)})[1/y] && \longrightarrow T^*[1/y] \\
\varphi_3 &: (B(j)[\tau_{j+1}, \dots, \tau_s]_{(-)})[1/y] && \longrightarrow T^*[1/y] \\
\varphi_4 &: (T[\tau_1, \dots, \tau_s]_{(-)})[1/y] && \longrightarrow T^*[1/y] \\
\varphi_5 &: T[\tau_1, \dots, \tau_s]_{(\mathbf{n}, \tau_1, \dots, \tau_s)} && \longrightarrow T^*[1/y].
\end{aligned}$$

Thus

$$\begin{aligned}
\psi : B \longrightarrow T^*[1/y] \text{ is } LF_k &\iff U \longrightarrow T^*[1/y] \text{ is } LF_k \\
&\iff \varphi' : U_0 \longrightarrow T^*[1/y] \text{ is } LF_k \\
&\iff \theta : B_n \longrightarrow T^*[1/y] \text{ is } LF_k \\
&\iff \varphi : B_0 \longrightarrow T^*[1/y] \text{ is } LF_k\,.
\end{aligned}$$

The respective flatness properties for φ_5 are equivalent to the conditions that $\tau_1, \dots, \tau_s$ be limit-intersecting, respectively, residually limit-intersecting, respectively, primarily limit-intersecting in y over T. Thus (3) $\Longrightarrow$ (1).

For (1) $\Longrightarrow$ (2), we go backwards: The statement of (1) for $\tau_1, \dots, \tau_s$ is equivalent to the respective flatness property for φ_5. This is equivalent to φ_4 and thus φ_3 having the respective flatness property. By (22.10.4), $B(j)[\tau_{j+1}, \dots, \tau_s]_{(-)} \longrightarrow T^*[1/y]$ has the appropriate flatness property. Also $B(j) \longrightarrow B(j)[\tau_{j+1}, \dots, \tau_s]_{(-)}$ is flat, and so $B(j) \longrightarrow T^*[1/y]$ has the appropriate flatness property. Thus $\tau_1, \dots, \tau_j$ are limit-intersecting, respectively, residually limit-intersecting, respectively, primarily limit-intersecting in y over T. Therefore $A(j) = B(j)$, and so $A(j) \longrightarrow T^*[1/y]$ has the appropriate flatness property. It follows that $\tau_{j+1}, \dots, \tau_s$ are limit-intersecting, respectively, residually limit-intersecting, respectively, primarily limit-intersecting in y over $A(j)$. □

22.3. A specific example where $B = A$ is non-Noetherian

Theorem 10.7 and Examples 10.9 yield examples where the constructed domains A and B are equal and are not Noetherian. We present in Theorem 22.19 a specific example of an excellent regular local domain $(R, \mathbf{m})$ of dimension three with $\mathbf{m} = (x, y, z)R$ and $\widehat{R} = \mathbb{Q}[[x, y, z]]$ such that there exists an element $\tau \in yR^*$, where

R^* is the (y)-adic completion of R, with τ limit-intersecting and residually limit-intersecting, but *not* primarily limit-intersecting in y over R. In this example we have $B = A$ and B is non-Noetherian.

THEOREM 22.19. *There exist an excellent regular local three-dimensional domain $(R, \mathbf{m})$ contained in $\mathbb{Q}[[x, y, z]]$, a power series ring in the indeterminates x, y, z over $\mathbb{Q}$, the rational numbers, with $\mathbf{m} = (x, y, z)R$, and an element τ in the (y)-adic completion R^* of R such that*

(22.19.1) *τ is residually limit-intersecting in y over R.*
(22.19.2) *τ is not primarily limit-intersecting in y over R.*
(22.19.3) *τ is limit-intersecting in y over R.*

In particular, the rings A and B constructed using τ and Notation 22.4.6 are equal, yet A and B fail to be Noetherian.

PROOF. We use the following elements of $\mathbb{Q}[[x, y, z]]$:

$$\gamma := e^x - 1 \in x\mathbb{Q}[[x]], \qquad \delta := e^{x^2} - 1 \in x\mathbb{Q}[[x]],$$
$$\sigma := \gamma + z\delta \in \mathbb{Q}[z]_{(z)}[[x]] \qquad \text{and} \qquad \tau := e^y - 1 \in y\mathbb{Q}[[y]].$$

For each n, we define the endpieces $\gamma_n, \delta_n, \sigma_n$ and τ_n as in (5.4), considering γ, δ, σ as series in x and τ as a series in y. Thus, for example,

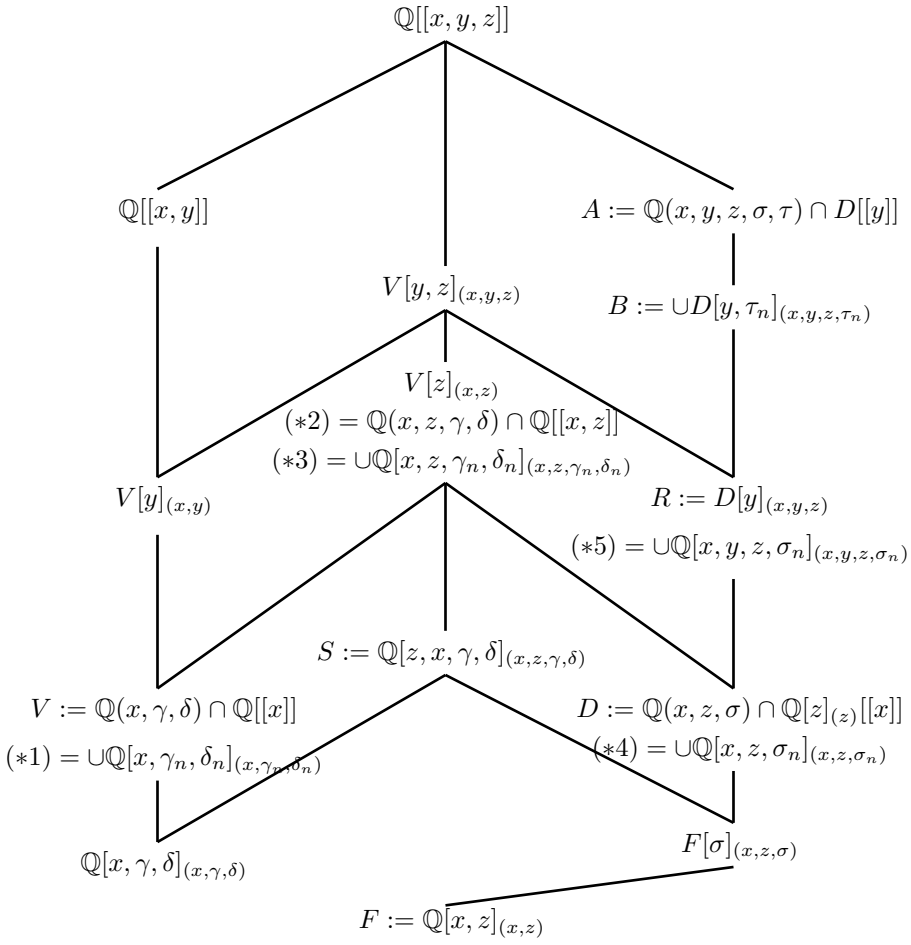
$$\gamma = \sum_{i=1}^{\infty} a_i x^i; \qquad \gamma_n = \sum_{i=n+1}^{\infty} a_i x^{i-n}, \qquad \text{and} \quad x^n\gamma_n + \sum_{i=1}^{n} a_i x^i = \gamma.$$

(Here $a_i := 1/i!$.) The δ_n, σ_n satisfy similar relations. Therefore for each positive integer n,

(1) $\mathbb{Q}[x, \gamma_{n+1}, \delta_{n+1}]_{(x,\gamma_{n+1},\delta_{n+1})}$ birationally dominates $\mathbb{Q}[x, \gamma_n, \delta_n]_{(x,\gamma_n,\delta_n)}$,
(2) $\mathbb{Q}[x, \gamma_n, \delta_n]_{(x,\gamma_n,\delta_n)}$ birationally dominates $\mathbb{Q}[x, \gamma, \delta]_{(x,\gamma,\delta)}$, and
(3) $\mathbb{Q}[x, z, \sigma_{n+1}]_{(x,z,\sigma_{n+1})}$ birationally dominates $\mathbb{Q}[x, z, \sigma_n]_{(x,z,\sigma_n)}$. □

For our proof of Theorem 22.19 we first establish that certain subrings of $\mathbb{Q}[[x, y, z]]$ can be expressed as directed unions:

CLAIM 22.20. *For $V := \mathbb{Q}(x, \gamma, \delta) \cap \mathbb{Q}[[x]]$ and $D := \mathbb{Q}(x, z, \sigma) \cap \mathbb{Q}[z]_{(z)}[[x]]$, the equalities (*1)-(*5) of the diagram below hold. Furthermore the ring $V[z]_{(x,z)}$ is excellent and the canonical local embedding $\psi : D \longrightarrow V[z]_{(x,z)}$ is a direct limit of the maps $\psi_n : \mathbb{Q}[x, z, \sigma_n]_{(x,z,\sigma_n)} \longrightarrow \mathbb{Q}[x, z, \gamma_n, \delta_n]_{(x,z,\gamma_n,\delta_n)}$, where $\psi(\sigma_n) = \gamma_n + z\delta_n$.*



The rings of the example

PROOF. (of Claim 22.20) The Noetherian Flatness Theorem 6.3 implies that the elements γ and δ are primarily limit-intersecting in x over $\mathbb{Q}[x]_{(x)}$ and thus we have (*1):

$$V := \mathbb{Q}(x,\gamma,\delta) \cap \mathbb{Q}[[x]] = \varinjlim(\mathbb{Q}[x,\gamma_n,\delta_n]_{(x,\gamma_n,\delta_n)}) = \bigcup \mathbb{Q}[x,\gamma_n,\delta_n]_{(x,\gamma_n,\delta_n)}.$$

Since V is a DVR of characteristic zero, V is excellent and (*2) holds:

$$V[z]_{(x,z)} = \mathbb{Q}(z,x,\gamma,\delta) \cap \mathbb{Q}[[z,x]].$$

Also $V[z]_{(x,z)}$ is excellent since it is a localization of a finitely generated extension of V. Item (*3) is clear from (*1), and so $V[z]_{(x,z)}$ is a directed union of the four-dimensional regular local domains given.

To establish (*4), observe that for each positive integer n, the map

$$\mathbb{Q}[x,z,\sigma_n]_{(x,z,\sigma_n)} \longrightarrow \mathbb{Q}[x,z,\gamma_n,\delta_n]_{(x,z,\gamma_n,\delta_n)}$$

is faithfully flat. Thus the induced map on the direct limits:

$$\psi_n : \varinjlim \mathbb{Q}[x,z,\sigma_n]_{(x,z,\sigma_n)} \longrightarrow \varinjlim \mathbb{Q}[x,z,\gamma_n,\delta_n]_{(x,z,\gamma_n,\delta_n)}$$

is also faithfully flat. Since $V = \varinjlim \mathbb{Q}[x, \gamma_n, \delta_n]_{(x,\gamma_n,\delta_n)}$, it follows that $V[z]_{(x,z)} = \varinjlim \mathbb{Q}[x, z, \gamma_n, \delta_n]_{(x,z,\gamma_n,\delta_n)}$ is faithfully flat over the limit $\varinjlim \mathbb{Q}[x, z, \sigma_n]_{(x,z,\sigma_n)}$. Since $V[z]_{(x,z)}$ is Noetherian, we conclude that $\varinjlim \mathbb{Q}[x, z, \sigma_n]_{(x,z,\sigma_n)}$ is Noetherian. Therefore ψ is a direct limit of ψ_n and

$$D = \varinjlim \mathbb{Q}[x, z, \sigma_n]_{(x,z,\sigma_n)} = \bigcup \mathbb{Q}[x, z, \sigma_n]_{(x,z,\sigma_n)}.$$

Now item (*5) follows:

$$R := D[y]_(x, y, z) = \varinjlim \mathbb{Q}[x, y, z, \sigma_n]_{(x,y,z,\sigma_n)} = \bigcup \mathbb{Q}[x, y, z, \sigma_n]_{(x,y,z,\sigma_n)}.$$

□

Claim 22.21. *The ring $D := \mathbb{Q}(x, z, \sigma) \cap \mathbb{Q}[z]_{(z)}[[x]]$ is excellent and $R := D[y]_{(x,y,z)}$ is a three-dimensional excellent regular local domain with maximal ideal $\mathbf{m} = (x, y, z)R$ and $\mathbf{m}$-adic completion $\widehat{R} = \mathbb{Q}[[x, y, x]]$.*

Proof. (of Claim 22.21) By Theorem 4.8 of Valabrega, the ring

$$D := \mathbb{Q}(x, z, \sigma) \cap \mathbb{Q}[z]_{(z)}[[x]]$$

is a two-dimensional regular local domain and the completion $\widehat{D}$ of D with respect to the powers of its maximal ideal is canonically isomorphic to $\mathbb{Q}[[x, z]]$.

We observe that with an appropriate change of notation, Theorem 10.10 applies to prove Claim 22.21.

Let $F = \mathbb{Q}[x, z]_{(x,z)}$ and let F^* denote the (x)-adic completion of F. Consider the local injective map

$$F[\sigma]_{(x,z,\sigma)} \xrightarrow{\phi} F[\gamma, \delta]_{(x,z,\gamma,\delta)} := S.$$

Let $\phi_x : F[\sigma]_{(x,z,\sigma)} \longrightarrow S_x$ denote the composition of ϕ followed by the canonical map of S to S_x. We have the setting of (**??**) and (10.10) where F plays the role of R and $V[z]_{(x,z)}$ plays the role of B.

By Theorem 10.10, to show D is excellent, it suffices to show that ϕ_x is a regular morphism. The map ϕ_x may be identified as the inclusion map

$$\begin{array}{ccc} \mathbb{Q}[z, x, t_1 + zt_2]_{(z,x,t_1+zt_2)} & \xrightarrow{\phi_x} & \mathbb{Q}[z, x, t_1, t_2]_{(z,x,t_1,t_2)}[1/x] \\ \downarrow \mu & & \downarrow \nu \\ \mathbb{Q}[z, x, \gamma + z\delta]_{(z,x,\gamma+z\delta)} & \xrightarrow{\phi_x} & \mathbb{Q}[z, x, \gamma, \delta]_{(z,x,\gamma,\delta)}[1/x] \end{array}$$

where μ and ν are the isomorphisms mapping $t_1 \to \gamma$ and $t_2 \to \delta$. Since $\mathbb{Q}[z, x, t_1, t_2] = \mathbb{Q}[z, x, t_1 + zt_2][t_2]$ is isomorphic to a polynomial ring in one variable over its subring $\mathbb{Q}[z, x, t_1 + zt_2]$, ϕ_x is a regular morphism, so by Theorem 10.10, D is excellent. This completes the proof of Claim 22.21. □

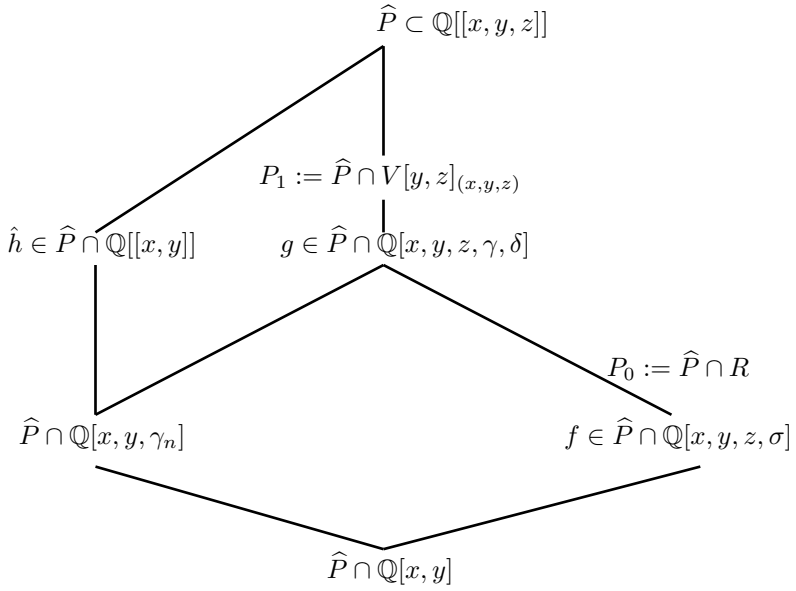
Claim 22.22. *The element $\tau := e^y - 1$ is in the (y)-adic completion R^* of $R := D[y]_{(x,y,z)}$, but τ is not primarily limit-intersecting in y over R and the ring B (constructed using τ) is not Noetherian.*

Proof. (of Claim 22.22) Consider the height-two prime ideal $\widehat{P} := (z, y - x)\widehat{R}$ of $\widehat{R}$. Now $y \notin \widehat{P}$, so $\widehat{P}\widehat{R}_y$ is a height-two prime ideal of $\widehat{R}_y$. Moreover, the ideal $Q := \widehat{P} \cap R[\tau]_{(\mathbf{m},\tau)}$ contains the element $\sigma - \tau$. Thus $\mathrm{ht}(Q) = 3$ and the canonical map $R[\tau]_{(\mathbf{m},\tau)} \longrightarrow \widehat{R}_y$ is not flat. The Noetherian Flatness Theorem 6.3 implies that τ is not primarily limit-intersecting in y over R, and B is not Noetherian. □

For the completion of the proof of Theorem 22.19, it remains to show that τ is residually limit-intersecting in y over R. We first establish the following claim.

CLAIM 22.23. *Let $\widehat{P}$ be a height-one prime ideal of $\widehat{R} = \mathbb{Q}[[x, y, z]]$, and suppose $\widehat{P} \cap \mathbb{Q}[[x, y]] \neq (0)$. Then the prime ideal $P_0 := \widehat{P} \cap R$ is extended from $\mathbb{Q}[x, y] \subseteq R$.*

PROOF. We may assume P_0 is distinct from (0), $x\widehat{R}$ and $y\widehat{R}$ since these are obviously extended. Since $\widehat{R}$ is faithfully flat over R, P_0 has height one. Similarly $P_1 := \widehat{P} \cap V[y, z]_{(x,y,z)}$ has height at most one since $\widehat{R}$ is also the completion of $V[y, z]_{(x,y,z)}$. We also have $P_1 \cap R = P_0$, so P_1 is nonzero and hence has height one. Since, for every $n \in \mathbb{N}$, $\sigma_n \in \mathbb{Q}[x, y, z, \sigma][1/x]$, the ring $R[1/x]$ is a localization of $\mathbb{Q}[x, y, z, \sigma][1/x]$. Thus $\widehat{P} \cap \mathbb{Q}[x, y, z, \sigma]$ has height one and contains an element f which generates P_0. Similarly, for every $n \in \mathbb{N}$, γ_n and δ_n are in $\mathbb{Q}[x, y, z, \gamma, \delta][1/x]$, which implies the ring $V[y, z]_{(x,y,z)}[1/x]$ is a localization of $\mathbb{Q}[x, y, z, \gamma, \delta][1/x]$. Thus $\widehat{P} \cap \mathbb{Q}[x, y, z, \gamma, \delta]$ has height one and contains a generator g for P_1. Let $\hat{h} \in \mathbb{Q}[[x, y]]$ be a generator of $\widehat{P} \cap \mathbb{Q}[[x, y]]$. Then $\hat{h}\widehat{R} = \widehat{P}$. The following diagram illustrates this situation:



Picture for proof of (22.23)

Subclaim 1: $g \in \widehat{P} \cap \mathbb{Q}[x, y, \gamma, \delta]$.

PROOF. (of Subclaim 1) : Write $g = g_0 + g_1 z + \cdots + g_r z^r$, where the $g_i \in \mathbb{Q}[x, y, \gamma, \delta]$. Since $g \in \widehat{P}$, we have $g = \hat{h}(x, y)\phi(x, y, z)$, for some $\phi(x, y, z) \in \mathbb{Q}[[x, y, z]]$. Since g is irreducible and $P_1 \neq zV[y, z]_{(x,y,z)}$, we have $g_0 \neq (0)$.

Setting $z = 0$, we have $g_0 = g(0) = \hat{h}(x, y)\phi(x, y, 0) \in \mathbb{Q}[[x, y]]$. Thus $g_0 \in \hat{h}\mathbb{Q}[[x, y]] \cap \mathbb{Q}[x, y, \gamma, \delta] \neq (0)$. Therefore $g\mathbb{Q}[x, y, \gamma, \delta, z] = g_0\mathbb{Q}[x, y, \gamma, \delta]$ is extended from $\mathbb{Q}[x, y, \gamma, \delta]$ and $g = g_0 \in \mathbb{Q}[x, y, \gamma, \delta]$, □

Subclaim 2: If we express f as $f = f_0 + f_1 z + \cdots + f_r z^r$, where the $f_i \in \mathbb{Q}[x, y, \gamma, \delta]$, then $f_0 \in \mathbb{Q}[x, y, \gamma]$.

PROOF. (of Subclaim 2) Since f is an element of $\mathbb{Q}[x, y, \sigma, z]$, we can write f as a polynomial

$$f = \sum a_{ij} z^i \sigma^j = \sum a_{ij} z^i (\gamma + z\delta)^j, \text{ where } a_{ij} \in \mathbb{Q}[x, y].$$

Setting $z = 0$, we have $f_0 = f(0) = \sum a_{0j} (\gamma)^j \in \mathbb{Q}[x, y, \gamma]$. □

PROOF. Completion of proof of Claim 22.23. Since $f \in \widehat{P} \cap \mathbb{Q}[x, y, z, \gamma, \delta]$, we have $f = dg$, for some $d \in \mathbb{Q}[x, y, \gamma, \delta, z]$. Regarding d as a polynomial in z with coefficients in $\mathbb{Q}[x, y, \gamma, \delta]$ and setting $z = 0$, gives $f_0 = f(0) = d(0)g \in \mathbb{Q}[x, y, \gamma, \delta]$. Thus f_0 is a multiple of g. Hence $g \in \mathbb{Q}[x, y, \gamma]$, by Subclaim 2.

Now again using that $f = dg$ and setting $z = 1$, we have $d(1)g = f(1) \in \mathbb{Q}[x, y, \gamma + \delta]$. This says that $f(1)$ is a multiple of the polynomial $g \in \mathbb{Q}[x, y, \gamma]$. Since γ and δ are algebraically independent over $\mathbb{Q}[x, y]$, this implies $f(1)$ has degree 0 in $\gamma + \delta$ and g has degree 0 in γ. Therefore $g \in \mathbb{Q}[x, y]$, $d \in \mathbb{Q}$, $0 \neq f \in \mathbb{Q}[x, y]$, and $P_0 = fR$ is extended from $\mathbb{Q}[x, y]$. □

It follows that τ is residually limit-intersecting provided we show the following:

CLAIM 22.24. *Suppose $\widehat{P}$ is a height-one prime ideal of $\widehat{R}$ with $y \notin \widehat{P}$ and* $\mathrm{ht}(\widehat{P} \cap R) = 1$. *Then the image $\bar{\tau}$ of τ in $\widehat{R}/\widehat{P}$ is algebraically independent over $R/(\widehat{P} \cap R)$.*

PROOF. (of Claim 22.24) Let $P_0 := \widehat{P} \cap R$ and let $\pi : \mathbb{Q}[[x, y, z]] \longrightarrow \mathbb{Q}[[x, y, z]]/\widehat{P}$; we use $\bar{}$ to denote the image under π. If $\widehat{P} = x\widehat{R}$, then we have the commutative diagram:

$$\begin{array}{ccccc} R/P_0 & \longrightarrow & (R/P_0)[\bar{\tau}] & \longrightarrow & \mathbb{Q}[[x, y, z]]/\widehat{P} \\ \cong\uparrow & & \cong\uparrow & & \cong\uparrow \\ \mathbb{Q}[y, z]_{(y,z)} & \longrightarrow & \mathbb{Q}[y, z]_{(y,z)}[\tau] & \longrightarrow & \mathbb{Q}[[y, z]] \ . \end{array}$$

Since τ is transcendental over $\mathbb{Q}[y, z]$, the result follows in this case. □

For the other height-one primes $\widehat{P}$ of $\widehat{R}$, we distinguish two cases:
case 1: $\widehat{P} \cap \mathbb{Q}[[x, y]] = (0)$.
Let $P_1 := V[y, z]_{(x,y,z)} \cap \widehat{P}$. We have the following commutative diagram of local injective morphisms:

$$\begin{array}{ccccc} R/P_0 & \longrightarrow & V[y, z]_{(x,y,z)}/P_1 & \longrightarrow & \mathbb{Q}[[x, y, z]]/\widehat{P} \\ & & \uparrow & & \uparrow \\ & & V[y]_{(x,y)} & \longrightarrow & \mathbb{Q}[[x, y]] \ , \end{array}$$

where $V[y, z]_{(x,y,z)}/P_1$ is algebraic over $V[y]_{(x,y)}$. Since $\tau \in \mathbb{Q}[[x, y]]$ is transcendental over $V[y]_{(x,y)}$, its image $\bar{\tau}$ in $\mathbb{Q}[[x, y, z]]/\widehat{P}$ is transcendental over $V[y, z]_{(x,y,z)}/P_1$ and thus is transcendental over R/P_0.
case 2: $\widehat{P} \cap \mathbb{Q}[[x, y]] \neq (0)$.
In this case, by Claim 22.23, the height-one prime $P_0 := \widehat{P} \cap R$ is extended from a prime ideal in $\mathbb{Q}[x, y]$. Let p be a prime element of $\mathbb{Q}[x, y]$ such that $(p) = \widehat{P} \cap \mathbb{Q}[x, y]$. We have the inclusions:

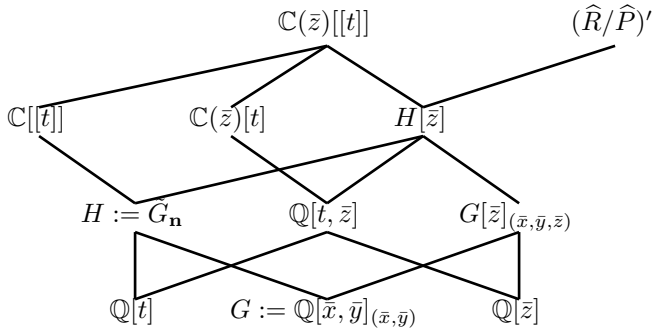
$$G := \mathbb{Q}[x, y]_{(x,y)}/(p) \hookrightarrow R/P_0 \hookrightarrow \widehat{R}/\widehat{P} \ ,$$

where $R/P_0 = \varinjlim \mathbb{Q}[x, y, z, \sigma_n]_{(x,y,z,\sigma_n)}/(p)$ has transcendence degree ≤ 1 over $G[\bar{z}]_{(\bar{x},\bar{y},\bar{z})}$. It suffices to show that $\bar{\sigma}$ and $\bar{\tau}$ are algebraically independent over the field of fractions $\mathbb{Q}(\bar{x}, \bar{y}, \bar{z})$ of $G[\bar{z}]$.

Let $\tilde{G}$ be the integral closure of G in its field of fractions and let $H := \tilde{G}_{\mathbf{n}}$ be a localization of $\tilde{G}$ at a maximal ideal $\mathbf{n}$ such that the completion $\widehat{H}$ of H is dominated by the integral closure $(\widehat{R}/\widehat{P})'$ of $\widehat{R}/\widehat{P}$.

Now $H[\bar{z}]$ has transcendence degree at least one over $\mathbb{Q}[\bar{z}]$. Also since $\widehat{P} \cap \mathbb{Q}[x, y] \neq 0$, the transcendence degree of $\mathbb{Q}[\bar{x}, \bar{y}]$ and so of H is at most one over $\mathbb{Q}$. Thus $H[\bar{z}]$ has transcendence degree exactly one over $\mathbb{Q}(\bar{z})$. There exists an element $t \in H$ that is transcendental over $\mathbb{Q}[\bar{z}]$ and is such that t generates the maximal ideal of the DVR H. Then H is algebraic over $\mathbb{Q}[t]$ and H may be regarded as a subring of $\mathbb{C}[[t]]$, where $\mathbb{C}$ is the complex numbers. In order to show that $\bar{\sigma}$ and $\bar{\tau}$ are algebraically independent over $G[\bar{z}]$, it suffices to show that $\bar{\sigma}$ and $\bar{\tau}$ are algebraically independent over $H[\bar{z}]$ and thus it suffices to show that these elements are algebraically independent over $\mathbb{Q}(t, \bar{z})$. Thus it suffices to show that $\bar{\sigma}$ and $\bar{\tau}$ are algebraically independent over $\mathbb{C}(t, \bar{z})$.

We have the setup shown in the following diagram:



By [**14**], if $\bar{x}, \bar{x}^2, \bar{y} \in t\mathbb{C}[[t]]$ are linearly independent over $\mathbb{Q}$, then:

$$\operatorname{trdeg}_{\mathbb{C}(t)}(\mathbb{C}(t)(\bar{x}, \bar{x}^2, \bar{y}, e^{\bar{x}}, e^{\bar{x}^2}, e^{\bar{y}})) \geq 3.$$

Since $\bar{x}, \bar{x}^2$ and $\bar{y}$ are in H, these elements are algebraic over $\mathbb{Q}(t)$. Therefore if $\bar{x}, \bar{x}^2, \bar{y}$ are linearly independent over $\mathbb{Q}$, then the exponential functions $e^{\bar{x}}, e^{\bar{x}^2}, e^{\bar{y}}$ are algebraically independent over $\mathbb{Q}(t)$ and hence $\bar{\sigma}$ and $\bar{\tau}$ are algebraically independent over $G(\bar{z})$.

We observe that if $\bar{x}, \bar{x}^2, \bar{y} \in tH$ are linearly dependent over $\mathbb{Q}$, then there exist $a, b, c \in \mathbb{Q}$ such that

$$lp = ax + bx^2 + cy \quad \text{in} \quad \mathbb{Q}[x, y].$$

where $l \in \mathbb{Q}[x, y]$. Since $(p) \neq (x)$, we have $c \neq 0$. Hence we may assume $c = 1$ and $lp = y - ax - bx^2$ with $a, b \in \mathbb{Q}$. Since $y - ax - bx^2$ is irreducible in $\mathbb{Q}[x, y]$, we may assume $l = 1$. Also a and b cannot both be 0 since $y \notin \widehat{P}$. Thus if $\bar{x}, \bar{x}^2, \bar{y} \in tH$ are linearly dependent over $\mathbb{Q}$, then we may assume

$$p = y - ax - bx^2 \quad \text{for some } a, b \in \mathbb{Q} \quad \text{not both 0.}$$

It remains to show that $\bar{\sigma}$ and $\bar{\tau}$ are algebraically independent over $G(\bar{z})$ provided that $p = y - ax - bx^2$, that is

$$\bar{y} = a\bar{x} + b\bar{x}^2, \quad \text{for } a, b \in \mathbb{Q}, \text{ not both } 0.$$

Suppose $h \in G[\bar{z}][u, v]$, where u, v are indeterminates and that $h(\bar{\sigma}, \bar{\tau}) = 0$. This implies

$$h(e^{\bar{x}} + \bar{z}e^{\bar{x}^2}, e^{a\bar{x}+b\bar{x}^2}) = 0.$$

We have $e^{a\bar{x}} = (e^{\bar{x}})^a$ and $e^{b\bar{x}^2}$ are algebraic over $G(\bar{z}, e^{\bar{x}}, e^{\bar{x}^2})$ since a and b are rational. By substituting $\bar{z} = 0$ we obtain an equation over G:

$$h(e^{\bar{x}}, e^{a\bar{x}+b\bar{x}^2}) = 0$$

which implies that $b = 0$ since $\bar{x}$ and $\bar{x}^2$ are linearly independent over $\mathbb{Q}$. Now the only case to consider is the case where $p = y + ax$. The equation we obtain then is:

$$h(e^{\bar{x}} + \bar{z}e^{\bar{x}^2}, e^{a\bar{x}}) = 0$$

which implies that h must be the zero polynomial, since $e^{\bar{x}^2}$ is transcendental over the algebraic closure of the field of fractions of $G[\bar{z}, e^{\bar{x}}]$. This completes the proof of Claim 22.24. Thus τ is residually limit-intersecting over R.

Since R is a UFD, the element τ is limit-intersecting over R by Remark 22.17. This completes the proof of Theorem 22.19. □

REMARK 22.25. With notation as in Theorem 22.19, let u, v be indeterminates over $\mathbb{Q}[[x, y, z]]$. Then the height-one prime ideal $\widehat{Q} = (u - \tau)$ in $\mathbb{Q}[[x, y, z, u]]$ is in the generic formal fiber of the excellent regular local ring $R[u]_{(x,y,z,u)}$ and the intersection domain

$$K(u) \cap \mathbb{Q}[[x, y, z, u]]/\widehat{Q} \cong K(\tau) \cap \widehat{R}\ ,$$

where K is the fraction field of R, fails to be Noetherian. In a similar fashion this intersection ring $K(\tau) \cap \widehat{R}$ may be identified with the following ring: Let $\widehat{U} = (u - \tau, v - \sigma)$ be the height-two prime ideal in $\mathbb{Q}[[x, y, z, u, v]]$ which is in the generic formal fiber of the polynomial ring $\mathbb{Q}[x, y, z, u, v]_{(x,y,z,u,v)}$. Then we have:

$$\mathbb{Q}(x, y, z, u, v) \cap ((\mathbb{Q}[[x, y, z, u, v]])/\widehat{U}) \cong K(\tau) \cap \widehat{R}\ ,$$

and as shown in Theorem 22.19, this ring is not Noetherian. We do not know an example of a height-one prime ideal $\widehat{W}$ in the generic formal fiber of a polynomial ring T for which the intersection ring $A = \mathcal{Q}(T) \cap (\widehat{T}/\widehat{W})$ fails to be Noetherian. In Chapter intsec we present an example of such an intersection ring A whose completion is not equal to $\widehat{T}$, however in this example the ring A is still Noetherian.

22.4. Several additional examples

Let $R = \mathbb{Q}[x, y]_{(x,y)}$, the localized polynomial ring in two variables x and y over the field $\mathbb{Q}$ of rational numbers. Then $\widehat{R} = \mathbb{Q}[[x, y]]$, the formal power series ring in x and y, is the $\mathbf{m} = (x, y)R$-adic completion of R. In Chapter 20, an element $\tau \in \widehat{\mathbf{m}} = (x, y)\widehat{R}$ is defined to be *residually algebraically independent over* R if τ is algebraically independent over R and for each height-one prime $\widehat{P}$ of $\widehat{R}$ such that $\widehat{P} \cap R \neq (0)$, the image of τ in $\widehat{R}/\widehat{P}$ is algebraically independent over the fraction field of $R/(\widehat{P} \cap R)$. It is shown in Theorem 20.27 of Chapter 20, that if τ is

residually algebraically independent over R and L is the field of fractions of $R[\tau]$, then $L \cap \widehat{R}$ is the localized polynomial ring $R[\tau]_{(\mathbf{m},\tau)}$.

In this section we present several examples of residually algebraically independent elements.

EXAMPLE 22.26. For $S := \mathbb{Q}[x,y,z]_{(x,y,z)}$, the construction of Theorem 12.16 yields an example of a height-one prime ideal $\widehat{P}$ of $\widehat{S} = \mathbb{Q}[[x,y,z]]$ in the generic formal fiber of S such that

$$\mathcal{Q}(S) \cap (\widehat{S}/\widehat{P}) = S.$$

PROOF. Let $\widehat{P} := (z - \tau) \subseteq \mathbb{Q}[[x,y,z]]$, where τ is as in Theorem 12.16. Then $\mathbb{Q}(x,y,z) \cap (\widehat{S}/\widehat{P})$ can be identified with the intersection $\mathbb{Q}(x,y,\tau) \cap \mathbb{Q}[[x,y]]$ of (6.1). Therefore

$$\mathbb{Q}(x,y,z) \cap (\widehat{S}/\widehat{P}) = S = \mathbb{Q}[x,y,z]_{(x,y,z)}.$$

□

With $S = \mathbb{Q}[x,y,z]_{(x,y,z)}$, every prime ideal of $\widehat{S} = \mathbb{Q}[[x,y,z]]$ that is maximal in the generic formal fiber of S has height 2. Thus the prime ideal $\widehat{P}$ is not maximal in the generic formal fiber of $S = \mathbb{Q}[x,y,z]_{(x,y,z)}$.

REMARK 22.27. Let $(R,\mathbf{m})$ be a localized polynomial ring over a field and let $\widehat{R}$ denote the $\mathbf{m}$-adic completion of R. It is observed in [**60**, Theorem 2.5] that there exists a one-to-one correspondence between prime ideals $\widehat{\mathbf{p}}$ of $\widehat{R}$ that are maximal in the generic formal fiber of R and DVRs C such that C birationally dominates R and $C/\mathbf{m}C$ is a finitely generated R-module. Example 22.26 demonstrates that this strong connection between the maximal ideals of the generic formal fiber of a localized polynomial ring R and certain birational extensions of R does not extend to prime ideals nonmaximal in the generic formal fiber R.

EXAMPLE 22.28. Again let $S = \mathbb{Q}[x,y,z]_{(x,y,z)}$. With a slight modification of Example 22.26, we exhibit a prime ideal $\widehat{P}$ in the generic formal fiber of S which *does* correspond to a nontrivial birational extension; that is, the intersection ring

$$A := \mathcal{Q}(S) \cap \widehat{S}/\widehat{P}$$

is essentially finitely generated over S.

PROOF. Let τ be the element from Theorem 12.16. Let $\widehat{P} = (z - x\tau) \subseteq \mathbb{Q}[[x,y,z]]$. Since τ is transcendental over $\mathbb{Q}(x,y,z)$, the prime ideal $\widehat{P}$ is in the generic formal fiber of S. The ring S can be identified with a subring of $\widehat{S}/\widehat{P} \cong \mathbb{Q}[[x,y]]$ by considering $S = \mathbb{Q}[x,y,x\tau]_{(x,y,x\tau)}$. By reasoning similar to that of Example 22.26,

$$\mathcal{Q}(S) \cap \mathbb{Q}[[x,y]] = \mathbb{Q}(x,y,\tau) \cap \mathbb{Q}[[x,y]] = \mathbb{Q}[x,y,\tau]_{(x,y,\tau)}.$$

The ring $\mathbb{Q}[x,y,\tau]_{(x,y,\tau)}$ is then the essentially finitely generated birational extension of S defined as $S[z/x]_{(x,y,z/x)}$. □

EXAMPLE 22.29. Let $\sigma \in x\mathbb{Q}[[x]]$ and $\rho \in y\mathbb{Q}[[y]]$ be as in Theorem 12.16. If $D := \mathbb{Q}(x,\sigma) \cap \mathbb{Q}[[x]] = \bigcup_{n=1}^{\infty} \mathbb{Q}[x,\sigma_n]_{(x,\sigma_n)}$ and $T := D[y]_{(x,y)}$, so T is regular local with completion $\widehat{T} = \mathbb{Q}[[x,y]]$, then the element ρ is primarily limit-intersecting in y over T.

PROOF. We show that the map $\varphi_y : T[\rho] \longrightarrow \mathbb{Q}[[x,y]][1/y]$ is LF_1; that is, the induced map $\varphi_{\widehat{P}} : T[\rho]_{\widehat{P}\cap T[\rho]} \longrightarrow \mathbb{Q}[[x,y]]_{\widehat{P}}$ is flat for every height-one prime ideal $\widehat{P}$ of $\mathbb{Q}[[x,y]]$ with $y \notin \widehat{P}$. It is equivalent to show for every height-one prime $\widehat{P}$ of $\mathbb{Q}[[x,y]]$ that $\widehat{P} \cap T[\rho]$ has height ≤ 1. If $\widehat{P} = (x)$, the statement is immediate, since ρ is algebraically independent over $\mathbb{Q}(y)$. Next we consider the case $\widehat{P} \cap Q[x,y,\sigma] = (0)$. Since $\mathbb{Q}(x,y,\sigma) = \mathbb{Q}(x,y,\sigma_n)$ for every positive integer n, $\widehat{P} \cap Q[x,y,\sigma] = (0)$ if and only if $\widehat{P} \cap Q[x,y,\sigma_n] = (0)$. Moreover, if this is true, then since the fraction field of $T[\rho]$ has transcendence degree one over $\mathbb{Q}(x,y,\sigma)$, then $\widehat{P} \cap T[\rho]$ has height ≤ 1. The remaining case is where $P := \widehat{P} \cap Q[x,y,\sigma] \neq (0)$ and $xy \notin \widehat{P}$. By Proposition 6.3, $\bar{\rho}$ is transcendental over $\bar{T} = T/(\widehat{P} \cap T)$, and this is equivalent to $\mathrm{ht}(\widehat{P} \cap T[\tau]) = 1$. □

Still referring to ρ, σ, σ_n as in Theorem 12.16 and Example 22.29, and using that σ is primarily limit-intersecting in y over T, we have:

$$A := \mathcal{Q}(T)(\rho) \cap \mathbb{Q}[[x,y]] = \varinjlim T[\rho_n]_{(x,y,\rho_n)} = \varinjlim \mathbb{Q}[x,y,\sigma_n,\rho_n]_{(x,y,\sigma_n,\rho_n)}$$

where the endpieces ρ_n are defined as in Section 5.4; viz., $\rho := \sum_{n=1}^{\infty} b_i y^i$ and $\rho_n = \sum_{i=n+1}^{\infty} b_i y^{i-n}$. The philosophy here is that sufficient "independence" of the algebraically independent elements σ and ρ allows us to explicitly describe the intersection ring A.

The previous examples have been over localized polynomial rings, where we are free to exchange variables. The next example shows, over a different regular local domain, that an element in the completion with respect to one regular parameter x may be residually limit-intersecting with respect to x whereas the corresponding element in the completion with respect to another regular parameter y may be transcendental but fail to be residually limit-intersecting.

EXAMPLE 22.30. There exists a regular local ring R with $\widehat{R} = \mathbb{Q}[[x,y]]$ such that $\sigma = e^x - 1$ is residually limit-intersecting in x over R, whereas $\gamma = e^y - 1$ fails to be limit-intersecting in y over R.

PROOF. Let $\{\omega_i\}_{i\in I}$ be a transcendence basis of $\mathbb{Q}[[x]]$ over $\mathbb{Q}(x)$ such that:

$$\{e^{x^n}\}_{n\in\mathbb{N}} \subseteq \{\omega_i\}_{i\in I}.$$

Let D be the discrete valuation ring:

$$D = \mathbb{Q}(x, \{\omega_i\}_{i\in I, \omega_i \neq e^x}) \cap \mathbb{Q}[[x]].$$

Obviously, $\mathbb{Q}[[x]]$ has transcendence degree 1 over D. The set $\{e^x\}$ is a transcendence basis of $\mathbb{Q}[[x]]$ over D. Let $R = D[y]_{(x,y)}$.

By Remark 22.12.1, the element $\sigma = e^x - 1$ is primarily limit-intersecting and hence residually limit-intersecting in x over D. Moreover, by Remark 22.12.2, σ is also primarily and hence residually limit-intersecting over $R := D[y]_{(x,y)}$. However, the element $\gamma = e^y - 1$ is not residually limit-intersecting in y over R. To see this, consider the height-one prime ideal $P := (y - x^2)\mathbb{Q}[[x,y]]$. The prime ideal $W := P \cap R[\tau]_{(x,y,\tau)}$ contains the element $\gamma - e^{x^2} - 1 = e^y - e^{x^2}$. Therefore W has height greater than one and γ is not residually limit-intersecting in y over R. □

Note that the intersection ring $\mathcal{Q}(R)(\tau) \cap \mathbb{Q}[[x,y]]$ is a regular local ring with completion $\mathbb{Q}[[x,y]]$ by Theorem 4.8, a theorem of Valabrega.

CHAPTER 23

Rings between excellent normal local domains and their completions

Let $(R, \mathbf{m})$ be an excellent normal local domain. Let y be a nonzero element in $\mathbf{m}$ and let R^* denote the (y)-adic completion of R. In this chapter we consider certain extension domains A inside R^* arising from Inclusion Construction 5.3 and Homomorphic Image Construction 17.2. We use test criteria given in Theorem 7.3, Theorem 7.4 and Corollary 7.5, involving the heights of certain prime ideals to determine flatness for the map φ defined in Equation 23.1.0. These characterizations of flatness involve the condition that certain fibers are Cohen-Macaulay and other fibers are regular.

We give in Theorem 23.12 and Remarks 23.14 necessary and sufficient conditions for an element $\tau \in yR^*$ to be primarily limit-intersecting in y over R; see Remark 23.2. If R is countable, we prove in Theorem 23.19 the existence of an infinite sequence of elements of yR^* that are primarily limit-intersecting in y over R. Using this result we establish the existence of a normal Noetherian local domain B such that: B dominates R; B has (y)-adic completion R^*; and B contains a height-one prime ideal $\mathbf{p}$ such that $R^*/\mathbf{p}R^*$ is not reduced. Thus B is not a Nagata domain and hence is not excellent; see Remark 3.38.

In Section 23.3 we observe that every Noetherian local ring containing an excellent local subring R and having the same completion as R has Cohen-Macaulay formal fibers. This applies to examples obtained by Inclusion Construction 5.3; see Corollary 23.23. It does not apply to examples obtained by Homomorphic Image Construction 17.2. In Remark 23.25, we discuss connections with a famous example of Ogoma.

We present in Section 23.4 integral domains B and A arising from Inclusion Construction 5.3 and C arising from Homomorphic Image Construction 17.2. In Theorems 23.27 and 23.28 we show that A and B are non-Noetherian and $B \subsetneq A$. We establish in Theorem 23.30 that the domain C is a two-dimensional Noetherian local domain, C is a homomorphic image of B and C has the property that its generic formal fiber is not Cohen-Macaulay.

23.1. Primarily limit-intersecting extensions and flatness

In this section, we consider properties of Inclusion Construction 5.3 under the assumptions of Setting 23.1.

SETTING 23.1. Let $(R, \mathbf{m})$ be an excellent normal local domain and let y be a nonzero element in $\mathbf{m}$. Let $(R^*, \mathbf{m}^*)$ be the (y)-adic completion of R and let $(\widehat{R}, \widehat{\mathbf{m}})$ be the $\mathbf{m}$-adic completion of R. Thus R^* and $\widehat{R}$ are normal Noetherian local domains and $\widehat{R}$ is the $\mathbf{m}^*$-adic completion of R^*. Let $\tau_1, \ldots, \tau_s$ be elements of yR^* that are algebraically independent over R, and set $U_0 = S := R[\tau_1, \ldots, \tau_s]$.

The field of fractions L of S is a subfield of the field of fractions $\mathcal{Q}(R^*)$ of R^*. Define $A := L \cap R^*$.

REMARK 23.2. The Noetherian Flatness Theorem 6.3 implies that $A = L \cap R^*$ is both Noetherian and a localization of a subring of $S[1/y]$ if and only if the extension φ is flat, where

$$\varphi : S \longrightarrow R^*[1/y] \tag{23.1.0}$$

By Definition 22.9.3, the elements $\tau_1, \ldots, \tau_s$ are *primarily limit-intersecting* in y over R if and only if φ is flat.

THEOREM 23.3. *Assume notation as in Setting 23.1. That is, $(R, \mathbf{m})$ is an excellent normal local domain, y is a nonzero element in $\mathbf{m}$, $(R^*, \mathbf{m}^*)$ is the (y)-adic completion of R, and the elements $\tau_1, \ldots, \tau_s \in yR^*$ are algebraically independent over R. Then the following statements are equivalent:*

1. *$S := R[\tau_1, \ldots, \tau_s] \hookrightarrow R^*[1/y]$ is flat. Equivalently, $\tau_1, \ldots, \tau_s$ are primarily limit-intersecting in y over R^*.*
2. *For P a prime ideal of S and Q^* a prime ideal of R^* minimal over PR^*, if $y \notin Q^*$, then $\operatorname{ht}(Q^*) = \operatorname{ht}(P)$.*
3. *If Q^* is a prime ideal of R^* with $y \notin Q^*$, then $\operatorname{ht}(Q^*) \geq \operatorname{ht}(Q^* \cap S)$.*

Moreover, if any of (1)-(3) hold, then $S \hookrightarrow R^*[1/y]$ has Cohen-Macaulay fibers.

PROOF. By Remark 23.2, we have the equivalence in item 1.

(1) $\Rightarrow$ (2): Let P be a prime ideal of S and let Q^* be a prime ideal of R^* that is minimal over PR^* and is such that $y \notin Q^*$. The assumption of item 1 implies flatness of the map:

$$\varphi_{Q^*} : S_{Q^* \cap S} \longrightarrow R^*_{Q^*}.$$

By Remark 2.31.10, we have $Q^* \cap S = P$, and by [**103**, Theorem 15.1], $\operatorname{ht} Q^* = \operatorname{ht} P$.

(2) $\Rightarrow$ (3): Let Q^* be a prime ideal of R^* with $y \notin Q^*$. Set $Q := Q^* \cap S$ and let $\mathbf{w}^*$ be a prime ideal of R^* that is minimal over QR^* and is contained in Q^*. Then $\operatorname{ht}(Q) = \operatorname{ht}(\mathbf{w}^*)$ by (2) since $y \notin \mathbf{w}^*$ and therefore $\operatorname{ht}(Q^*) \geq \operatorname{ht}(Q)$.

(3) $\Rightarrow$ (1): Let Q^* be a prime ideal of R^* with $y \notin Q^*$. Then for every prime ideal $\mathbf{w}^*$ of R^* contained in Q^*, we also have $y \notin \mathbf{w}^*$, and by (3), $\operatorname{ht}(\mathbf{w}^*) \geq \operatorname{ht}(\mathbf{w}^* \cap S)$. Therefore, by Theorem 7.4, $\varphi_{Q^*} : S_{Q^* \cap S} \longrightarrow R^*_{Q^*}$ is flat with Cohen-Macaulay fibers. □

With notation as in Setting 23.1, the map $R^* \hookrightarrow \widehat{R}$ is flat. Hence the corresponding statements in Theorem 23.3 with R^* replaced by $\widehat{R}$ also hold. We record this as

COROLLARY 23.4. *Assume notation as in Setting 23.1. Then the following statements are equivalent:*

1. *$S := R[\tau_1, \ldots, \tau_s] \hookrightarrow \widehat{R}[1/y]$ is flat.*
2. *For P a prime ideal of S and $\widehat{Q}$ a prime ideal of $\widehat{R}$ minimal over PR, if $y \notin \widehat{Q}$, then $\operatorname{ht}(\widehat{Q}) = \operatorname{ht}(P)$.*
3. *If $\widehat{Q}$ is a prime ideal of $\widehat{R}$ with $y \notin \widehat{Q}$, then $\operatorname{ht}(\widehat{Q}) \geq \operatorname{ht}(\widehat{Q} \cap S)$.*

Moreover, if any of (1)-(3) hold, then $S \hookrightarrow \widehat{R}[1/y]$ has Cohen-Macaulay fibers.

As another corollary to Theorem 23.3, we have the following result:

COROLLARY 23.5. *With the notation of Theorem 23.3, assume that* $\widehat{R}[1/y]$ *is flat over* S. *Let* $P \in \operatorname{Spec} S$ *with* $\operatorname{ht}(P) \geq \dim(R)$. *Then*

(1) *For every* $\widehat{Q} \in \operatorname{Spec} \widehat{R}$ *minimal over* $P\widehat{R}$ *we have* $y \in \widehat{Q}$.
(2) *Some power of* y *is in* $P\widehat{R}$.

PROOF. Clearly items 1 and 2 are equivalent. To prove these hold, suppose that $y \notin \widehat{Q}$. By Theorem 23.3.2, $\operatorname{ht}(P) = \operatorname{ht}(\widehat{Q})$. Since $\dim(R) = \dim(\widehat{R})$, we have $\operatorname{ht}(\widehat{Q}) \geq \dim(\widehat{R})$. But then $\operatorname{ht}(\widehat{Q}) = \dim(\widehat{R})$ and $\widehat{Q}$ is the maximal ideal of $\widehat{R}$. This contradicts the assumption that $y \notin \widehat{Q}$. We conclude that $y \in \widehat{Q}$. □

Theorem 23.3, together with results from Chapter 6, gives the following corollary.

COROLLARY 23.6. *Assume notation as in Setting 23.1, and consider the following conditions:*

(1) A *is Noetherian and is a localization of a subring of* $S[1/y]$.
(2) $S \hookrightarrow R^*[1/y]$ *is flat.*
(3) $S \hookrightarrow R^*[1/y]$ *is flat with Cohen-Macaulay fibers.*
(4) *For every* $Q^* \in \operatorname{Spec}(R^*)$ *with* $y \notin Q^*$, *we have* $\operatorname{ht}(Q^*) \geq \operatorname{ht}(Q^* \cap S)$.
(5) A *is Noetherian.*
(6) $A \hookrightarrow R^*$ *is flat.*
(7) $A \hookrightarrow R^*[1/y]$ *is flat.*
(8) $A \hookrightarrow R^*[1/y]$ *is flat with Cohen-Macaulay fibers.*

Conditions (1)-(4) are equivalent, conditions (5)-(8) are equivalent and (1)-(4) imply (5)-(8).

PROOF. Item 1 is equivalent to item 2 by Noetherian Flatness Theorem 17.13, item 2 is equivalent to item 3 and item 7 is equivalent to item 8 by Theorem 7.4, and item 2 is equivalent to item 4 by Theorem 23.3.

It is obvious that item 1 implies item 5. By Construction Properties Theorem 5.14.3, the ring R^* is the y-adic completion of A, and so item 5 is equivalent to item 6. By Lemma 6.2).1, item 6 is equivalent to item 7. □

REMARKS 23.7. **(i)** With the notation of Corollary 23.6, if $\dim A = 2$, it follows that condition (7) of Corollary 23.6 holds. Since R^* is normal, so is A. Thus if $Q^* \in \operatorname{Spec} R^*$ with $y \notin Q^*$, then $A_{Q^* \cap A}$ is either a DVR or a field. The map $A \to R^*_{Q^*}$ factors as $A \to A_{Q^* \cap A} \to R^*_{Q^*}$. Since $R^*_{Q^*}$ is a torsionfree and hence flat $A_{Q^* \cap A}$-module, it follows that $A \to R^*_{Q^*}$ is flat. Therefore $A \hookrightarrow R^*[1/y]$ is flat and A is Noetherian.
(ii) There exist examples where $\dim A = 2$ and conditions (5)-(8) of Corollary 23.6 hold, but yet conditions (1)-(4) fail to hold; see Theorem 12.3.

QUESTION 23.8. With the notation of Corollary 23.6, suppose for every prime ideal Q^* of R^* with $y \notin Q^*$ that $\operatorname{ht}(Q^*) \geq \operatorname{ht}(Q^* \cap A)$. Does it follow that R^* is flat over A or, equivalently, that A is Noetherian?

Theorem 23.3 also extends to give equivalences for the *locally flat in height k* property; see Definitions 22.1.

THEOREM 23.9. *Assume notation as in Setting 23.1. That is,* $(R, \mathbf{m})$ *is an excellent normal local domain,* y *is a nonzero element in* $\mathbf{m}$, $(R^*, \mathbf{m}^*)$ *is the* (y)*-adic*

completion of R, and the elements $\tau_1, \ldots, \tau_s \in yR^*$ are algebraically independent over R. Then the following statements are equivalent:

(1) *$S := R[\tau_1, \ldots, \tau_s] \hookrightarrow \widehat{R}[1/y]$ is LF_k.*
(2) *If P is a prime ideal of S and $\widehat{Q}$ is a prime ideal of $\widehat{R}$ minimal over $P\widehat{R}$ and if, moreover, $y \notin \widehat{Q}$ and $\operatorname{ht}(\widehat{Q}) \leq k$, then $\operatorname{ht}(\widehat{Q}) = \operatorname{ht}(P)$.*
(3) *If $\widehat{Q}$ is a prime ideal of $\widehat{R}$ with $y \notin \widehat{Q}$ and $\operatorname{ht}(\widehat{Q}) \leq k$, then $\operatorname{ht}(\widehat{Q}) \geq \operatorname{ht}(\widehat{Q} \cap S)$.*

PROOF. (1) $\Rightarrow$ (2): Let P be a prime ideal of S and let $\widehat{Q}$ be a prime ideal of $\widehat{R}$ that is minimal over $P\widehat{R}$ with $y \notin \widehat{Q}$ and $\operatorname{ht}(\widehat{Q}) \leq k$. The assumption of item 1 implies flatness for the map:

$$\varphi_{\widehat{Q}} : S_{\widehat{Q} \cap S} \longrightarrow \widehat{R}_{\widehat{Q}},$$

and we continue as in Theorem 23.3.

(2) $\Rightarrow$ (3): Let $\widehat{Q}$ be a prime ideal of $\widehat{R}$ with $y \notin \widehat{Q}$ and $\operatorname{ht}(\widehat{Q}) \leq k$. Set $Q := \widehat{Q} \cap S$ and let $\widehat{W}$ be a prime ideal of $\widehat{R}$ which is minimal over $Q\widehat{R}$, and so that $\widehat{W} \subseteq \widehat{Q}$. Then $\operatorname{ht}(Q) = \operatorname{ht}(\widehat{W})$ by item 2 since $y \notin \widehat{W}$ and therefore $\operatorname{ht}(\widehat{Q}) \geq \operatorname{ht}(Q)$.

(3) $\Rightarrow$ (1): Let $\widehat{Q}$ be a prime ideal of $\widehat{R}$ with $y \notin \widehat{Q}$ and $\operatorname{ht}(\widehat{Q}) \leq k$. Then for every prime ideal $\widehat{W}$ contained in $\widehat{Q}$, we also have $y \notin \widehat{W}$ and $\operatorname{ht}(\widehat{W}) \geq \operatorname{ht}(\widehat{W} \cap S)$, by item 3. To complete the proof it suffices to show that $\varphi_{\widehat{Q}} : S_{\widehat{Q} \cap S} \longrightarrow \widehat{R}_{\widehat{Q}}$ is flat, and this is a consequence of Theorem 7.4. □

23.2. Existence of primarily limit-intersecting extensions

In this section, we establish the existence of primary limit-intersecting elements over countable excellent normal local domains. To do this, we use the following prime avoidance lemma that is analogous to Lemma 20.18, but avoids the hypothesis of Lemma 20.18 that T is complete in its $\mathbf{n}$-adic topology. See the articles [**24**], [**145**], [**159**] and the book [**91**, Lemma 14.2] for other prime avoidance results involving countably infinitely many prime ideals.

LEMMA 23.10. *Let $(T, \mathbf{n})$ be a Noetherian local domain that is complete in the (y)-adic topology, where y is a nonzero element of $\mathbf{n}$. Let $\mathcal{U}$ be a countable set of prime ideals of T such that $y \notin P$ for each $P \in \mathcal{U}$, and fix an arbitrary element $t \in \mathbf{n} \setminus \mathbf{n}^2$. Then there exists an element $a \in y^2T$ such that $t - a \notin \bigcup\{P : P \in \mathcal{U}\}$.*

PROOF. We may assume there are no inclusion relations among the $P \in \mathcal{U}$. We enumerate the prime ideals in $\mathcal{U}$ as $\{P_i\}_{i=1}^{\infty}$. We choose $b_2 \in T$ so that $t - b_2 y \notin P_1$ as follows: (i) if $t \in P_1$, let $b_2 = 1$. Since $y \notin P_1$, we have $t - y^2 \notin P_1$. (ii) if $t \notin P_1$, let b_2 be a nonzero element of P_1. Then $t - b_2 y^2 \notin P_1$. Assume by induction that we have found $b_2, \ldots, b_n$ in T such that

$$t - cy^2 \ := \ t - b_2 y^2 - \cdots - b_n y^n \ \notin \ P_1 \cup \cdots \cup P_{n-1}.$$

We choose $b_{n+1} \in T$ so that $t - cy^2 - b_{n+1} y^{n+1} \notin \bigcup_{i=1}^{n} P_i$ as follows: (i) if $t - cy^2 \in P_n$, let $b_{n+1} \in (\prod_{i=1}^{n-1} P_i) \setminus P_n$. (ii) if $t - cy^2 \notin P_n$, let b_{n+1} be any nonzero element in $\prod_{i=1}^{n} P_i$. Hence in either case there exists $b_{n+1} \in T$ so that

$$t - b_2 y^2 - \cdots - b_{n+1} y^{n+1} \ \notin P_1 \cup \cdots \cup P_n.$$

Since T is complete in the (y)-adic topology, the Cauchy sequence

$$\{b_2 y^2 + \cdots + b_n y^n\}_{n=2}^{\infty}$$

has a limit $a \in \mathbf{n}^2$. Since T is Noetherian and local, every ideal of T is closed in the (y)-adic topology. Hence, for each integer $n \geq 2$, we have

$$t - a \;=\; (t - b_2y^2 - \cdots - b_ny^n) \;-\; (b_{n+1}y^{n+1} + \cdots),$$

where $t - b_2y^2 - \cdots - b_ny^n \notin P_{n-1}$ and $(b_{n+1}y^{n+1} + \cdots) \in P_{n-1}$. We conclude that $t - a \notin \bigcup_{i=1}^{\infty} P_i$. □

We use the following setting to describe necessary and sufficient conditions for an element to be primarily limit-intersecting.

SETTING 23.11. Let $(R, \mathbf{m})$ be a d-dimensional excellent normal local domain with $d \geq 2$, let y be a nonzero element of $\mathbf{m}$ and let R^* denote the (y)-adic completion of R. Let t be a variable over R, let $S := R[t]_{(\mathbf{m},t)}$, and let S^* denote the I-adic completion of S, where $I := (y,t)S$. Then $S^* = R^*[[t]]$ is a $(d+1)$-dimensional normal Noetherian local domain with maximal ideal $\mathbf{n}^* := (\mathbf{m}, t)S^*$. For each element $a \in y^2S^*$, we have $S^* = R^*[[t]] = R^*[[t-a]]$. Let $\lambda_a : S^* \to R^*$ denote the canonical homomorphism $S^* \to S^*/(t-a)S^* = R^*$, and let $\tau_a = \lambda_a(t) = \lambda_a(a)$. Consider the set

$$\mathcal{U} \;:=\; \{P^* \in \operatorname{Spec} S^* \mid \operatorname{ht}(P^* \cap S) = \operatorname{ht} P^*, \text{ and } y \notin P^* \}.$$

Since $S \hookrightarrow S^*$ is flat and thus satisfies the Going-down property, the set $\mathcal{U}$ can also be described as the set of all $P^* \in \operatorname{Spec} S^*$ such that $y \notin P^*$ and P^* is minimal over PS^* for some $P \in \operatorname{Spec} S$, see [**103**, Theorem 15.1]

THEOREM 23.12. *With the notation of Setting 23.11, the element τ_a is primarily limit-intersecting in y over R if and only if $t - a \notin \bigcup\{P^* \mid P^* \in \mathcal{U}\}$.*

PROOF. Consider the commutative diagram:

$$\begin{array}{ccccccc}
 & & S = R[t]_{(\mathbf{m},t)} & \xrightarrow{\subseteq} & S^* = R^*[[t]] & \xrightarrow{\subseteq} & S^*[1/y] \\
 & & {\scriptstyle\lambda_0}\downarrow & & {\scriptstyle\lambda_a}\downarrow & & \\
R & \xrightarrow{\subseteq} & R_1 = R[\tau_a]_{(\mathbf{m},\tau_a)} & \longrightarrow & R^* & \xrightarrow{\subseteq} & R^*[1/y].
\end{array}$$

Diagram 23.12.0

The map λ_0 denotes the restriction of λ_a to S.

Assume that τ_a is primarily limit-intersecting in y over R. Then τ_a is algebraically independent over R and λ_0 is an isomorphism. If $t - a \in P^*$ for some $P^* \in \mathcal{U}$, we prove that $\varphi : R_1 \to R^*[1/y]$ is not flat. Let $Q^* := \lambda_a(P^*)$. We have $\operatorname{ht} Q^* = \operatorname{ht} P^* - 1$, and $y \notin P^*$ implies $y \notin Q^*$. Let $P := P^* \cap S$ and $Q := Q^* \cap R_1$. Commutativity of Diagram 23.12.0 and λ_0 an isomorphism imply that $\operatorname{ht} P = \operatorname{ht} Q$. Since $P^* \in \mathcal{U}$, we have $\operatorname{ht} P = \operatorname{ht} P^*$. It follows that $\operatorname{ht} Q > \operatorname{ht} Q^*$. This implies that $\varphi : R_1 \to R^*[1/y]$ is not flat.

For the converse, assume that $t - a \notin \bigcup\{P^* \mid P^* \in \mathcal{U}\}$. Since $a \in y^2S^*$ and S^* is complete in the (y,t)-adic topology, we have $S^* = R^*[[t]] = R^*[[t-a]]$. Thus

$$\mathfrak{p} \;:=\; \ker(\lambda_a) \;=\; (t - \tau_a)S^* \;=\; (t-a)S^*$$

is a height-one prime ideal of S^*. Since $y \in R$ and $\mathfrak{p} \cap R = (0)$, we have $y \notin \mathfrak{p}$.

Since $t - a$ is outside every element of $\mathcal{U}$, we have $\mathfrak{p} \notin \mathcal{U}$. Since $\mathfrak{p}$ does not fit the condition of $\mathcal{U}$, we have $\operatorname{ht}(\mathfrak{p} \cap S) \neq \operatorname{ht} \mathfrak{p} = 1$, and so, by the faithful flatness of

$S \hookrightarrow S^*$, $\mathfrak{p} \cap S = (0)$. Therefore the map $\lambda_0 : S \to R_1$ has trivial kernel, and so λ_0 is an isomorphism. Thus τ_a is algebraically independent over R.

Since R is excellent and R_1 is a localized polynomial ring over R, the hypotheses of Corollary 7.5 are satisfied. It follows that the element τ_a is primarily limit-intersecting in y over R provided that $\operatorname{ht}(Q_1^* \cap R_1) \leq \operatorname{ht} Q_1^*$ for every prime ideal $Q_1^* \in \operatorname{Spec}(R^*[1/y])$, or, equivalently, if for every $Q^* \in \operatorname{Spec} R^*$ with $y \notin Q^*$, we have $\operatorname{ht}(Q^* \cap R_1) \leq \operatorname{ht} Q^*$. Thus, to complete the proof of Theorem 23.12, it suffices to prove Claim 23.13. □

CLAIM 23.13. *For every prime ideal $Q^* \in \operatorname{Spec} R^*$ with $y \notin Q^*$, we have*

$$\operatorname{ht}(Q^* \cap R_1) \leq \operatorname{ht} Q^*.$$

PROOF. (of Claim 23.13) Since $\dim R^* = d$ and $y \notin Q^*$, we have $\operatorname{ht} Q^* = r \leq d - 1$. Since the map $R \hookrightarrow R^*$ is flat, we have $\operatorname{ht}(Q^* \cap R) \leq \operatorname{ht} Q^* = r$. Suppose that $Q := Q^* \cap R_1$ has height at least $r + 1$ in $\operatorname{Spec} R_1$. Since R_1 is a localized polynomial ring in one variable over R and $\operatorname{ht}(Q \cap R) \leq r$, we have $\operatorname{ht}(Q) = r + 1$. Let $P := \lambda_0^{-1}(Q) \in \operatorname{Spec} S$. Then $\operatorname{ht} P = r + 1$ and $y \notin P$.

Let $P^* := \lambda_a^{-1}(Q^*)$. Since the prime ideals of S^* that contain $t - a$ and have height $r + 1$ are in one-to-one correspondence with the prime ideals of R^* of height r, we have $\operatorname{ht} P^* = r + 1$. By the commutativity of the diagram, we also have $y \notin P^*$ and $P \subseteq P^* \cap S$, and so

$$r + 1 = \operatorname{ht} P \leq \operatorname{ht}(P^* \cap S) \leq \operatorname{ht} P^* = r + 1,$$

where the last inequality holds because the map $S \hookrightarrow S^*$ is flat. It follows that $P = P^* \cap S$, and so $P^* \in \mathcal{U}$. This contradicts the fact that $t - a \notin P_1^*$ for each $P_1^* \in \mathcal{U}$. Thus we have $\operatorname{ht}(Q^* \cap R_1) \leq r = \operatorname{ht} Q^*$, as asserted in Claim 23.13. This completes the proof of Theorem 23.12. □

Theorem 23.12 yields a necessary and sufficient condition for an element of R^* that is algebraically independent over R to be primarily limit-intersecting in y over R.

REMARKS 23.14. Assume notation as in Setting 23.11.

(1) For each $a \in y^2 S^*$ as in Setting 23.11, we have $(t - a)S^* = (t - \tau_a)S^*$. Hence $t - a \notin \bigcup\{P^* \mid P^* \in \mathcal{U}\} \iff t - \tau_a \notin \bigcup\{P^* \mid P^* \in \mathcal{U}\}$.
(2) If $a \in R^*$, then the commutativity of Diagram 23.12.0 implies that $\tau_a = a$.
(3) For $\tau \in R^*$, we have $\tau = a_0 + a_1 y + \tau'$, where a_0 and a_1 are in R and $\tau' \in y^2 R^*$.
   (a) The rings $R[\tau]$ and $R[\tau']$ are equal. Hence τ is primarily limit-intersecting in y over R if and only if τ' is primarily limit-intersecting in y over R.
   (b) Assume $\tau \in R^*$ is algebraically independent over R. Then τ is primarily limit-intersecting in y over R if and only if $t - \tau' \notin \bigcup\{P^* \mid P^* \in \mathcal{U}\}$.

   Item 3b follows from Theorem 23.12 by setting $a = \tau'$ and applying item 3a and item 2.

We use Theorem 23.12 and Lemma 23.10 to prove Theorem 23.15.

THEOREM 23.15. *Let $(R, \mathbf{m})$ be a countable excellent normal local domain with dimension $d \geq 2$, and let y be a nonzero element in $\mathbf{m}$. Let R^* denote the (y)-adic completion of R. Then there exists an element $\tau \in yR^*$ that is primarily limit-intersecting in y over R.*

PROOF. As in Setting 23.11, let

$$\mathcal{U} \;:=\; \{P^* \in \operatorname{Spec} S^* \mid \operatorname{ht}(P^* \cap S) = \operatorname{ht} P^*, \;\text{ and } y \notin P^* \;\}.$$

Since the ring S is countable and Noetherian, the set $\mathcal{U}$ is countable. Lemma 20.18 implies that there exists an element $a \in y^2S^*$ such that $t - a \notin \bigcup\{P^* \mid P^* \in \mathcal{U}\}$. By Theorem 23.12, the element τ_a is primarily limit-intersecting in y over R. □

To establish the existence of more than one primarily limit-intersecting element we use the following setting.

SETTING 23.16. Let $(R, \mathbf{m})$ be a d-dimensional excellent normal local domain, let y be a nonzero element of $\mathbf{m}$ and let R^* denote the (y)-adic completion of R. Let $t_1, \dots, t_{n+1}$ be indeterminates over R, and let S_n and S_{n+1} denote the localized polynomial rings

$$S_n \;:=\; R[t_1, \dots, t_n]_{(\mathbf{m}, t_1, \dots, t_n)} \quad \text{and} \quad S_{n+1} \;:=\; R[t_1, \dots, t_{n+1}]_{(\mathbf{m}, t_1, \dots, t_{n+1})}.$$

Let S_n^* denote the I_n-adic completion of S_n, where $I_n := (y, t_1, \dots, t_n)S_n$. Then $S_n^* = R^*[[t_1, \dots, t_n]]$ is a $(d+n)$-dimensional normal Noetherian local domain with maximal ideal $\mathbf{n}^* = (\mathbf{m}, t_1, \dots, t_n)S_n^*$. Assume that $\tau_1, \dots, \tau_n \in yR^*$ are primarily limit-intersecting in y over R, and define $\lambda : S_n^* \to R^*$ to be the R^*-algebra homomorphism such that $\lambda(t_i) = \tau_i$, for $1 \leq i \leq n$.

Since $S_n^* = R^*[[t_1 - \tau_1, \dots, t_n - \tau_n]]$, we have $\mathbf{p}_n := \ker \lambda = (t_1 - \tau_1, \dots, t_n - \tau_n)S_n^*$. Consider the commutative diagram:

$$\begin{array}{ccccccc}
 & & S_n = R[t_1, \dots, t_n]_{(\mathbf{m}, t_1, \dots, t_n)} & \xrightarrow{\subseteq} & S_n^* = R^*[[t_1, \dots, t_n]] & \xrightarrow{\subseteq} & S_n^*[1/y] \\
 & & {\scriptstyle \lambda_0,\, \cong}\big\downarrow & & {\scriptstyle \lambda}\big\downarrow & & \\
R & \xrightarrow{\subseteq} & R_n = R[\tau_1, \dots, \tau_n]_{(\mathbf{m}, \tau_1, \dots, \tau_n)} & \xrightarrow{\varphi_0} & R^* & \xrightarrow{\alpha} & R^*[1/y].
\end{array}$$

Let S_{n+1}^* denote the I_{n+1}-adic completion of S_{n+1}, where $I_{n+1} := (y, t_1, \dots, t_{n+1})S_{n+1}$. For each element $a \in y^2S_{n+1}^*$, we have

$$S_{n+1}^* = S_n^*[[t_{n+1}]] = S_n^*[[t_{n+1} - a]]. \tag{23.16.1}$$

Let $\lambda_a : S^* \to R^*$ denote the composition

$$S_{n+1}^* \;=\; S_n^*[[t_{n+1}]] \;\longrightarrow\; \tfrac{S_n^*[[t_{n+1}]]}{(t_{n+1}-a)} \;=\; S_n^* \xrightarrow{\lambda} R^*,$$

and let $\tau_a := \lambda_a(t_{n+1}) = \lambda_a(a)$. We have $\ker \lambda_a = (\mathbf{p}_n, t_{n+1} - a)S_{n+1}^*$. Consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & S_n & \xrightarrow{\subseteq} & S_n^* & \xrightarrow{\subseteq} & S_{n+1}^* & \longrightarrow & S_{n+1}^*[1/y] \\
 & & {\scriptstyle \lambda_0,\, \cong}\big\downarrow & & {\scriptstyle \lambda}\big\downarrow & & {\scriptstyle \lambda_a}\big\downarrow & & \big\downarrow \\
R & \xrightarrow{\subseteq} & R_n & \xrightarrow{\varphi_0} & R^* & \xrightarrow{=} & R^* & \longrightarrow & R^*[1/y].
\end{array}$$

Diagram 23.16.2

Let

$$\mathcal{U} := \{P^* \in \operatorname{Spec} S^*_{n+1} \mid P^* \cap S_{n+1} = P,\ y \notin P \text{ and } P^* \text{ is minimal over } (P, \mathbf{p}_n)S^*_{n+1}\}.$$

Notice that $y \notin P^*$ for each $P^* \in \mathcal{U}$, since $y \in R$ implies $\lambda_a(y) = y$.

THEOREM 23.17. *With the notation of Setting 23.16, the elements $\tau_1, \ldots, \tau_n, \tau_a$ are primarily limit-intersecting in y over R if and only if $t_{n+1} - a \notin \bigcup\{P^* \mid P^* \in \mathcal{U}\}$.*

PROOF. Assume that $\tau_1, \ldots, \tau_n, \tau_a$ are primarily limit-intersecting in y over R. Then $\tau_1, \ldots, \tau_n, \tau_a$ are algebraically independent over R. Consider the following commutative diagram:

$$\begin{array}{ccccc} & & S_{n+1} = R[t_1, \ldots, t_{n+1}]_{(\mathbf{m}, t_1, \ldots, t_{n+1})} & \xrightarrow{\subseteq} & S^*_{n+1} = R^*[[t_1, \ldots, t_{n+1}]] \\ & & \lambda_1 \big\downarrow & & \lambda_a \big\downarrow \\ R & \xrightarrow{\subseteq} & R_{n+1} = R[\tau_1, \ldots, \tau_a]_{(\mathbf{m}, \tau_1, \ldots, \tau_a)} & \longrightarrow & R^*. \end{array}$$

Diagram 23.17.0

The map λ_1 is the restriction of λ_a to S_{n+1}, and is an isomorphism since $\tau_1, \ldots, \tau_n, \tau_a$ are algebraically independent over R.

If $t_{n+1} - a \in P^*$ for some $P^* \in \mathcal{U}$, we prove that $\varphi : R_{n+1} \to R^*[1/y]$ is not flat, a contradiction to our assumption that $\tau_1, \ldots, \tau_n, \tau_a$ are primarily limit-intersecting. Since $P^* \in \mathcal{U}$, we have $\mathbf{p}_n \subset P^*$. Then $t_{n+1} - a \in P^*$ implies $\ker \lambda_a \subset P^*$. Let $\lambda_a(P^*) := Q^*$. Then $\lambda_a^{-1}(Q^*) = P^*$ and $\operatorname{ht} P^* = n + 1 + \operatorname{ht} Q^*$. Since $P^* \in \mathcal{U}$, we have $y \notin P^*$. The commutativity of Diagram 23.17.0 implies that $y \notin Q^*$. Let $P := P^* \cap S_{n+1}$ and let $Q := Q^* \cap R_{n+1}$. Commutativity of Diagram 23.17.0 and λ_0 an isomorphism imply that $\operatorname{ht} P = \operatorname{ht} Q$. Since P^* is a minimal prime of $(P, \mathbf{p}_n)S^*_{n+1}$ and $\mathbf{p}_n$ is n-generated and S^*_{n+1} is Noetherian and catenary, we have $\operatorname{ht} P^* \leq \operatorname{ht} P + n$. Hence $\operatorname{ht} P \geq \operatorname{ht} P^* - n$. Thus

$$\operatorname{ht} Q \ = \ \operatorname{ht} P \ \geq \ \operatorname{ht} P^* - n \ = \ \operatorname{ht} Q^* + n + 1 - n \ = \ \operatorname{ht} Q^* + 1.$$

The fact that $\operatorname{ht} Q > \operatorname{ht} Q^*$ implies that the map $R_{n+1} \to R^*[1/y]$ is not flat.

For the converse, we have

Assumption 23.17.1: $\quad t_{n+1} - a \ \notin \ \bigcup \{\ P^* \mid P^* \in \mathcal{U}\ \}$.

Since $\lambda_a : S^*_{n+1} \to R^*$ is an extension of $\lambda : S^*_n \to R^*$ as in Diagram 23.16.2, we have $\ker \lambda_a \cap S_n = (0)$. Let $\mathbf{p} := (t_{n+1} - a)S^*_{n+1} = (t_{n+1} - \tau_a)S^*_{n+1}$. As in Equation 23.16.1, we have

$$S^*_{n+1} \ = \ R^*[[t_1, \ldots, t_{n+1}]] \ = \ R^*[[t_1 - \tau_1, \ldots, t_n - \tau_n, t_{n+1} - a]].$$

Thus $P^* := (\mathbf{p}_n, \mathbf{p})S^*_{n+1}$ is a prime ideal of height $n+1$ and $P^* \cap R^* = (0)$. It follows that $y \notin P^*$. We show that $P^* \cap S_{n+1} = (0)$. Assume that $P = P^* \cap S_{n+1} \neq (0)$. Since $\operatorname{ht} P* = n+1$, P^* is minimal over $(P, \mathbf{p}_n)S^*_{n+1}$, and so $P^* \in \mathcal{U}$, a contradiction to Assumption 23.17.1. Therefore $P^* \cap S_{n+1} = (0)$. It follows that $\mathbf{p} \cap S_{n+1} = (0)$ since $\mathbf{p} \subset P^*$. Thus $\ker \lambda_1 = (0)$, and so λ_1 in Diagram 23.17.0 is an isomorphism. Therefore τ_a is algebraically independent over R_n.

Since R is excellent and R_{n+1} is a localized polynomial ring in $n+1$ variables over R, the hypotheses of Corollary 7.5 are satisfied. It follows that the elements $\tau_1, \ldots, \tau_n, \tau_a$ are primarily limit-intersecting in y over R if for every $Q^* \in \operatorname{Spec} R^*$

with $y \notin Q^*$, we have $\operatorname{ht}(Q^* \cap R_{n+1}) \leq \operatorname{ht} Q^*$. Thus, to complete the proof of Theorem 23.17, it suffices to prove Claim 23.18. □

CLAIM 23.18. *Let $Q^* \in \operatorname{Spec} R^*$ with $y \notin Q^*$ and $\operatorname{ht} Q^* = r$. Then*

$$\operatorname{ht}(Q^* \cap R_{n+1}) \leq \ r.$$

PROOF. (of Claim 23.18) Let $Q_1 := Q^* \cap R_{n+1}$ and let $Q_0 := Q^* \cap R_n$. Suppose $\operatorname{ht} Q_1 > r$. Notice that $r < d$, since $d = \dim R^*$ and $y \notin Q^*$.

Since $\tau_1, \ldots, \tau_n$ are primarily limit-intersecting in y over R, the extension

$$R_n := R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)} \hookrightarrow R^*[1/y]$$

from Diagram 23.16.2 is flat. Thus $\operatorname{ht} Q_0 \leq r$ and $\operatorname{ht} Q_0 \leq \operatorname{ht} L^*$ for every prime ideal L^* of R^* with $Q_0 R^* \subseteq L^* \subseteq Q^*$. Since R_{n+1} is a localized polynomial ring in the indeterminate τ_a over R_n, we have that $\operatorname{ht} Q_1 \leq \operatorname{ht} Q_0 + 1 = r + 1$. Thus $\operatorname{ht} Q_1 = r + 1$ and $\operatorname{ht} Q_0 = r$. It follows that Q^* is a minimal prime of $Q_0 R^*$.

Let $h(\tau_a)$ be a polynomial in

$$(Q^* \cap R_n[\tau_a]) \ \setminus \ (Q^* \cap R_n) R_{n+1}.$$

It follows that $Q^* \cap R_{n+1} := Q_1$ is a minimal prime of the ideal $(Q^* \cap R_n, h(\tau_a)) R_{n+1}$.

With notation from Diagram 23.16.2, define

$$P_0 \ := \ \lambda_0^{-1}(Q_0) \ \text{ and } \ P_0^* \ := \lambda^{-1}(Q^*).$$

Since λ_0 is an isomorphism, P_0 is a prime ideal of S_n with $\operatorname{ht} P_0 = r$. Moreover, we have the following:

(1) $P_0^* \cap S_n = P_0$ (by commutativity in Diagram 23.16.2),
(2) $y \notin P_0^*$ (by item 1),
(3) P_0^* is a minimal prime of $(P_0, \mathbf{p}_n) S_n^*$ (since $S_n^*/\mathbf{p}_n = R^*$ in Diagram 23.16.2, and Q^* is a minimal prime of $Q_0 R^*$),
(4) $\operatorname{ht} P_0^* = n + r$ (by the correspondence between prime ideals of S_n^* containing $\mathbf{p}_n$ and prime ideals of R^*).

Consider the commutative diagram below with the left and right ends identified:

$$\begin{array}{ccccccccc}
S_{n+1}^* & \longleftarrow & S_n^* & \longleftarrow & S_n & \longrightarrow & S_{n+1} & \xrightarrow{\ \theta\ } & S_{n+1}^* \\
{\scriptstyle \lambda_a}\downarrow & & {\scriptstyle \lambda}\downarrow & & {\scriptstyle \lambda_0, \cong}\downarrow & & {\scriptstyle \lambda_1, \cong}\downarrow & & {\scriptstyle \lambda_a}\downarrow \\
R^* & \longleftarrow & R^* & \longleftarrow & R_n & \longrightarrow & R_{n+1} & \longrightarrow & R^*,
\end{array}$$

Diagram 23.18.0

where λ, λ_0 and λ_1 are as in Diagrams 23.16.2 and 23.17.0, and so λ_a restricted to S_n^* is λ. Let $h(t_{n+1}) = \lambda_1^{-1}(h(\tau_a))$ and set

$$P_1 \ := \ \lambda_1^{-1}(Q_1) \ \in \ \operatorname{Spec}(S_{n+1}), \ \text{ and } \ P^* \ := \ \lambda_a^{-1}(Q^*) \in \operatorname{Spec}(S_{n+1}^*).$$

Then P_1 is a minimal prime of $(P_0, h(t_{n+1})) S_{n+1}$, since Q_1 is a minimal prime of $(Q_0, h(\tau_a)) R_{n+1}$. Since $Q_1 \subseteq Q^*$, we have $h(t_{n+1}) \in P^*$ and $P_1 S_{n+1}^* \subseteq P^*$ because $\lambda_a(h(t_{n+1})) = \lambda_1(h(t_{n+1})) = h(\tau_a) \in Q_1$ and $\lambda_a(P_1) = \lambda_1(P_1) = Q_1$. By the correspondence between prime ideals of S_{n+1}^* containing $\ker(\lambda_a) = \mathbf{p}_{n+1}$ and prime ideals of R^*, we see

$$\operatorname{ht} P^* = \operatorname{ht} Q^* + n + 1 = r + n + 1.$$

Since $\lambda_a(P_0^*) \subseteq Q^*$, we have $P_0^* \subseteq P^*$, but $h(t_{n+1}) \notin P_0$ implies $h(t_{n+1}) \notin P_0^* S_{n+1}^*$. Therefore

$$(P_0, \mathbf{p}_n)S_{n+1}^* \subseteq P_0^* S_{n+1}^* \subsetneq (P_0^*, h(t_{n+1}))S_{n+1}^* \subseteq P^*.$$

By items 3 and 4 above, $\operatorname{ht} P_0^* = n + r$ and P_0^* is a minimal prime of $(P_0, \mathbf{p}_n)S_n^*$. Since $\operatorname{ht} P^* = n+r+1$, it follows that P^* is a minimal prime of $(P_0, h(t_{n+1}), \mathbf{p}_n)S_{n+1}^*$. Since $(P_0, h(t_{n+1}), \mathbf{p}_n)S_{n+1}^* \subseteq (P_1, \mathbf{p}_n)S_{n+1}^* \subseteq P^*$, we have P^* is a minimal prime of $(P_1, \mathbf{p}_n)S_{n+1}^*$. But then, by Assumption 23.17.1, $t_{n+1} - a \notin P^*$, a contradiction. This contradiction implies that $\operatorname{ht} Q_1 = r$. This completes the proof of Claim 23.18 and thus also the proof of Theorem 23.17. □

We use Theorem 23.15, Theorem 23.17 and Lemma 23.10 to prove in Theorem 23.19 the existence over a countable excellent normal local domain of dimension at least two of an infinite sequence of primarily limit-intersecting elements.

THEOREM 23.19. *Let R be a countable excellent normal local domain with dimension $d \geq 2$, let y be a nonzero element in the maximal ideal $\mathbf{m}$ of R, and let R^* be the (y)-adic completion of R. Let n be a positive integer. Then*

1. *If the elements $\tau_1, \ldots, \tau_n \in yR^*$ are primarily limit-intersecting in y over R, then there exists an element $\tau_a \in yR^*$ such that $\tau_1, \ldots, \tau_n, \tau_a$ are primarily limit-intersecting in y over R.*
2. *There exists an infinite sequence $\tau_1, \ldots, \tau_n, \ldots \in yR^*$ of elements that are primarily limit-intersecting in y over R.*

PROOF. Since item 1 implies item 2, it suffices to prove item 1. Theorem 23.15 implies the existence of an element $\tau_1 \in yR^*$ that is primarily limit-intersecting in y over R. As in Setting 23.16, let

$$\mathcal{U} := \{P^* \in \operatorname{Spec} S_{n+1}^* \mid P^* \cap S_{n+1} = P \in \mathcal{S} \text{ and } P^* \text{ is minimal over } (P, \mathbf{p}_n)S_{n+1}^*\}.$$

Since the ring S_{n+1} is countable and Noetherian, the set $\mathcal{U}$ is countable. Lemma 20.18 implies that there exists an element $a \in y^2 S_{n+1}^*$ such that

$$t_{n+1} - a \notin \bigcup \{P^* \mid P^* \in \mathcal{U}\}.$$

By Theorem 23.17, the elements $\tau_1, \ldots, \tau_n, \tau_a$ are primarily limit-intersecting in y over R. □

Using Theorem 23.15, we establish in Theorem 23.20, for every countable excellent normal local domain R of dimension $d \geq 2$, the existence of a primarily limit-intersecting element $\eta \in yR^*$ such that the constructed Noetherian domain

$$B = A = R^* \cap \mathcal{Q}(R[\eta])$$

is not a Nagata domain and hence is not excellent.

THEOREM 23.20. *Let R be a countable excellent normal local domain of dimension $d \geq 2$, let y be a nonzero element in the maximal ideal $\mathbf{m}$ of R, and let R^* be the (y)-adic completion of R. There exists an element $\eta \in yR^*$ such that*

1. *η is primarily limit-intersecting in y over R.*
2. *The associated intersection domain $A := R^* \cap \mathcal{Q}(R[\eta])$ is equal to its approximation domain B.*
3. *The ring A has a height-one prime ideal $\mathbf{p}$ such that $R^*/\mathbf{p}R^*$ is not reduced.*

Thus the integral domain $A = B$ associated to η is a normal Noetherian local domain that is not a Nagata domain and hence is not excellent.

PROOF. Since $\dim R \geq 2$, there exists $x \in \mathbf{m}$ such that $\mathrm{ht}(x, y)R = 2$. By Theorem 23.15, there exists $\tau \in yR^*$ such that τ is primarily limit-intersecting in y over R. Hence the extension $R[\tau] \longrightarrow R^*[1/y]$ is flat. Let $n \in \mathbb{N}$ with $n \geq 2$, and let $\eta := (x + \tau)^n$. Since τ is algebraically independent over R, the element η is also algebraically independent over R. Moreover, the polynomial ring $R[\tau]$ is a free $R[\eta]$-module with $1, \tau, \ldots, \tau^{n-1}$ as a free module basis. Hence the map $R[\eta] \longrightarrow R^*[1/y]$ is flat. It follows that η is primarily limit-intersecting in y over R. Therefore the intersection domain $A := R^* \cap \mathcal{Q}(R[\eta])$ is equal to its associated approximation domain B and is a normal Noetherian domain with (y)-adic completion R^*. Since η is a prime element of the polynomial ring $R[\eta]$ and $B[1/y]$ is a localization of $R[\eta]$, it follows that $\mathbf{p} := \eta B$ is a height-one prime ideal of B. Since $\tau \in R^*$, and $\eta = (x + \tau)^n$, the ring $R^*/\mathbf{p}R^*$ contains nonzero nilpotent elements. Since a Nagata local domain is analytically unramified, it follows that the normal Noetherian domain B is not a Nagata ring, [**103**, page 264] or [**117**, (32.2)]. □

Let d be an integer with $d \geq 2$. In Examples 10.9 we give extensions that satisfy LF_{d-1} but do not satisfy LF_d; see Definition 22.1. These extensions are weakly flat but are not flat. In our setting these examples have the intersection domain A equal to its approximation domain B but A is not Noetherian In Theorem 23.21, we present a more general construction of examples with these properties.

THEOREM 23.21. *Let $(R, \mathbf{m})$ be a countable excellent normal local domain. Assume that $\dim R = d + 1 \geq 3$, that $(x_1, \ldots, x_d, y)R$ is an $\mathbf{m}$-primary ideal, and that R^* is the (y)-adic completion of R. Then there exists $f \in yR^*$ such that f is algebraically independent over R and the map $\varphi : R[f] \longrightarrow R^*[1/y]$ is weakly flat but not flat. Indeed, φ satisfies LF_{d-1}, but fails to satisfy LF_d. Thus the intersection domain $A := \mathcal{Q}(R[f]) \cap R^*$ is equal to its approximation domain B, but A is not Noetherian.*

PROOF. By Theorem 23.19, there exist elements $\tau_1, \ldots, \tau_d \in yR^*$ that are primarily limit-intersecting in y over R. Let

$$f \;:=\; x_1\tau_1 \;+ \cdots +\; x_d\tau_d.$$

Using that $\tau_1, \ldots, \tau_d$ are algebraically independent over R, we regard f as a polynomial in the polynomial ring $T := R[\tau_1, \ldots, \tau_d]$. Let $S := R[f]$. For $Q \in \mathrm{Spec}\, R^*[1/y]$ and $P := Q \cap T$, consider the composition φ_Q

$$S \;\longrightarrow\; T_P \;\longrightarrow\; R^*[1/y]_Q.$$

Since $\tau_1, \ldots, \tau_d$ are primarily limit-intersecting in y over R, the map $T \hookrightarrow R^*[1/y]$ is flat. Thus the map φ_Q is flat if and only if the map $S \longrightarrow T_P$ is flat. Let $\mathbf{p} := P \cap R$.

Assume that P is a minimal prime of $(x_1, \ldots, x_d)T$. Then $\mathbf{p}$ is a minimal prime of $(x_1, \ldots, x_d)R$. Since T is a polynomial ring over R, we have $P = \mathbf{p}T$ and $\mathrm{ht}(\mathbf{p}) = d = \mathrm{ht}\, P$. Notice that $(\mathbf{p}, f)S = P \cap S$ and $\mathrm{ht}(\mathbf{p}, f)S = d + 1$. Since a flat extension satisfies the Going-down property, the map $S \longrightarrow T_P$ is not flat. Hence φ does not satisfy LF_d.

Assume that $\operatorname{ht} P \leq d-1$. Then $(x_1, \ldots, x_d)T$ is not contained in P. Hence $(x_1, \ldots, x_d)R$ is not contained in $\mathbf{p}$. Consider the sequence

$$S \;=\; R[f] \;\hookrightarrow\; R_{\mathbf{p}}[f] \;\xrightarrow{\psi}\; R_{\mathbf{p}}[\tau_1, \ldots, \tau_d] \;\hookrightarrow\; T_P,$$

where the first and last injections are localizations. Since the nonconstant coefficients of f generate the unit ideal of $R_{\mathbf{p}}$, the map ψ is flat; see Theorem 7.23. Thus φ satisfies LF_{d-1}.

We conclude that the intersection domain $A = R^* \cap \mathcal{Q}(R[f])$ is equal to its approximation domain B and is not Noetherian. □

23.3. Cohen-Macaulay formal fibers and Ogoma's example

In Corollary 23.23 we observe that if R is excellent, then every Noetherian example A obtained via Inclusion Construction 5.3 has Cohen-Macaulay formal fibers. We observe in Remark 23.25 that this implies the non-Noetherian property of a certain integral domain B that has Ogoma's example as a homomorphic image.

The following is an analogue of [**103**, Theorem 32.1(ii)]. The distinction is that we are considering regular fibers rather than geometrically regular fibers.

PROPOSITION 23.22. *Suppose R, S, and T are Noetherian commutative rings and suppose we have maps $R \to S$ and $S \to T$ and the composite map $R \to T$. Assume*

(i) *$R \to T$ is flat with regular fibers,*
(ii) *$S \to T$ is faithfully flat.*

Then $R \to S$ is flat with regular fibers.

As an immediate consequence of Theorem 7.4 and Proposition 23.22, we have the following implication concerning Cohen-Macaulay formal fibers.

COROLLARY 23.23. *Every Noetherian local ring B containing an excellent local subring R and having the same completion as R has Cohen-Macaulay formal fibers. Thus the ring A of Setting 23.1 has Cohen-Macaulay formal fibers whenever A is Noetherian.*

REMARK 23.24. (Cohen-Macaulay formal fibers) Corollary 23.23 implies that every Noetherian local ring B that has as its completion $\widehat{B}$ the formal power series ring $k[[x_1, \ldots, x_d]]$ and that contains the polynomial ring $k[x_1, \ldots, x_d]$ has Cohen-Macaulay formal fibers. In connection with Cohen-Macaulay formal fibers, Luchezar Avramov pointed out to us that every homomorphic image of a regular local ring has formal fibers that are complete intersections and therefore Cohen-Macaulay [**51**, (3.6.4), page 118]. Also every homomorphic image of a Cohen-Macaulay local ring has formal fibers that are Cohen-Macaulay [**103**, page 181]. It is interesting that while regular local rings need not have regular formal fibers, they must have Cohen-Macaulay formal fibers.

REMARK 23.25. (Ogoma's example) Corollary 23.23 sheds light on Ogoma's famous example [**123**] of a Nagata local domain of dimension three whose generic formal fiber is not equidimensional.

Ogoma's construction begins with a countable field k of infinite but countable transcendence degree over the field $\mathbb{Q}$ of rational numbers. Let x, y, z, w be variables

over k, and let $R = k[x,y,z,w]_{(x,y,z,w)}$ be the localized polynomial ring. By a clever enumeration of the prime elements in R, Ogoma constructs three power series $g,h,\ell \in \widehat{R} = k[[x,y,z,w]]$ that satisfy the following conditions:

(a) g,h,ℓ are algebraically independent over $k(x,y,z,w) = \mathcal{Q}(R)$.
(b) g,h,ℓ are part of a regular system of parameters for $\widehat{R} = k[[x,y,z,w]]$.
(c) If $\widehat{P} = (g,h,\ell)\widehat{R}$, then $\widehat{P} \cap R = (0)$, i.e., $\widehat{P}$ is in the generic formal fiber of R.
(d) If $I = (gh, g\ell)\widehat{R}$ and $C = \mathcal{Q}(R) \cap (\widehat{R}/I)$, then C is a Nagata local domain[1] with completion $\widehat{C} = \widehat{R}/I$.
(e) It is then obvious that the completion $\widehat{C} = \widehat{R}/I$ of C has a minimal prime $g\widehat{R}/I$ of dimension 3 and a minimal prime $(h,\ell)\widehat{R}/I$ of dimension 2. Thus C fails to be formally equidimensional. Therefore C is not universally catenary [**103**, Theorem 31.7] and provides a counterexample to the catenary chain condition.

Since C is not universally catenary, C is not a homomorphic image of a regular local ring. There exists a local integral domain B that dominates R, has completion $\widehat{R} = k[[x,y,z,w]]$, and contains an ideal J such that $C = B/J$. If B were Noetherian, then B would be a regular local ring and $C = B/J$ would be universally catenary. Thus B is necessarily non-Noetherian.

Theorem 7.4 provides a different way to deduce that the ring B is non-Noetherian. To see this, we consider more details about the construction of B. The ring B is defined as a nested union of rings:

Let $\lambda_1 = gh$ and $\lambda_2 = g\ell$ and define:

$$B = \bigcup_{n=1}^{\infty} R[\lambda_{1n}, \lambda_{2n}]_{(x,y,z,w,\lambda_{1n},\lambda_{2n})} \subseteq k[[x,y,z,w]]$$

where the λ_{in} are endpieces of the λ_i. The construction is done in such a way that the λ's are in every completion of R with respect to a nonzero principal ideal. By the construction of the power series g,h,ℓ, for every nonzero element $f \in R$ the ring B/fB is essentially of finite type over the field k. This implies that the maximal ideal of B is generated by x,y,z,w and that the completion with respect to the maximal ideal of B is the formal power series ring $\widehat{R} = k[[x,y,z,w]]$. Let $K = k(x,y,z,w)$, then $K \otimes_R B$ is a localization of the polynomial ring in two variables $K[\lambda_1,\lambda_2]$. Recall that $I = (\lambda_1,\lambda_2)\widehat{R}$ and $\widehat{P} = (g,h,\ell)\widehat{R}$. Let $J = I \cap B$. Since $\widehat{P} \cap R = (0)$ we see that $J = \widehat{P} \cap B$ is a prime ideal such that $J(K \otimes_R B)$ is a localization of the prime ideal $(\lambda_1,\lambda_2)K[\lambda_1,\lambda_2]$. Thus

$$J(K \otimes_R \widehat{R}) \;=\; (\lambda_1,\lambda_2)(K \otimes_R \widehat{R})$$

and $\widehat{P}$ is in the formal fiber of B/J. Since $(\widehat{R}/I)_{\widehat{P}}$ is not Cohen-Macaulay, Corollary 23.23 implies that B is not Noetherian.

There is another intermediate ring between R and its completion $k[[x,y,z,w]]$ that carries information about C. This is the intersection ring:

$$A = k(x,y,z,w,\lambda_1,\lambda_2) \cap k[[x,y,z,w]].$$

[1]Ogoma [**123**, page 158] actually constructs C as a directed union of birational extensions of R. He proves that C is Noetherian and that $\widehat{C} = \widehat{R}/I$. It follows that $C = \mathcal{Q}(R) \cap (\widehat{R}/I)$. Heitmann observes in [**84**] that C is already normal.

It is shown in [**60**, Claim 4.3] that the maximal ideal of A is generated by x, y, z, w, and is shown in [**60**, Claim 4.4] that A is non-Noetherian.

23.4. Examples not having Cohen-Macaulay fibers

In this section we adapt the two forms of the basic construction technique to obtain three rings A, B and C that we describe in detail. The setting is somewhat similar to that of Ogoma's example. It is simpler in the sense that it is fairly easy to see that the ring C that corresponds to the ring C in Ogoma's example is Noetherian. Also C is a birational extension of a polynomial ring in 3 variables over a field. On the other hand this setting seems more complicated, since for A and B (which are the two obvious choices of intermediate rings) the ring B maps surjectively onto C, while A does not.

Setting and Notation 23.26. Let k be a field, and x, y, z variables over k. Let $\tau_1, \tau_2 \in xk[[x]]$ be formal power series in x which are algebraically independent over $k(x)$. Suppose that

$$\tau_i = \sum_{n=1}^{\infty} a_{in} x^n, \qquad \text{with } a_{in} \in k, \qquad \text{for } i = 1, 2.$$

The intersection ring $V := k(x, \tau_1, \tau_2) \cap k[[x]]$ is a discrete valuation domain which is a nested union of localized polynomial rings in 3 variables over k: $V = \bigcup_{n=1}^{\infty} k[x, \tau_{1n}, \tau_{2n}]_{(x, \tau_{1n}, \tau_{2n})}$, where τ_{1n}, τ_{2n} are the endpieces:

$$\tau_{in} = \sum_{j=n}^{\infty} a_{ij} x^{j-n+1}, \qquad \text{for all } n \in \mathbb{N} \quad \text{and } i = 1, 2.$$

We now define a 3-dimensional regular local ring D such that: (i) D is a localization of a nested union of polynomial rings in 5 variables, (ii) D has maximal ideal $(x, y, z)D$ and completion $\widehat{R} = k[[x, y, z]]$, and (iii) D dominates the localized polynomial ring $R := k[x, y, z]_{(x,y,z)}$:

$$D := V[y, z]_{(x,y,z)} = U_{(x,y,z)D \cap U}, \qquad \text{where } U := \bigcup_{n=1}^{\infty} k[x, y, z, \tau_{1n}, \tau_{2n}]. \tag{23.4.1.1}$$

Moreover, $D = k(x, y, z, \tau_1, \tau_2) \cap \widehat{R}$ (see Polynomial Example Theorem 9.2).

We consider the following elements of $\widehat{R}$:

$$s := y + \tau_1, \quad t := z + \tau_2, \quad \rho := s^2 = (y + \tau_1)^2 \quad \text{and} \quad \sigma := st = (y + \tau_1)(z + \tau_2).$$

The elements s and t are algebraically independent over $k(x, y, z)$ as are also the elements ρ and σ. The endpieces of ρ and σ are given as

$$\rho_n := \frac{1}{x^n}((y + \tau_1)^2 - (y + \sum_{j=1}^{n} a_{1j} x^j)^2)$$

$$\sigma_n := \frac{1}{x^n}((y + \tau_1)(z + \tau_2) - (y + \sum_{j=1}^{n} a_{1j} x^j)(z + \sum_{j=1}^{n} a_{2j} x^j).$$

The ideal $I := (\rho, \sigma)\widehat{R}$ has height 1 and is the product of two prime ideals $I = P_1 P_2$ where $P_1 := s\widehat{R}$ and $P_2 := (s, t)\widehat{R}$. Observe that P_1 and P_2 are the associated prime ideals of I, and that P_1 and P_2 are in the generic formal fiber of R.

We now define rings A and C as follows:

$$A := \mathcal{Q}(R)(\rho,\sigma) \cap k[[x,y,z]], \qquad C := \mathcal{Q}(R) \cap (k[[x,y,z]]/I). \tag{23.4.1.2}$$

In analogy with the rings D and U of (23.4.1.1), we have rings $B \subseteq D$ and $W \subseteq U$ defined as follows:

(23.4.1.3)

$$B := \bigcup_{n=1}^{\infty} R[\rho_n,\sigma_n]_{(x,y,z,\rho_n,\sigma_n)} = W_{(x,y,z)B\cap W}, \text{ where } W := \bigcup_{n=1}^{\infty} k[x,y,z,\rho_n,\sigma_n].$$

It is clear that A, B and C are local domains and $B \subseteq A$ with A birationally dominating B. Moreover $(x,y,z)B$ is the maximal ideal of B.

We show in Theorems 23.27 and 23.28 that A and B are non-Noetherian and that $B \subsetneq A$. In Theorem 23.30, we show that C is a Noetherian local domain with completion $\widehat{C} = \widehat{R}/I$ such that C has a non-Cohen-Macaulay formal fiber.

THEOREM 23.27. *With the notation of (23.26), the local integral domains $B \subseteq A$ both have completion $\widehat{R}$ with respect to the powers of their maximal ideals. Also:*

1. *We have $P_1 \cap B = P_2 \cap B$,*
2. *B is a UFD,*
3. *$\operatorname{ht}(P_1 \cap B) > \operatorname{ht}(P_1) = 1$,*
4. *B fails to have Cohen-Macaulay formal fibers, and*
5. *B is non-Noetherian.*

PROOF. It follows from Construction Properties Theorem 5.14 that $\widehat{R}$ is the completion of both A and B.

For item 1, it suffices to show $P_1 \cap W = P_2 \cap W$. It is clear that $P_1 \cap W \subseteq P_2 \cap W$. Let $v \in P_2 \cap W$. Then there is an integer $n \in \mathbb{N}$ such that $x^n v \in k[x,y,z,\rho,\sigma]$. Thus

$$x^n v = \sum b_{ij}\rho^i\sigma^j, \qquad \text{where } b_{ij} \in k[x,y,z], \qquad \text{for all } i,j \in \mathbb{N}.$$

Since $P_2 \cap k[x,y,z] = (0)$ and since $\rho,\sigma \in P_2$ we have that $b_{00} = 0$. This implies that $v \in P_1$. Thus item 1 holds.

For item 2, since $B/xB = R/xR$, the ideal $xB = \mathbf{q}$ is a principal prime ideal in B. Since B is dominated by $\widehat{R}$, we have $\cap_{n=1}^{\infty}\mathbf{q}^n = (0)$. Hence $B_{\mathbf{q}}$ is a DVR. Moreover, by construction, B_x is a localization of $(B_0)_x$, where $B_0 := R[\rho,\sigma]_{(x,y,z,\rho,\sigma)}$, and $(B_0)_x$ is a UFD. Therefore $B = B_x \cap B_{\mathbf{q}}$ is a UFD by Theorem 2.21.

For item 3, we have B_x is a localization of the ring $(B_0)_x$ and the ideal $J = (\rho,\sigma)B_0$ is a prime ideal of height 2. Let $Q = P_1 \cap B$; then $x \notin Q$ and $B_Q = (B_0)_J$. Therefore $\operatorname{ht} Q = 2$. Since $P_1 = s\widehat{R}$ has height one, this proves item 3.

Item 3 implies item 5, since $\operatorname{ht}(P_1 \cap B) > \operatorname{ht} P_1$ implies that $B \to \widehat{R}$ fails to satisfy the Going-down property, so $\widehat{R}$ is not flat over B and B is not Noetherian.

For item 4, as we saw above, $Qk[[x,y,z]]_{P_2} = (\rho,\sigma)_{P_2} = I_{P_2}$. Thus $\widehat{R}_{P_2}/I\widehat{R}_{P_2}$ is a formal fiber of B. Since $k[[x,y,z]]/I = k[[x,s,t]]/(s^2,st)$, we see that $P_2/I = (s,t)\widehat{R}/(s^2,st)\widehat{R}$ is an embedded associated prime of the ring $k[[x,y,z]]/I$. Hence $(k[[x,y,z]]/I)_{P_2}$ is not Cohen-Macaulay and the embedding $B \longrightarrow k[[x,y,z]]$ fails to have Cohen-Macaulay formal fibers. This also implies that B is non-Noetherian by Corollary 23.6. □

THEOREM 23.28. *With the notation of Setting 23.26 we have:*

1. *A is a local Krull domain with maximal ideal $(x,y,z)A$ and completion $\widehat{R}$,*

(2) $P_1 \cap A \subsetneq P_2 \cap A$, *so* $B \subsetneq A$,
(3) A *is non-Noetherian.*

PROOF. For item 1, it follows from Construction Properties Theorem 5.14 that $(x, y, z)A$ is the maximal ideal of A. By definition, A is the intersection of a field with the Krull domain $\widehat{R}$; thus A is a Krull domain.

For item 2, let $Q_i := P_i \cap A$, for $i = 1, 2$. Observe that

$$\sigma^2/\rho \;=\; (z + \tau_2)^2 \;=\; t^2 \;\in\; (Q_2 \;\setminus\; B) \;\setminus\; Q_1.$$

For item 3, assume A is Noetherian. Then A is a regular local ring and the embedding $A \longrightarrow \widehat{R} = k[[x, y, z]]$ is flat. In particular, A is a UFD and the ideal $P := s\widehat{R} \cap A = P_1 \cap A$ is a prime ideal of height one in A. Thus P is principal. We have that $\rho = s^2 \in P$ and $\sigma^2 = \rho(\sigma^2/\rho)$, therefore $st = \sigma \in P$. Let v be a generator of P. Then $v = sa$ where a is a unit in $D \subseteq k[[x, y, z]]$. We write:

$$v = sa = h(\rho, \sigma)/g(\rho, \sigma), \quad \text{where} \quad h(\rho, \sigma),\ g(\rho, \sigma) \;\in\; k[x, y, z][\rho, \sigma]. \tag{23.28.1}$$

Now $a \in D = U_{(x,y,z)D \cap U}$, so $a = g_1/g_2$, where $g_1, g_2 \in k[x, y, z, \tau_{1n}, \tau_{2n}]$, for some $n \in \mathbb{N}$, and g_2 as a power series in $k[[x, y, z]]$ has nonzero constant term. There exists $m \in \mathbb{N}$ such that $x^m g_1 := f_1$ and $x^m g_2 := f_2$ are in the polynomial ring $k[x, y, z, \tau_1, \tau_2] = k[x, y, z][s, t]$. We regard $f_2(s, t)$ as a polynomial in s and t with coefficients in $k[x, y, z]$. We have $f_2 k[[x, y, z]] = x^m k[[x, y, z]] = x^m k[[x, s, t]]$. Therefore $f_2 \notin (s, t)k[[x, s, t]]$. It follows that the constant term of $f_2(s, t) \in k[x, y, z][s, t]$ is a nonzero element of $k[x, y, z]$. Since we have

$$a = \frac{x^m g_1}{x^m g_2} = \frac{f_1}{f_2}, \tag{23.28.2}$$

and a is a unit of D, the constant term of $f_1(s, t) \in k[x, y, z][s, t]$ is also nonzero. Equations 23.28.1 and 23.28.2 together yield

$$s f_1(s, t) h(s^2, st) = f_2(s, t) g(s^2, st). \tag{23.28.3}$$

The term of lowest total degree in s and t on the left hand side of Equation 23.28.3 has odd degree, while the term of lowest total degree in s and t on the right hand side has even degree, a contradiction. Therefore the assumption that A is Noetherian leads to a contradiction. We conclude that A is not Noetherian. □

REMARKS 23.29. (i) Although A is not Noetherian, the proof of Theorem 23.28 does not rule out the possibility that A is a UFD. The proof does show that if A is a UFD, then $\operatorname{ht}(P_1 \cap A) > \operatorname{ht}(P_1)$. It would be interesting to know whether the non-flat map $A \to \widehat{A} = \widehat{R}$ has the property that $\operatorname{ht}(\widehat{Q} \cap A) \leq \operatorname{ht}(\widehat{Q})$, for each $\widehat{Q} \in \operatorname{Spec} \widehat{R}$. It would also be interesting to know the dimension of A.
(ii) We observe the close connection of the integral domains $A \subseteq D$ of Setting 23.26. The extension of fields $\mathcal{Q}(A) \subseteq \mathcal{Q}(D)$ has degree two and $A = \mathcal{Q}(A) \cap D$, yet A is non-Noetherian, while D is Noetherian.

THEOREM 23.30. *With the notation of (23.26),* C *is a two-dimensional Noetherian local domain having completion* $\widehat{R}/I$ *and the generic formal fiber of* C *is not Cohen-Macaulay.*

PROOF. It follows from Construction Properties Theorem 17.11, that the completion of C is $\widehat{R}/I$. Hence if C is Noetherian, then $\dim(C) = \dim(\widehat{R}/I) = 2$. To

show that C is Noetherian, by the Noetherian Flatness Theorem 17.13, it suffices to show that the canonical map φ is flat, where:

$$\begin{aligned} R = k[x,y,z]_{(x,y,z)} \xrightarrow{\varphi} (\widehat{R}/I)[1/x] &= (k[[x,y,z]]/I)[1/x] \\ &= (k[[x,s,t]]/(s^2,st)k[[x,s,t]])[1/x]. \end{aligned}$$

Thus it suffices to show for every prime ideal $\widehat{Q}$ of $\widehat{R}$ with $x \notin \widehat{Q}$ that the map

$$\varphi_{\widehat{Q}} : R \longrightarrow \widehat{R}_{\widehat{Q}}/I\widehat{R}_{\widehat{Q}} = (\widehat{R}/I)_{\widehat{Q}}$$

is flat. We may assume $I = P_1P_2 \subseteq \widehat{Q}$.

If $\widehat{Q} = P_2 = (s,t)\widehat{R}$, then $\varphi_{\widehat{Q}}$ is flat since $P_2 \cap R = (0)$.

If $\widehat{Q} \neq P_2$, then $P_2\widehat{R}_{\widehat{Q}} = \widehat{R}_{\widehat{Q}}$, because $\operatorname{ht} P_2 = 2$. Hence $I\widehat{R}_{\widehat{Q}} = P_1\widehat{R}_{\widehat{Q}} = s\widehat{R}_{\widehat{Q}}$. Thus we need to show

$$\varphi_{\widehat{Q}} : R \longrightarrow \widehat{R}_{\widehat{Q}}/s\widehat{R}_{\widehat{Q}} = (\widehat{R}/s\widehat{R})_{\widehat{Q}}$$

is flat. To see that $\varphi_{\widehat{Q}}$ is flat, we observe that, since $R \subseteq D_{\widehat{Q}\cap D} \subseteq \widehat{R}_{\widehat{Q}}$ and $s\widehat{R} \cap R = (0)$, the map $\varphi_{\widehat{Q}}$ factors through a homomorphic image of $D = V[y,z]_{(x,y,z)}$. That is, $\varphi_{\widehat{Q}}$ is the composition of the following maps:

$$R \xrightarrow{\gamma} (D/sD)_{D\cap\widehat{Q}} \xrightarrow{\psi_{\widehat{Q}}} (\widehat{R}/s\widehat{R})_{\widehat{Q}}.$$

Since D is Noetherian, the map $\psi_{\widehat{Q}}$ is faithfully flat. Thus it remains to show that γ is flat. Since $x \notin \widehat{Q}$, the ring $(D/sD)_{D\cap\widehat{Q}}$ is a localization of $(D/sD)[1/x]$. Thus it is a localization of the polynomial ring:

$$k[x,y,z,\tau_1,\tau_2]/sk[x,y,z,\tau_1,\tau_2] = k[x,y,z,s,t]/sk[x,y,z,s,t],$$

which is clearly flat over R. Thus C is Noetherian.

Now $P_2/I = \mathbf{p}$ is an embedded associated prime of (0) of $\widehat{C}$ so $\widehat{C}_{\mathbf{p}}$ is not Cohen-Macaulay. Since $\mathbf{p} \cap C = (0)$ the generic formal fiber of C is not Cohen-Macaulay. □

PROPOSITION 23.31. *The canonical map $B \longrightarrow \widehat{R}/I$ factors through C. We have $B/Q \cong C$, where $Q = I \cap B = s\widehat{R} \cap B$. On the other hand, the canonical map $A \longrightarrow \widehat{R}/I$ fails to factor through C.*

PROOF. We have canonical maps $B \to \widehat{R}/I$ and $C \to \widehat{R}/I$. We define a map $\phi : B \twoheadrightarrow C$ such that the following diagram commutes:

$$\begin{array}{ccc} B & \longrightarrow & \widehat{R} \\ {\scriptstyle\phi}\downarrow & & \downarrow \\ C & \longrightarrow & \widehat{R}/I. \end{array} \tag{23.4.6.1}$$

We write C as a nested union as is done in Chapter 6 (17.13):

$$C = \bigcup_{n=1}^{\infty} R[\bar{\rho_n}, \bar{\sigma_n}]_{(x,y,z,\bar{\rho_n},\bar{\sigma_n})}$$

where $\bar{\rho_n}, \bar{\sigma_n}$ are the n^{th} frontpieces of ρ and σ:

$$\bar{\rho_n} = \frac{1}{x^n}(y + \sum_{j=1}^{n} a_{1j}x^j)^2 \qquad \text{and} \qquad \bar{\sigma_n} = \frac{1}{x^n}(y + \sum_{j=1}^{n} a_{1j}x^j)(z + \sum_{j=1}^{n} a_{2j}x^j).$$

Then

$$B = \bigcup_{n=1}^{\infty} R[\rho_n, \sigma_n]_{(x,y,z,\rho_n,\sigma_n)} \qquad \text{and} \qquad C = \bigcup_{n=1}^{\infty} R[\bar{\rho_n}, \bar{\sigma_n}]_{(x,y,z,\bar{\rho_n},\bar{\sigma_n})}.$$

It is clear that for each $n \in \mathbb{N}$ there is a surjection

$$R[\rho_n, \sigma_n]_{(x,y,z,\rho_n,\sigma_n)} \longrightarrow R[\bar{\rho_n}, \bar{\sigma_n}]_{(x,y,z,\bar{\rho_n},\bar{\sigma_n})}$$

which maps $\rho_n \mapsto \bar{\rho_n}$ and $\sigma_n \mapsto \bar{\sigma_n}$ and which extends to a surjective homomorphism ϕ on the directed unions such that diagram (23.4.6.1) commutes. This shows that $C \cong B/Q$ is a homomorphic image of B.

In order to see the canonical map $\zeta : A \longrightarrow \widehat{R}/I$ fails to factor through C, we note that $I \cap D = (\rho, \sigma)D$ and so ζ factors through D:

$$A \xrightarrow{\gamma} D/(\rho, \sigma)D \xrightarrow{\delta} \widehat{R}/I$$

where δ is injective. The map γ sends the element $\sigma^2/\rho = t^2$ to the residue class of $t^2 = (z + \tau_2)^2$ in $D/(\rho, \sigma)D$. This element is algebraically independent over R, which shows that the ring $A/\ker(\gamma)$ is transcendental over R. Since C is a birational extension of R, the map $A \longrightarrow \widehat{R}/I$ fails to factor through C. □

Exercise

(1) Let $(R, \mathbf{m})$ be a Noetherian local ring, let y be an element in $\mathbf{m}$ and let R^* be the (y)-adic completion of R. Let S be the localized polynomial ring $R[t]_{(\mathbf{m},t)}$ and let S^* denote the I-adic completion completion of S, where $I = (y, t)S$. Let a be an element in the ideal yR^*.
   (a) Prove that R^* is complete in the (a)-adic topology on R^*, and that S^* is complete in the $(t - a)$-adic topology on S^*.
   (b) Prove that S^* is the formal power series ring $R^*[[t]]$.
   (c) Prove that $R^*[[t]] = R^*[[t - a]]$. Thus S^* is the formal power series ring in $t - a$ over R^*, as is used in the proof of Theorem 23.12.

   Comment: Item a is a special case of Exercise 2 of [**103**, p. 63].

   Suggestion: For item c, prove that every element of S^* has a unique expression as a power series in t over R^* and also a unique expression as a power series in $t - a$ over R^*.

CHAPTER 24

Weierstrass techniques for generic fiber rings

Let k be a field, let m and n be positive integers, and let $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$ be sets of independent variables over k. We define the rings A, B and C as follows:

$$A := k[X]_{(X)},\ B := k[[X]]\,[Y]_{(X,Y)} \quad \text{and} \quad C := k[Y]_{(Y)}[[X]]. \tag{24.1.0}$$

That is, A is the usual localized polynomial ring in the variables of X. The rings B and C are "mixed polynomial-power series rings", formed from k using X, the power series variables, and Y, the polynomial variables, in two different ways: For the ring B we take polynomials in Y with coefficients in the power series ring $k[[X]]$ and for C we take power series in the X variables over the localized polynomial ring $k[Y]_{[Y]}$. We have the following local embeddings.

$$\begin{aligned} &A := k[X]_{(X)} \hookrightarrow \widehat{A} := k[[X]], \quad \widehat{A} \hookrightarrow \widehat{B} = \widehat{C} = k[[X,Y]] \quad \text{and} \\ &B := k[[X]]\,[Y]_{(X,Y)} \hookrightarrow C := k[Y]_{(Y)}[[X]] \hookrightarrow \widehat{B} = \widehat{C} = k[[X]]\,[[Y]]. \end{aligned}$$

There is a canonical inclusion map $B \hookrightarrow C$, and the ring C has infinite transcendence degree over B, even if $m = n = 1$. In Chapter 26 we consider this embedding further and we analyze the associated spectral map.

In this chapter, we develop techniques using the Weierstrass Preparation Theorem. We use these techniques in Chapter 25 to describe the prime ideals maximal in generic fiber rings associated to the polynomial-power series rings A, B, and C. In particular, in Chapter 25, we prove every prime ideal P in $k[[X]]$ that is maximal with respect to $P \cap A = (0)$ has $\operatorname{ht} P = n - 1$. For every prime ideal P of $k[[X]][[Y]]$ such that P is maximal with respect to either $P \cap B = (0)$ or $P \cap C = (0)$, we prove $\operatorname{ht}(P) = n + m - 2$. In addition we prove each prime ideal P of $k[[X,Y]]$ that is maximal with respect to $P \cap k[[X]] = (0)$ has $\operatorname{ht} P = m$ or $n + m - 2$; see Theorem 24.3.

24.1. Terminology, Background and Results

We begin with definitions and notation for generic formal fiber rings.

NOTATION 24.1. Let $(R, \mathbf{m})$ be a Noetherian local domain and let $\widehat{R}$ be the $\mathbf{m}$-adic completion of R. The *generic formal fiber ring* of R is the localization $(R \setminus (0))^{-1}\widehat{R}$ of $\widehat{R}$ with respect to the multiplicatively closed set of nonzero elements of R. Let $\operatorname{Gff}(R)$ denote the generic formal fiber ring of R.

The *formal fibers* of R are the fibers of the map $\operatorname{Spec} \widehat{R} \to \operatorname{Spec} R$. For a prime ideal P of R, the formal fiber over P is $\operatorname{Spec}(\,(R_P/PR_P) \otimes_R \widehat{R}\,)$, or equivalently $\operatorname{Spec}((R \setminus P)^{-1}(\widehat{R}/P\widehat{R}))$; see Discussion 3.22 and Definition 3.34. Let $\operatorname{Gff}(R/P)$

denote the generic formal fiber ring of R/P. Since $\widehat{R}/P\widehat{R}$ is the completion of R/P, the formal fiber over P is $\operatorname{Spec}(\operatorname{Gff}(R/P))$.

Let $R \hookrightarrow S$ be an injective homomorphism of commutative rings. If R is an integral domain, the *generic fiber ring* of the map $R \hookrightarrow S$ is the localization $(R \setminus (0))^{-1}S$ of S.

The formal fibers encode important information about the structure of R. For example, R is excellent provided it is universally catenary and has geometrically regular formal fibers [**51**, (7.8.3), page 214]; see Definition 13.22.

We give some historical remarks regarding dimensions of generic formal fiber rings and heights of the maximal ideals of these rings:

REMARKS 24.2. (1) Let $(R, \mathbf{m})$ be a Noetherian local domain. In [**102**] Matsumura remarks that, as the ring R gets closer to its $\mathbf{m}$-adic completion $\widehat{R}$, it is natural to think that the dimension of the generic formal fiber ring $\operatorname{Gff}(R)$ gets smaller. He proves that the generic formal fiber ring of A has dimension $\dim A - 1$, and the generic formal fiber rings of B and C have dimension $\dim B - 2 = \dim C - 2$ in [**102**]. Matsumura speculates as to whether $\dim R - 1, \dim R - 2$ and 0 are the only possible values for $\dim(\operatorname{Gff}(R))$ in [**102**, p. 261].

(2) In answer to Matsumura's question Rotthaus establishes the following result in [**134**]: Let n be a positive integer. Then there exist excellent regular local rings R such that $\dim R = n$ and such that the generic formal fiber ring of R has dimension t, where the value of t may be taken to be any integer between 0 and $n - 1$.

(3) Let $(R, \mathbf{m})$ be an n-dimensional universally catenary Noetherian local domain. Loepp and Rotthaus in [**95**] compare the dimension of the generic formal fiber ring of R with that of the localized polynomial ring $R[x]_{(\mathbf{m},x)}$. Matsumura shows in [**102**] that the dimension of the generic formal fiber ring $\operatorname{Gff}(R[x]_{(\mathbf{m},x)})$ is either n or $n-1$. Loepp and Rotthaus in [**95**, Theorem 2] prove that $\dim(\operatorname{Gff}(R[x]_{(\mathbf{m},x)})) = n$ implies that $\dim(\operatorname{Gff} R) = n-1$. They show by example that in general the converse is not true, and they give sufficient conditions for the converse to hold.

(4) Let (T, M) be a complete Noetherian local domain that contains a field of characteristic zero. Assume that T/M has cardinality at least the cardinality of the real numbers. By adapting techniques developed by Heitmann in [**82**], in the articles [**93**] and [**94**], Loepp proves, among other things, for every prime ideal p of T with $p \neq M$, there exists an excellent regular local ring A that has completion T and has generic formal fiber ring $\operatorname{Gff}(A) = T_p$. By varying the height of p, this yields examples where the dimension of the generic formal fiber ring is any integer t with $0 \leq t < \dim T$. Loepp also shows for these examples that for each nonzero prime P of A, there exists a unique prime q of T with $q \cap A = P$ and $q = PT$.

(5) In the case where R is countable, Heinzer, Rotthaus and Sally show in [**60**, Proposition 4.10, page 36] that:

(a) The generic formal fiber ring $\operatorname{Gff}(R)$ is a Jacobson ring in the sense that each prime ideal of $\operatorname{Gff}(R)$ is an intersection of maximal ideals of $\operatorname{Gff}(R)$.
(b) $\dim(\widehat{R}/P) = 1$ for each prime ideal $P \in \operatorname{Spec}\widehat{R}$ that is maximal with respect to $P \cap R = (0)$.
(c) If $\widehat{R}$ is equidimensional, then $\operatorname{ht} P = n-1$ for each prime ideal $P \in \operatorname{Spec}\widehat{R}$ that is maximal with respect to $P \cap R = (0)$.
(d) If $Q \in \operatorname{Spec}\widehat{R}$ with $\operatorname{ht} Q \geq 1$, then there exists a prime ideal $P \subset Q$ such that $P \cap R = (0)$ and $\operatorname{ht}(Q/P) = 1$.

If the field k is countable, it follows from this result that all ideals maximal in the generic formal fiber ring of A have the same height.

(6) In Matsumura's article [**102**] from item 1 above, he does not address the question of whether all ideals maximal in the generic formal fiber rings for A, B and C have the same height. In general, for an excellent regular local ring R it can happen that $\mathrm{Gff}(R)$ contains maximal ideals of different heights; see the article [**134**, Corollary 3.2] of Rotthaus.

(7) Charters and Loepp in [**25**, Theorem 3.1] extend Rotthaus's result of item 6: Let (T, M) be a complete Noetherian local ring and let G be a nonempty subset of $\operatorname{Spec} T$ such that the number of maximal elements of G is finite. They prove there exists a Noetherian local domain A whose completion is T and whose generic formal fiber is exactly G if G satisfies the following conditions:

(a) $M \notin G$ and G contains the associated primes of T,
(b) If $P \subset Q$ are in $\operatorname{Spec} T$ and $Q \in G$, then $P \in G$, and
(c) Every $Q \in G$ meets the prime subring of T in (0).

If T contains the ring of integers and, in addition to items 1, 2, and 3, one also has

(d) T is equidimensional, and
(e) T_P is a regular local ring for each maximal element P of G,

then Charters and Loepp prove there exists an excellent local domain A whose completion is T and whose generic formal fiber is exactly G; see [**25**, Theorem 4.1]. Since the maximal elements of the set G may be chosen to have different heights, this result provides many examples where the generic formal fiber ring contains maximal ideals of different heights.

The Weierstrass techniques developed in this chapter enable us to prove the following theorem in Chapter 25:

Maximal Generic Fibers Theorem 24.3. *Let k be a field, let m and n be positive integers, and let $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$ be sets of independent variables over k. Then, for each of the rings $A := k[X]_{(X)}, B := k[[X]]\,[Y]_{(X,Y)}$ and $C := k[Y]_{(Y)}[[X]]$, every prime ideal maximal in the generic formal fiber ring has the same fixed height; more precisely*:

(1) *If P is a prime ideal of $\widehat{A}$ maximal with respect to $P \cap A = (0)$, then* $\operatorname{ht}(P) = n - 1$.
(2) *If P is a prime ideal of $\widehat{B}$ maximal with respect to $P \cap B = (0)$, then* $\operatorname{ht}(P) = n + m - 2$.
(3) *If P is a prime ideal of $\widehat{C}$ maximal with respect to $P \cap C = (0)$, then* $\operatorname{ht}(P) = n + m - 2$.
(4) *In addition, there are at most two possible values for the height of a maximal ideal of the generic fiber ring $(\widehat{A} \setminus (0))^{-1}\widehat{C}$ of the inclusion map $\widehat{A} \hookrightarrow \widehat{C}$.*
    (a) *If $n \geq 2$ and P is a prime ideal of $\widehat{C}$ maximal with respect to $P \cap \widehat{A} = (0)$, then either $\operatorname{ht} P = n + m - 2$ or $\operatorname{ht} P = m$.*
    (b) *If $n = 1$, then all ideals maximal in the generic fiber ring $(\widehat{A} \setminus (0))^{-1}\widehat{C}$ have height m.*

We were motivated to consider generic fiber rings for the embeddings displayed above because of questions related to Chapters 26 and 27 and ultimately because of the following question posed by Melvin Hochster and Yongwei Yao.

QUESTION 24.4. Let R be a complete Noetherian local domain. Can one describe or somehow classify the local maps of R to a complete Noetherian local domain S such that $U^{-1}S$ is a field, where $U = R \setminus (0)$, i.e., such that the generic fiber of $R \hookrightarrow S$ is trivial?

REMARK 24.5. By Cohen's structure theorems [**28**], [**117**, (31.6)], a complete Noetherian local domain R is a finite integral extension of a complete regular local domain R_0. If R has the same characteristic as its residue field, then R_0 is a formal power series ring over a field; see Remarks 3.12. The generic fiber $R \hookrightarrow S$ is trivial if and only if the generic fiber of $R_0 \hookrightarrow S$ is trivial.

A local ring R is called "equicharacteristic", if the ring and its residue field have the same characteristic; see Definition 3.11.1. If the equicharacteristic local ring has characteristic zero, then we say R is "equicharacteristic zero" or "of equal characteristic zero". Such a ring contains the field of rational numbers; see Exercise 24.1.

Thus, as Hochster and Yao remark, there is a natural way to construct such extensions in the case where the local ring R has characteristic zero and contains the rational numbers; consider

$$R = k[[x_1, ..., x_n]] \hookrightarrow T = L[[x_1, ..., x_n, y_1, ..., y_m]] \to T/P = S, \tag{24.4.0}$$

where k is a subfield of L, the x_i, y_j are formal indeterminates, and P is a prime ideal of T maximal with respect to being disjoint from the image of $R \setminus \{0\}$. Such prime ideals P correspond to the maximal ideals of the generic fiber $(R \setminus (0))^{-1}T$. The composite extension $T \hookrightarrow S$ satisfies the condition of Question 24.4.

In Theorem 25.6, we answer Question 24.4 in the special case where the extension arises from the embedding in Sequence 24.4.0 with the field $L = K$. We prove in this case that the dimension of the extension ring S must be either 2 or n.

We introduce the following terminology for the condition of Question 24.4 with a more general setting:

DEFINITION 24.6. For R and S integral domains with R a subring of S, we say that S is a *trivial generic fiber* extension of R, or a *TGF* extension of R, if every nonzero prime ideal of S has nonzero intersection with R. If $R \overset{\varphi}{\hookrightarrow} S$, then φ is also called a *trivial generic fiber extension* or *TGF extension*.

As in Remark 24.5, every extension $R \hookrightarrow T$ from an integral domain R to a commutative ring T yields a TGF extension by considering a composition

$$R \hookrightarrow T \to T/P = S, \tag{24.6.0}$$

where $P \in \operatorname{Spec} T$ is maximal with respect to $P \cap R = (0)$. Thus the generic fiber ring and so also Theorem 24.3 give information regarding TGF extensions in the case where the smaller ring is a mixed mixed polynomial-power series ring.

In addition, Theorem 24.3 is useful in the study of Sequence 24.4.0, because the map in Sequence 24.4.0 factors through:

$$R = k[[x_1, \ldots, x_n]] \hookrightarrow k[[x_1, \ldots, x_n]]\,[y_1, \ldots, y_m] \hookrightarrow T = L[[x_1, \ldots, x_n, y_1, \ldots, y_n]].$$

The second extension of this sequence is TGF if $n = m = 1$ and $k = L$; see Exercise 1 of this chapter. We study TGF extensions in Chapters 26 and 27.

Section 24.2 contains implications of Weierstrass' Preparation Theorem to the prime ideals of power series rings. We first prove a technical proposition regarding a

change of variables that provides a "nice" generating set for a given prime ideal P of a power series ring; then in Theorem 24.11 we prove that, in certain circumstances, a larger prime ideal can be found with the same contraction as P to a certain subring. In Section 24.3 we use Valabrega's, Theorem 4.8, concerning subrings of a two-dimensional regular local domain.

In Sections 25.1 and 25.2, we prove parts 2 and 3 of Theorem 24.3 stated above. We apply Theorem 24.11 in Section 25.3 to prove part 1 of Theorem 24.3, and in Section 25.4 we prove part 4.

24.2. Variations on a theme of Weierstrass

We apply the Weierstrass Preparation Theorem 24.7 below to examine the structure of a given prime ideal P in the power series ring $\widehat{A} = k[[X]]$, where $X = \{x_1, \ldots, x_n\}$ is a set of n variables over the field k. Here $A = k[X]_{(X)}$ is the localized polynomial ring in these variables. Our procedure is to make a change of variables that yields a regular sequence in P of a nice form.

We recall the statement of the Weierstrass Preparation Theorem.

THEOREM 24.7. (Weierstrass) [**165**, Theorem 5, p. 139; Corollary 1, p. 145] *Let $(R, \mathbf{m})$ be a complete Noetherian local ring, let $f \in R[[x]]$ be a formal power series and let $\overline{f}$ denote the image of f in $(R/\mathbf{m})[[x]]$. Assume that $\overline{f} \neq 0$ and that* $\operatorname{ord} \overline{f} = s > 0$. *There exists a unique ordered pair (u, F) such that u is a unit in $R[[x]]$ and $F \in R[x]$ is a distinguished monic polynomial of degree s such that $f = uF$. Here $F = x^s + a_{s-1}x^{s-1} + \cdots + a_0 \in R[x]$ is* **distinguished** *if $a_i \in \mathbf{m}$ for $0 \leq i \leq s-1$.*

We often write "By Weierstrass", where we use Theorem 24.7.

COROLLARY 24.8. *The ideal $fR[[x]]$ is extended from $R[x]$ and $R[[x]]/(f)$ is a free R-module of rank s. Every $g \in R[[x]]$ is of the form $g = qf + r$, where $q \in R[[x]]$ and $r \in R[x]$ is a polynomial with* $\deg r \leq s-1$.

NOTATION 24.9. By a *change of variables*, we mean a finite sequence of 'polynomial' change of variables of the type described below, where $X = \{x_1, \ldots, x_n\}$ is a set of n variables over the field k. For example, with $e_i, f_i \in \mathbb{N}$, consider

$$x_1 \mapsto x_1 + x_n^{e_1} = z_1, \qquad x_2 \mapsto x_2 + x_n^{e_2} = z_2, \qquad \ldots,$$
$$x_{n-1} \mapsto x_{n-1} + x_n^{e_{n-1}} = z_{n-1}, \qquad x_n \mapsto x_n = z_n,$$

followed by:

$$z_1 \mapsto z_1 = t_1, \qquad z_2 \mapsto z_2 + z_1^{f_2} = t_2, \qquad \ldots,$$
$$z_{n-1} \mapsto z_{n-1} + z_1^{f_{n-1}} = t_{n-1}, \qquad x_n \mapsto z_n + z_1^{f_n} = t_n.$$

Thus a change of variables defines an automorphism of $\widehat{A}$ that restricts to an automorphism of A.

We also consider a change of variables for subrings of A and $\widehat{A}$. For example, if $A_1 = k[x_2, \ldots, x_n] \subseteq A$ and $S = k[[x_2, \ldots, x_n]] \subseteq \widehat{A}$, then by a *change of variables inside* A_1 and S, we mean a finite sequence of automorphisms of A and $\widehat{A}$ of the type described above on $x_2, \ldots, x_n$ that leave the variable x_1 fixed. In this case we obtain an automorphism of $\widehat{A}$ that restricts to an automorphism on each of S, A and A_1.

PROPOSITION 24.10. *Let* $\widehat{A} := k[[X]] = k[[x_1, \ldots, x_n]]$ *and let* $P \in \operatorname{Spec} \widehat{A}$ *with* $x_1 \notin P$ *and* $\operatorname{ht} P = r$, *where* $1 \leq r \leq n-1$. *There exists a change of variables* $x_1 \mapsto z_1 := x_1$ (x_1 *is fixed*), $x_2 \mapsto z_2, \ldots, x_n \mapsto z_n$ *and a regular sequence* $f_1, \ldots, f_r \in P$ *so that, upon setting* $Z_1 = \{z_1, \ldots, z_{n-r}\}$, $Z_2 = \{z_{n-r+1}, \ldots, z_n\}$ *and* $Z = Z_1 \cup Z_2$, *we have*

$$\begin{aligned}
&f_1 \in k[[Z_1]]\,[z_{n-r+1}, \ldots, z_{n-1}]\,[z_n] && \text{is monic as a polynomial in } z_n\\
&f_2 \in k[[Z_1]]\,[z_{n-r+1}, \ldots, z_{n-2}]\,[z_{n-1}] && \text{is monic as a polynomial in } z_{n-1}, \text{ etc}\\
&\quad\vdots\\
&f_r \in k[[Z_1]]\,[z_{n-r+1}] && \text{is monic as a polynomial in } z_{n-r+1}.
\end{aligned}$$

In addition:

(1) P *is a minimal prime of the ideal* $(f_1, \ldots, f_r)\widehat{A}$.
(2) *The* (Z_2)*-adic completion of* $k[[Z_1]]\,[Z_2]_{(Z)}$ *is identical to the* $(f_1, \ldots, f_r)$*-adic completion and both equal* $\widehat{A} = k[[X]] = k[[Z]]$.
(3) *If* $P_1 := P \cap k[[Z_1]]\,[Z_2]_{(Z)}$, *then* $P_1\widehat{A} = P$, *that is,* P *is extended from* $k[[Z_1]]\,[Z_2]_{(Z)}$.
(4) *The ring extension:*

$$k[[Z_1]] \hookrightarrow k[[Z_1]]\,[Z_2]_{(Z)}/P_1 \cong k[[Z]]/P$$

is finite (and integral).

PROOF. Since $\widehat{A}$ is a unique factorization domain, there exists a nonzero prime element f in P. The power series f is therefore not a multiple of x_1, and so f must contain a monomial term $x_2^{i_2} \ldots x_n^{i_n}$ with a nonzero coefficient in k. This nonzero coefficient in k may be assumed to be 1. There exists an automorphism $\sigma : \widehat{A} \to \widehat{A}$ defined by the change of variables:

$$x_1 \mapsto x_1 \qquad x_2 \mapsto t_2 := x_2 + x_n^{e_2} \quad \ldots \quad x_{n-1} \mapsto t_{n-1} := x_{n-1} + x_n^{e_{n-1}} \qquad x_n \mapsto x_n,$$

with $e_2, \ldots, e_{n-1} \in \mathbb{N}$ chosen suitably so that f written as a power series in the variables $x_1, t_2, \ldots, t_{n-1}, x_n$ contains a term $a_n x_n^{s_n}$, where s_n is a positive integer, and $a_n \in k$ is nonzero. We assume that the integer s_n is minimal among all integers i such that a term $a x_n^i$ occurs in f with a nonzero coefficient $a \in k$; we further assume that the coefficient $a_n = 1$. By Weierstrass, that is, Theorem 24.7, we have that:

$$f = m\epsilon,$$

where $m \in k[[x_1, t_2, \ldots, t_{n-1}]]\,[x_n]$ is a monic polynomial in x_n of degree s_n and ϵ is a unit in $\widehat{A}$. Since $f \in P$ is a prime element, $m \in P$ is also a prime element. Using Weierstrass again, every element $g \in P$ can be written as:

$$g = mh + q,$$

where $h \in k[[x_1, t_2, \ldots, t_{n-1}, x_n]] = \widehat{A}$ and $q \in k[[x_1, t_2, \ldots, t_{n-1}]]\,[x_n]$ is a polynomial in x_n of degree less than s_n. Note that

$$k[[x_1, t_2, \ldots, t_{n-1}]] \hookrightarrow k[[x_1, t_2, \ldots, t_{n-1}]]\,[x_n]/(m)$$

is an integral (finite) extension. Thus the ring $k[[x_1, t_2, \ldots, t_{n-1}]]\,[x_n]/(m)$ is complete. Moreover, the two ideals $(x_1, t_2, \ldots, t_{n-1}, m) = (x_1, t_2, \ldots, t_{n-1}, x_n^{s_n})$ and

$(x_1, t_2, \ldots, t_{n-1}, x_n)$ of $B_0 := k[[x_1, t_2, \ldots, t_{n-1}]]\,[x_n]$ have the same radical. Therefore $\widehat{A}$ is the (m)-adic and the (x_n)-adic completion of B_0 and P is extended from B_0.

This implies the statement for $r = 1$, with $f_1 = m$, $z_n = x_n$, $z_1 = x_1$, $z_2 = t_2$, $\ldots$, $z_{n-1} = t_{n-1}$, $Z_1 = \{x_1, t_2, \ldots, t_{n-1}\}$ and $Z_2 = \{z_n\} = \{x_n\}$. In particular, when $r = 1$, P is minimal over $m\widehat{A}$, so $P = m\widehat{A}$.

For $r > 1$ we continue by induction on r. Let $P_0 := P \cap k[[x_1, t_2, \ldots, t_{n-1}]]$. Since $m \notin k[[x_1, t_2, \ldots, t_{n-1}]]$ and P is extended from $B_0 := k[[x_1, t_2, \ldots, t_{n-1}]]\,[x_n]$, then $P \cap B_0$ has height r and $\operatorname{ht} P_0 = r - 1$. Since $x_1 \notin P$, we have $x_1 \notin P_0$, and by the induction hypothesis there is a change of variables $t_2 \mapsto z_2, \ldots, t_{n-1} \mapsto z_{n-1}$ of $k[[x_1, t_2, \ldots, t_{n-1}]]$ and elements $f_2, \ldots, f_r \in P_0$ so that:

$$\begin{aligned}
&f_2 \in k[[x_1, z_2 \ldots, z_{n-r}]]\,[z_{n-r+1}, \ldots, z_{n-2}]\,[z_{n-1}] && \text{is monic in } z_{n-1}\\
&f_3 \in k[[x_1, z_2 \ldots, z_{n-r}]]\,[z_{n-r+1}, \ldots, z_{n-3}]\,[z_{n-2}] && \text{is monic in } z_{n-2}, \text{ etc}\\
&\quad\vdots\\
&f_r \in k[[x_1, z_2, \ldots, z_{n-r}]]\,[z_{n-r+1}] && \text{is monic in } z_{n-r+1},
\end{aligned}$$

and $f_2, \ldots, f_r$ satisfy the assertions of Proposition 24.10 for P_0.

It follows that $m, f_2, \ldots, f_r$ is a regular sequence of length r and that P is a minimal prime of the ideal $(m, f_2, \ldots, f_r)\widehat{A}$. Set $z_n = x_n$. We now prove that m may be replaced by a polynomial $f_1 \in k[[x_1, z_2, \ldots, z_{n-r}]]\,[z_{n-r+1}, \ldots, z_n]$. Write

$$m = \sum_{i=0}^{s_n} a_i z_n,$$

where the $a_i \in k[[x_1, z_2, \ldots, z_{n-1}]]$. For each $i < s_n$, apply Weierstrass to a_i and f_2 in order to obtain:

$$a_i = f_2 h_i + q_i,$$

where h_i is a power series in $k[[x_1, z_2, \ldots, z_{n-1}]]$ and $q_i \in k[[x_1, z_2, \ldots, z_{n-2}]]\,[z_{n-1}]$ is a polynomial in z_{n-1}. With $q_{s_n} = 1 = a_{s_n}$, we define

$$m_1 = \sum_{i=0}^{s_n} q_i z_n^i.$$

Now $(m_1, f_2, \ldots, f_r)\widehat{A} = (m, f_2, \ldots, f_r)\widehat{A}$ and we may replace m by m_1 which is a polynomial in z_{n-1} and z_n. To continue, for each $i < s_n$, write:

$$q_i = \sum_{j,k} b_{ij} z_{n-1}^j \quad \text{with} \quad b_{ij} \in k[[x_1, z_2, \ldots, z_{n-2}]].$$

For each b_{ij}, we apply Weierstrass to b_{ij} and f_3 to obtain:

$$b_{ij} = f_3 h_{ij} + q_{ij},$$

where $q_{ij} \in k[[x_1, z_2, \ldots, z_{n-3}]]\,[z_{n-2}]$. Set

$$m_2 = \sum_{i,j} q_{ij} z_{n-1}^j z_n^i \in k[[x_1, z_2, \ldots, z_{n-3}]]\,[z_{n-2}, z_{n-1}, z_n]$$

with $q_{s_n 0} = 1$. It follows that $(m_2, f_2, \ldots, f_r)\widehat{A} = (m, f_2, \ldots, f_r)\widehat{A}$. Continuing this process by applying Weierstrass to the coefficients of $z_{n-2}^k z_{n-1}^j z_n^i$ and f_4, we establish the existence of a polynomial $f_1 \in k[[Z_1]]\,[z_{n-r+1}, \ldots, z_n]$ that is monic

in z_n so that $(f_1, f_2, \ldots, f_r)\widehat{A} = (m, f_2, \ldots, f_r)\widehat{A}$. Therefore P is a minimal prime of $(f_1, \ldots, f_r)\widehat{A}$.

The extension

$$k[[Z_1]] \longrightarrow k[[Z_1]]\,[Z_2]/(f_1, \ldots, f_r)$$

is integral and finite. Thus the ring $k[[Z_1]]\,[Z_2]/(f_1, \ldots, f_r)$ is complete. This implies $\widehat{A} = k[[x_1, z_2, \ldots, z_n]]$ is the $(f_1, \ldots, f_r)$-adic (and the (Z_2)-adic) completion of $k[[Z_1]]\,[Z_2]_{(Z)}$ and that P is extended from $k[[Z_1]]\,[Z_2]_{(Z)}$. This completes the proof of Proposition 24.10. □

The following theorem is the technical heart of this section.

THEOREM 24.11. *Let k be a field and let y and $X = \{x_1, \ldots, x_n\}$ be variables over k. Assume that V is a discrete valuation domain with completion $\widehat{V} = k[[y]]$ and that $k[y] \subseteq V \subseteq k[[y]]$. Also assume that the field $k((y)) = k[[y]]\,[1/y]$ has uncountable transcendence degree over the quotient field $\mathcal{Q}(V)$ of V. Set $R_0 := V[[X]]$ and $R = \widehat{R}_0 = k[[y, X]]$. Let $P \in \operatorname{Spec} R$ be such that:*

(i) *$P \subseteq (X)R$ (so $y \notin P$), and*
(ii) *$\dim(R/P) > 2$.*

Then there is a prime ideal $Q \in \operatorname{Spec} R$ such that

(1) *$P \subset Q \subset XR$,*
(2) *$\dim(R/Q) = 2$, and*
(3) *$P \cap R_0 = Q \cap R_0$.*

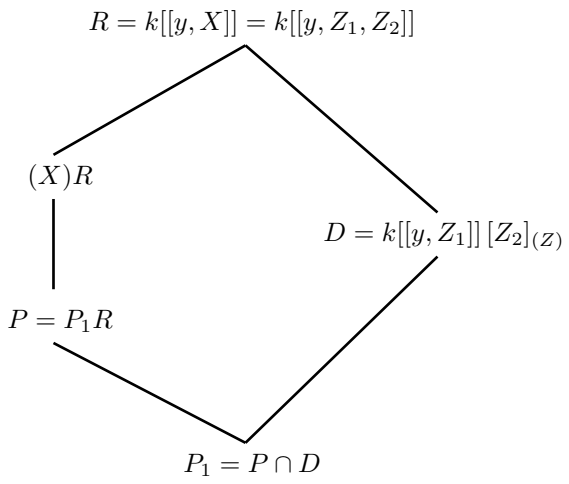
In particular, $P \cap k[[X]] = Q \cap k[[X]]$.

PROOF. Assume that P has height r. Since $\dim(R/P) > 2$, we have $0 \leq r < n - 1$. If $r > 0$, then there exist a transformation $x_1 \mapsto z_1, \ldots, x_n \mapsto z_n$ and elements $f_1, \ldots, f_r \in P$, by Proposition 24.10, so that the variable y is fixed, and

$$\begin{aligned}
&f_1 \in k[[y, z_1, \ldots, z_{n-r}]]\,[z_{n-r+1}, \ldots, z_n] \text{ is monic in } z_n,\\
&f_2 \in k[[y, z_1, \ldots, z_{n-r}]]\,[z_{n-r+1}, \ldots, z_{n-1}] \text{ is monic in } z_{n-1} \text{ etc,}\\
&\quad\vdots\\
&f_r \in k[[y, z_1, \ldots, z_{n-r}]]\,[z_{n-r+1}] \text{ is monic in } z_{n-r+1},
\end{aligned}$$

and the assertions of Proposition 24.10 are satisfied. In particular, P is a minimal prime of $(f_1, \ldots, f_r)R$. Let $Z_1 = \{z_1, \ldots, z_{n-r}\}$ and $Z_2 = \{z_{n-r+1}, \ldots, z_{n-1}, z_n\}$. By Proposition 24.10, if $D := k[[y, Z_1]]\,[Z_2]_{(Z)}$ and $P_1 := P \cap D$, then $P_1 R = P$.

The following diagram shows these rings and ideals.



Note that $f_1, \ldots, f_r \in P_1$. Let $g_1, \ldots, g_s \in P_1$ be other generators such that $P_1 = (f_1, \ldots, f_r, g_1, \ldots, g_s)D$. Then $P = P_1 R = (f_1, \ldots, f_r, g_1, \ldots, g_s)R$. For each $(i) := (i_1, \ldots, i_n) \in \mathbb{N}^n$ and j, k with $1 \leq j \leq r$, $1 \leq k \leq s$, let $a_{j,(i)}$, $b_{k,(i)}$ denote the coefficients in $k[[y]]$ of the f_j, g_k, so that

$$f_j = \sum_{(i)\in\mathbb{N}^n} a_{j,(i)} z_1^{i_1} \ldots z_n^{i_n}\,, \qquad g_k = \sum_{(i)\in\mathbb{N}^n} b_{k,(i)} z_1^{i_1} \ldots z_n^{i_n} \in k[[y]]\,[[Z]].$$

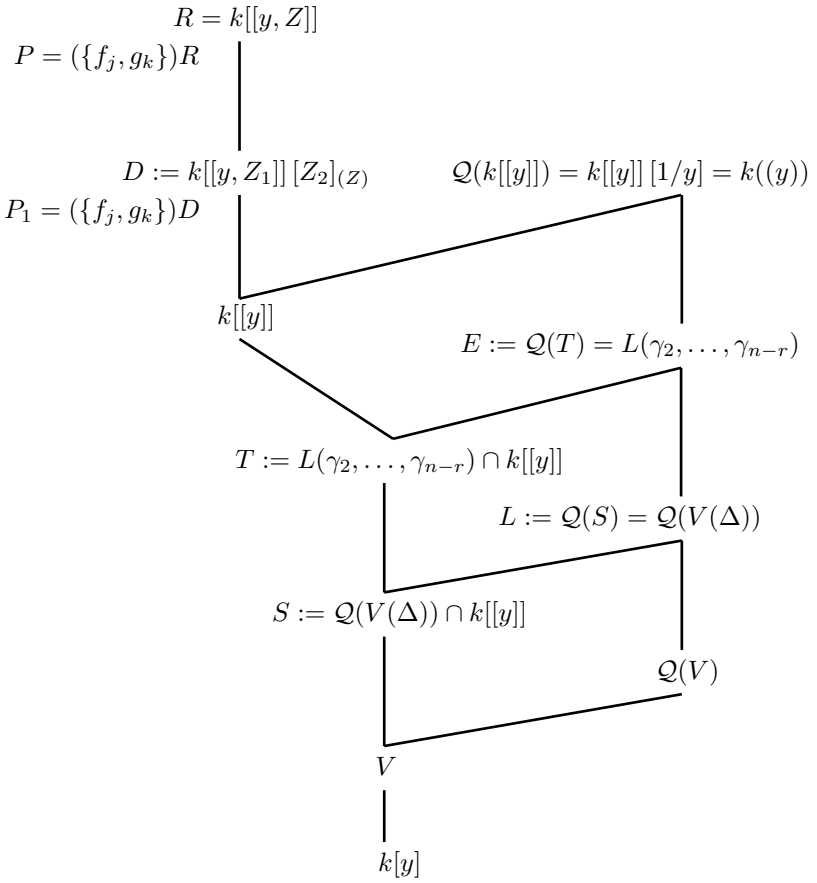
Define

$$\Delta := \begin{cases} \{a_{j,(i)}, b_{k,(i)}\} \subseteq k[[y]], & \text{for } r > 0 \\ \emptyset, & \text{for } r = 0. \end{cases}$$

A key observation here is that in either case the set Δ is countable.

To continue the proof, we consider $S := \mathcal{Q}(V(\Delta)) \cap k[[y]]$, a discrete valuation domain, and its field of quotients $L := \mathcal{Q}(V(\Delta))$. Since Δ is a countable set, the field $k((y))$ is (still) of uncountable transcendence degree over L. Let $\gamma_2, \ldots, \gamma_{n-r}$ be elements of $k[[y]]$ that are algebraically independent over L. We define $T := L(\gamma_2, \ldots, \gamma_{n-r}) \cap k[[y]]$ and $E := \mathcal{Q}(T) = L(\gamma_2, \ldots, \gamma_{n-r})$.

The diagram below shows the prime ideals P and P_1 and the containments among the relevant rings.



Let $P_2 := P \cap S[[Z_1]]\,[Z_2]_{(Z)}$. Since $f_1, \ldots, f_r, g_1, \ldots, g_s \in S[[Z_1]]\,[Z_2]_{(Z)}$, we have $P_2R = P$. Since $P \subseteq (x_1, \ldots, x_n)R = (Z)R$, there is a prime ideal $\widetilde{P}$ in $L[[Z]]$ that is minimal over $P_2L[[Z]]$. Since $L[[Z]]$ is flat over $S[[Z]]$, $\widetilde{P} \cap S[[Z]] = P_2S[[Z]]$. Note that $L[[X]] = L[[Z]]$ is the $(f_1, \ldots, f_r)$-adic (and the (Z_2)-adic) completion of $L[[Z_1]][Z_2]_{(Z)}$. In particular,

$$L[[Z_1]]\,[Z_2]/(f_1, \ldots, f_r) = L[[Z_1]]\,[[Z_2]]/(f_1, \ldots, f_r)$$

and this also holds with the field L replaced by its extension field E.

Since $L[[Z]]/\widetilde{P}$ is a homomorphic image of $L[[Z]]/(f_1, \ldots, f_r)$, it follows that $L[[Z]]/\widetilde{P}$ is integral (and finite) over $L[[Z_1]]$. This yields the commutative diagram:

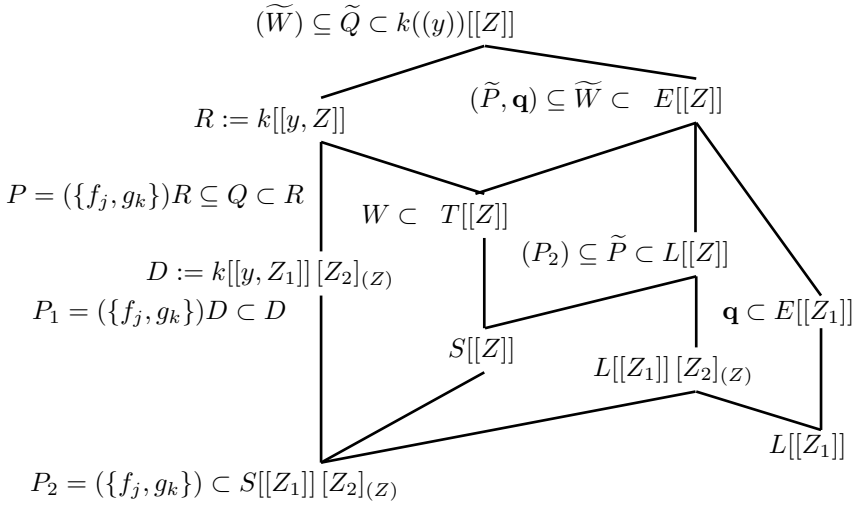
$$\begin{array}{ccc} E[[Z_1]] & \longrightarrow & E[[Z_1]]\,[[Z_2]]/\widetilde{P}E[[Z]] \\ \uparrow & & \uparrow \\ L[[Z_1]] & \longrightarrow & L[[Z_1]]\,[[Z_2]]/\widetilde{P} \end{array} \tag{24.11.0}$$

with injective integral (finite) horizontal maps. Recall that E is the subfield of $k((y))$ obtained by adjoining $\gamma_2, \ldots, \gamma_{n-r}$ to the field L. Thus the vertical maps in Diagram 24.11.0 are faithfully flat.

Let $\mathbf{q} := (z_2 - \gamma_2 z_1, \ldots, z_{n-r} - \gamma_{n-r} z_1)E[[Z_1]] \in \operatorname{Spec}(E[[Z_1]])$ and let $\widetilde{W}$ be a minimal prime of the ideal $(\widetilde{P}, \mathbf{q})E[[Z]]$. Since

$$f_1, \ldots, f_r, z_2 - \gamma_2 z_1, \ldots, z_{n-r} - \gamma_{n-r} z_1$$

is a regular sequence in $T[[Z]]$ the prime ideal $W := \widetilde{W} \cap T[[Z]]$ has height $n-1$. Let $\widetilde{Q}$ be a minimal prime of $\widetilde{W}k((y))[[Z]]$ and let $Q := \widetilde{Q} \cap R$. Then $W = Q \cap T[[Z]]$, $P \subset Q \subset ZR = XR$, and pictorially we have:



Notice that $\mathbf{q}$ is a prime ideal of height $n - r - 1$. Also, since $k((y))[[Z]]$ is flat over $k[[y, Z]] = R$, we have $\operatorname{ht} Q = n - 1$ and $\dim(R/Q) = 2$. We clearly have $P_2 \subseteq W \cap S[[Z_1]]\,[Z_2]_{(Z)}$.

Claim 24.12. $\mathbf{q} \cap L[[Z_1]] = (0)$.

To show this we argue as in [**102**]: Suppose that

$$h = \sum_{m \in \mathbb{N}} H_m \ \in \ \mathbf{q} \cap L[[z_1, \ldots, z_{n-r}]],$$

where $H_m \in L[z_1, \ldots, z_{n-r}]$ is a homogeneous polynomial of degree m:

$$H_m = \sum_{|(i)|=m} c_{(i)} z_1^{i_1} \ldots z_{n-r}^{i_{n-r}},$$

where $(i) := (i_1, \ldots, i_{n-r}) \in \mathbb{N}^{n-r}$, $|(i)| := i_1 + \cdots + i_{n-r}$ and $c_{(i)} \in L$. Consider the E-algebra homomorphism $\pi : E[[Z_1]] \to E[[z_1]]$ defined by $\pi(z_1) = z_1$ and $\pi(z_i) = \gamma_i z_1$ for $2 \leq i \leq n - r$. Then $\ker \pi = \mathbf{q}$, and for each $m \in \mathbb{N}$:

$$\pi(H_m) = \pi\Big(\sum_{|(i)|=m} c_{(i)} z_1^{i_1} \ldots z_{n-r}^{i_{n-r}}\Big) = \sum_{|(i)|=m} c_{(i)} \gamma_2^{i_2} \ldots \gamma_{n-r}^{i_{n-r}} z_1^m$$

and

$$\pi(h) = \sum_{m \in \mathbb{N}} \pi(H_m) = \sum_{m \in \mathbb{N}} \sum_{|(i)|=m} c_{(i)} \gamma_2^{i_2} \ldots \gamma_{n-r}^{i_{n-r}} z_1^m.$$

Since $h \in \mathbf{q}$, $\pi(h) = 0$. Since $\pi(h)$ is a power series in $E[[z_1]]$, each of its coefficients is zero, that is, for each $m \in \mathbb{N}$,

$$\sum_{|(i)|=m} c_{(i)} \gamma_2^{i_2} \dots \gamma_{n-r}^{i_{n-r}} = 0.$$

Since the γ_i are algebraically independent over L, each $c_{(i)} = 0$. Therefore $h = 0$, and so $\mathbf{q} \cap L[[Z_1]] = (0)$. This proves Claim 24.12.

Using the commutativity of Diagram 24.11.0 and that the horizonal maps of this diagram are integral extensions, we deduce that $(\widetilde{W} \cap E[[Z_1]]) = \mathbf{q}$, and $\mathbf{q} \cap L[[Z_1]] = (0)$ implies $\widetilde{W} \cap L[[Z_1]] = (0)$. We conclude that $Q \cap S[[Z]] = P \cap S[[Z]]$ and therefore $Q \cap R_0 = P \cap R_0$. □

We record the following corollary.

COROLLARY 24.13. *Let k be a field, let $X = \{x_1, \dots, x_n\}$ and y be independent variables over k, and let $R = k[[y, X]]$. Assume $P \in \operatorname{Spec} R$ is such that:*

(i): $P \subseteq (x_1, \dots, x_n)R$ *and*
(ii): $\dim(R/P) > 2$.

Then there is a prime ideal $Q \in \operatorname{Spec} R$ so that

(1) $P \subset Q \subset (x_1, \dots, x_n)R$,
(2) $\dim(R/Q) = 2$, *and*
(3) $P \cap k[y]_{(y)}[[X]] = Q \cap k[y]_{(y)}[[X]]$.

In particular, $P \cap k[[x_1, \dots, x_n]] = Q \cap k[[x_1, \dots, x_n]]$.

PROOF. With notation as in Theorem 24.11, let $V = k[y]_{(y)}$. □

24.3. Subrings of the power series ring $k[[z, t]]$

In this section we establish properties of certain subrings of the power series ring $k[[z, t]]$ that will be useful in considering the generic formal fiber of localized polynomial rings over the field k.

NOTATION 24.14. Let k be a field and let z and t be independent variables over k. Consider countably many power series:

$$\alpha_i(z) = \sum_{j=0}^{\infty} a_{ij} z^j \in k[[z]]$$

with coefficients $a_{ik} \in k$. Let s be a positive integer and let $\omega_1, \dots, \omega_s \in k[[z, t]]$ be power series in z and t, say:

$$\omega_i = \sum_{j=0}^{\infty} \beta_{ij} t^j, \quad \text{where} \quad \beta_{ij}(z) = \sum_{k=0}^{\infty} b_{ijk} z^k \in k[[z]] \quad \text{and} \quad b_{ijk} \in k,$$

for each i with $1 \leq i \leq s$. Consider the subfield $k(z, \{\alpha_i\}, \{\beta_{ij}\})$ of $k((z))$ and the discrete rank-one valuation domain

$$V := k(z, \{\alpha_i\}, \{\beta_{ij}\}) \cap k[[z]]. \tag{24.14.0}$$

The completion of V is $\widehat{V} = k[[z]]$. Assume that $\omega_1, \dots, \omega_r$ are algebraically independent over $\mathcal{Q}(V)(t)$ and that the elements $\omega_{r+1}, \dots, \omega_s$ are algebraic over the

field $\mathcal{Q}(V)(t,\{\omega_i\}_{i=1}^r)$. Notice that the set $\{\alpha_i\}\cup\{\beta_{ij}\}$ is countable, and that also the set of coefficients of the α_i and β_{ij}

$$\Delta := \{a_{ij}\}\cup\{b_{ijk}\}$$

is a countable subset of the field k. Let k_0 denote the prime subfield of k and let F denote the algebraic closure in k of the field $k_0(\Delta)$. The field F is countable and the power series $\alpha_i(z)$ and $\beta_{ij}(z)$ are in $F[[z]]$. Consider the subfield $F(z,\{\alpha_i\},\{\beta_{ij}\})$ of $F((z))$ and the discrete rank-one valuation domain

$$V_0 := F(z,\{\alpha_i\},\{\beta_{ij}\})\cap F[[z]].$$

The completion of V_0 is $\widehat{V}_0 = F[[z]]$. Since $\mathcal{Q}(V_0)(t)\subseteq\mathcal{Q}(V)(t)$, the elements $\omega_1,\dots,\omega_r$ are algebraically independent over the field $\mathcal{Q}(V_0)(t)$.

Consider the subfield $E_0 := \mathcal{Q}(V_0)(t,\omega_1,\dots,\omega_r)$ of $\mathcal{Q}(V_0[[t]])$ and the subfield $E := \mathcal{Q}(V)(t,\omega_1,\dots,\omega_r)$ of $\mathcal{Q}(V[[t]])$. A result of Valabrega, Theorem 4.8, implies that the integral domains:

$$D_0 := E_0\cap V_0[[t]] \qquad \text{and} \qquad D := E\cap V[[t]] \tag{24.14.1}$$

are two-dimensional regular local rings with completions $\widehat{D}_0 = F[[z,t]]$ and $\widehat{D} = k[[z,t]]$, respectively. Moreover, $\mathcal{Q}(D_0) = E_0$ is a countable field.

PROPOSITION 24.15. *Let D_0 be as defined in Equation 24.14.1. Then there exists a power series $\gamma\in zF[[z]]$ such that the prime ideal $(t-\gamma)F[[z,t]]\cap D_0 = (0)$, that is, $(t-\gamma)F[[z,t]]$ is in the generic formal fiber of D_0.*

PROOF. Since D_0 is countable there are only countably many prime ideals in D_0 and since D_0 is Noetherian there are only countably many prime ideals in $\widehat{D}_0 = F[[z,t]]$ that lie over a nonzero prime of D_0. There are uncountably many primes in $F[[z,t]]$, which are generated by elements of the form $t-\sigma$ for some $\sigma\in zF[[z]]$. Thus there must exist an element $\gamma\in zF[[z]]$ with $(t-\gamma)F[[z,t]]\cap D_0 = (0)$. □

For $\omega_i = \omega_i(t) = \sum_{j=0}^{\infty}\beta_{ij}t^j$ as in Notation 24.14 and γ an element of $zk[[z]]$, let $\omega_i(\gamma)$ denote the following power series in $k[[z]]$:

$$\omega_i(\gamma) := \sum_{j=0}^{\infty}\beta_{ij}\gamma^j \in k[[z]].$$

PROPOSITION 24.16. *Let V and D be as defined in Equations 24.14.0 and 24.14.1. For an element $\gamma\in zk[[z]]$ the following conditions are equivalent*:

(i): $(t-\gamma)k[[z,t]]\cap D = (0)$.
(ii): *The elements $\gamma,\omega_1(\gamma),\dots,\omega_r(\gamma)$ are algebraically independent over $\mathcal{Q}(V)$.*

PROOF. (i) ⇒ (ii): Assume by way of contradiction that $\{\gamma,\omega_1(\gamma),\dots,\omega_r(\gamma)\}$ is an algebraically dependent set over $\mathcal{Q}(V)$ and let $d_{(k)}\in V$ be finitely many elements such that

$$\sum_{(k)} d_{(k)}\omega_1(\gamma)^{k_1}\dots\omega_r(\gamma)^{k_r}\gamma^{k_{r+1}} = 0$$

is a nontrivial equation of algebraic dependence for $\gamma,\omega_1(\gamma),\dots,\omega_r(\gamma)$, where each $(k) = (k_1\dots,k_r,k_{r+1})$ is an $(r+1)$-tuple of nonnegative integers. It follows that

$$\sum_{(k)} d_{(k)}\omega_1^{k_1}\dots\omega_r^{k_r}t^{k_{r+1}} \in (t-\gamma)k[[z,t]]\cap D = (0).$$

Since $\omega_1, \ldots, \omega_r$ are algebraically independent over $\mathcal{Q}(V)(t)$, we have $d_{(k)} = 0$ for all (k), a contradiction. This completes the proof that $(i) \Rightarrow (ii)$.

(ii) $\Rightarrow$ (i): If $(t-\gamma)k[[z,t]] \cap D \neq (0)$, then there exists a nonzero element

$$\tau = \sum_{(k)} d_{(k)} \omega_1^{k_1} \ldots \omega_r^{k_r} t^{k_{r+1}} \in (t-\gamma)k[[z,t]] \cap V[t, \omega_1, \ldots, \omega_r].$$

But this implies that

$$\tau(\gamma) = \sum_{(k)} d_{(k)} \omega_1(\gamma)^{k_1} \ldots \omega_r(\gamma)^{k_r} \gamma^{k_{r+1}} = 0.$$

Since $\gamma, \omega_1(\gamma), \ldots, \omega_r(\gamma)$ are algebraically independent over $\mathcal{Q}(V)$, it follows that all the coefficients $d_{(k)} = 0$, a contradiction to the assumption that τ is nonzero. □

Let $\gamma \in zF[[z]]$ be as in Proposition 24.15 with $(t-\gamma)F[[z,t]] \cap D_0 = (0)$. Then:

PROPOSITION 24.17. *With notation as above, we have* $(t-\gamma)k[[z,t]] \cap D = (0)$, *that is,* $(t-\gamma)k[[z,t]]$ *is in the generic formal fiber of* D.

PROOF. Let $L := F(\{t_i\}_{i \in I})$, where $\{t_i\}_{i \in I}$ is a transcendence basis of k over F. Then k is algebraic over L. Let $\{\alpha_i\}, \{\beta_{ij}\} \subset F[[z]]$ be as in (5.1) and define

$$V_1 = L(z, \{\alpha_i\}, \{\beta_{ij}\}) \cap L[[z]] \quad \text{and} \quad D_1 = \mathcal{Q}(V_1)(t, \omega_1, \ldots, \omega_r) \cap L[[z,t]].$$

Then V_1 is a discrete rank-one valuation domain with completion $L[[z]]$ and D_1 is a two-dimensional regular local domain with completion $\widehat{D}_1 = L[[z,t]]$. Note that $\mathcal{Q}(V)$ and $\mathcal{Q}(D)$ are algebraic over $\mathcal{Q}(V_1)$ and $\mathcal{Q}(D_1)$, respectively. Since $(t-\gamma)k[[z,t]] \cap L[[z,t]] = (t-\gamma)L[[z,t]]$, it suffices to prove that $(t-\gamma)L[[z,t]] \cap D_1 = (0)$. By Proposition 24.16, it suffices to show that $\gamma, \omega_1(\gamma), \ldots, \omega_r(\gamma)$ are algebraically independent over $\mathcal{Q}(V_1)$. The commutative diagram

$$\begin{array}{ccc} F[[z]] & \xrightarrow{\{t_i\} \text{algebraically ind.}} & L[[z]] \\ \uparrow & & \uparrow \\ \mathcal{Q}(V_0) & \xrightarrow{\text{transcendence basis } \{t_i\}} & \mathcal{Q}(V_1) \end{array}$$

implies that the set $\{\gamma, \omega_1(\gamma), \ldots, \omega_r(\gamma)\} \cup \{t_i\}$ is algebraically independent over $\mathcal{Q}(V_0)$. Therefore $\{\gamma, \omega_1(\gamma), \ldots, \omega_r(\gamma)\}$ is algebraically independent over $\mathcal{Q}(V_1)$. This completes the proof of Proposition 24.17. □

REMARK 24.18. If $\omega_{r+1}, \ldots, \omega_s$ is algebraic over $\mathcal{Q}(V)(\omega_1, \ldots, \omega_r)$ as in (5.1) and we define

$$\widetilde{D} := \mathcal{Q}(V)(t, \omega_1, \ldots, \omega_s) \cap V[[t]],$$

then again by Valabrega's Theorem 4.8, $\widetilde{D}$ is a two-dimensional regular local domain with completion $k[[z,t]]$. Moreover, $\mathcal{Q}(\widetilde{D})$ is algebraic over $\mathcal{Q}(D)$ and $(t-\gamma)k[[z,t]] \cap D = (0)$ implies that $(t-\gamma)k[[z,t]] \cap \widetilde{D} = (0)$.

Exercise

(1) Prove that a local ring that has residue field of characteristic zero contains the field of rational numbers.

CHAPTER 25

Generic fiber rings of mixed polynomial-power series rings

Our primary project in this chapter is to prove Theorem 24.3 concerning generic fiber rings for extensions of the polynomial-power series rings A, B and C defined in Chapter 24; see Equation 24.1.0 and Notation 24.1. By Theorem 24.3, all ideals maximal in each of the generic formal fiber rings for A, B and C have the same height. These results are proved using the techniques developed in Chapter 24. Matsumura proves in [**102**] that the generic formal fiber ring of A has dimension $n-1 = \dim A - 1$, and the generic formal fiber rings of B and C have dimension $n+m-2 = \dim B - 2 = \dim C - 2$. Matsumura does not consider in [**102**] the question of whether all the maximal ideals in these generic formal fiber ring have the same height.

For a local extension $R \hookrightarrow S$ of Noetherian local integral domains, Theorem 25.12 gives sufficient conditions in order that all maximal ideals in $\mathrm{Gff}(S)$ have height $h = \dim \mathrm{Gff}(R)$. Using Theorem 25.12, we show in Theorem 25.10 that all prime ideals maximal in the generic formal fiber of a local domain essentially finitely generated over a field have the same height. For certain Noetherian local extensions S of the rings B and C, we show in Theorem 25.16 that the maximal ideals of $\mathrm{Gff}(S)$ all have height $n+m-2$.

In Sections 25.1 and 25.2, we prove parts 2 and 3 of Theorem 24.3 stated in Chapter 24. In Section 25.3 we prove part 1 of Theorem 24.3, by using the results of Section 24.3, and in Section 25.4 we prove part 4. Theorems 25.12, 25.10 and 25.16 are in Section 25.5.

25.1. Weierstrass implications for the ring $B = k[[X]]\,[Y]_{(X,Y)}$

As before, k denotes a field, n and m are positive integers, and $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$ denote sets of variables over k. Let

$$B := k[[X]]\,[Y]_{(X,Y)} = k[[x_1, \ldots, x_n]]\,[y_1, \ldots, y_m]_{(x_1,\ldots,x_n,y_1,\ldots,y_m)}.$$

The completion of B is $\widehat{B} = k[[X,Y]]$.

THEOREM 25.1. *With the notation as above, every ideal Q of $\widehat{B} = k[[X,Y]]$ maximal with the property that $Q \cap B = (0)$ is a prime ideal of height $n+m-2$.*

PROOF. Suppose first that Q is such an ideal. Then clearly Q is prime. Matsumura shows in [**102**, Theorem 3] that the dimension of the generic formal fiber of B is at most $n+m-2$. Therefore $\operatorname{ht} Q \leq n+m-2$.

Now suppose $P \in \operatorname{Spec} \widehat{B}$ is an arbitrary prime ideal of height $r < n+m-2$ with $P \cap B = (0)$. We construct a prime $Q \in \operatorname{Spec} \widehat{B}$ with $P \subset Q$, $Q \cap B = (0)$,

and $\operatorname{ht} Q = n + m - 2$. This will show that all prime ideals maximal in the generic fiber have height $n + m - 2$.

For the construction of Q we consider first the case where $P \not\subseteq X\widehat{B}$. Then there exists a prime element $f \in P$ that contains a term $\theta := y_1^{i_1} \cdots y_m^{i_m}$, where the i_j's are nonnegative integers and at least one of the i_j is positive. Notice that $m \geq 2$ for otherwise with $y = y_1$ we have $f \in P$ contains a term y^i. By Weierstrass, that is, by Theorem 24.7, it follows that $f = g\epsilon$, where $g \in k[[X]][y]$ is a nonzero monic polynomial in y and ϵ is a unit of $\widehat{B}$. But $g \in P$ and $g \in B$ implies $P \cap B \neq (0)$, a contradiction to our assumption that $P \cap B = (0)$.

For convenience we now assume that the last exponent i_m appearing in θ above is positive. We apply a change of variables: $y_m \to t_m := y_m$ and, for $1 \leq \ell < m$, let $y_\ell \to t_\ell := y_\ell + {t_m}^{e_\ell}$, where the e_ℓ are chosen so that f, expressed in the variables $t_1, \ldots, t_m$, contains a term t_m^q, for some positive integer q. This change of variables induces an automorphism of B. By Weierstrass $f = g_1 h$, where h is a unit in $\widehat{B}$ and $g_1 \in k[[X, t_1, \ldots, t_{m-1}]][t_m]$ is monic in t_m. Set $P_1 = P \cap k[[X, t_1, \ldots, t_{m-1}]]$. If $P_1 \subseteq Xk[[X, t_1, \ldots, t_{m-1}]]$, we stop the procedure and take $s = m - 1$ in what follows. If $P_1 \not\subseteq Xk[[X, t_1, \ldots, t_{m-1}]]$, then there exists a prime element $\tilde{f} \in P_1$ that contains a term ${t_1}^{j_1} \cdots {t_{m-1}}^{j_{m-1}}$, where the j_k's are nonnegative integers and at least one of the j_k is positive. We then repeat the procedure using the prime ideal P_1. That is, we replace $t_1, \ldots t_{m-1}$ with a change of variables so that a prime element of P_1 contains a term monic in some one of the new variables. After a suitable finite iteration of changes of variables, we obtain an automorphism of $\widehat{B}$ that restricts to an automorphism of B and maps $y_1, \ldots, y_m \mapsto z_1, \ldots, z_m$. Moreover, there exist a positive integer $s \leq m - 1$ and elements $g_1, \ldots g_{m-s} \in P$ such that

$$\begin{aligned} g_1 &\in k[[X, z_1, \ldots, z_{m-1}]][z_m] && \text{is monic in } z_m \\ g_2 &\in k[[X, z_1, \ldots, z_{m-2}]][z_{m-1}] && \text{is monic in } z_{m-1}, \text{ etc} \\ &\vdots \\ g_{m-s} &\in k[[X, z_1, \ldots, z_s]][z_{s+1}] && \text{is monic in } z_{s+1}, \end{aligned}$$

and such that, for $R_s := k[[X, z_1, \ldots, z_s]]$ and $P_s := P \cap R_s$, we have $P_s \subseteq XR_s$.

As in the proof of Proposition 24.10, we replace the regular sequence $g_1, \ldots g_{m-s}$ by a regular sequence $f_1, \ldots, f_{m-s}$ so that:

$$\begin{aligned} f_1 &\in R_s[z_{s+1}, \ldots, z_m] && \text{is monic in } z_m \\ f_2 &\in R_s[z_{s+1}, \ldots, z_{m-1}] && \text{is monic in } z_{m-1}, \text{ etc} \\ &\vdots \\ f_{m-s} &\in R_s[z_{s+1}] && \text{is monic in } z_{s+1}. \end{aligned}$$

and $(g_1, \ldots g_{m-s})\widehat{B} = (f_1, \ldots, f_{m-s})\widehat{B}$.

Let $G := k[[X, z_1, \ldots, z_s]][z_{s+1}, \ldots, z_m] = R_s[z_{s+1}, \ldots, z_m]$. By Proposition 24.10, P is extended from G. Let $\mathbf{q} := P \cap G$ and extend $f_1, \ldots, f_{m-s}$ to a generating system of $\mathbf{q}$, say, $\mathbf{q} = (f_1, \ldots, f_{m-s}, h_1, \ldots, h_t)G$. For integers k, ℓ with $1 \leq k \leq m - s$ and $1 \leq \ell \leq t$, express the f_k and h_ℓ in G as power series in

$\widehat{B} = k[[z_1]][[z_2, \ldots, z_m]]\,[[X]]$ with coefficients in $k[[z_1]]$:

$$f_k = \sum a_{k(i)(j)} z_2^{i_2} \ldots z_m^{i_m} x_1^{j_1} \ldots x_n^{j_n} \quad \text{and} \quad h_\ell = \sum b_{\ell(i)(j)} z_2^{i_2} \ldots z_m^{i_m} x_1^{j_1} \ldots x_n^{j_n}\,,$$

where $a_{k(i)(j)}, b_{\ell(i)(j)} \in k[[z_1]]$, $(i) = (i_2, \ldots, i_m)$ and $(j) = (j_1, \ldots, j_n)$. The set $\Delta = \{a_{k(i)(j)}, b_{\ell(i)(j)}\}$ is countable. We define $V := k(z_1, \Delta) \cap k[[z_1]]$. Then V is a discrete valuation domain with completion $k[[z_1]]$ and $k((z_1))$ has uncountable transcendence degree over $\mathcal{Q}(V)$. Let $V_s := V[[X, z_2, \ldots, z_s]] \subseteq R_s$. Notice that $R_s = \widehat{V_s}$, the completion of V_s. Also $f_1, \ldots, f_{m-s} \in V_s[z_{s+1}, \ldots, z_m] \subseteq G$ and $(f_1, \ldots, f_{m-s})G \cap R_s = (0)$. Furthermore the extension

$$V_s := V[[X, z_2, \ldots, z_s]] \hookrightarrow V_s[z_{s+1}, \ldots, z_m]/(f_1, \ldots, f_{m-s})$$

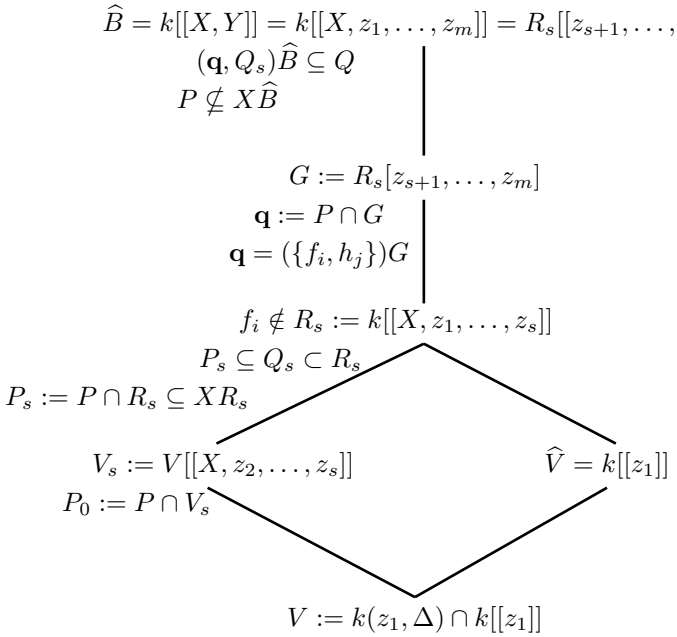
is finite. Set $P_0 := P \cap V_s$. Then $P_0 \subseteq XR_s \cap V_s = XV_s$.

Consider the commutative diagram:

$$\begin{array}{ccc} R_s := k[[X, z_1, \ldots, z_s]] & \longrightarrow & R_s[[z_{s+1}, \ldots, z_m]]/(f_1, \ldots, f_{m-s}) \\ \uparrow & & \uparrow \\ V_s := V[[X, z_2, \ldots, z_s]] & \longrightarrow & V_s[z_{s+1}, \ldots, z_m]/(f_1, \ldots, f_{m-s})\,. \end{array} \tag{25.1}$$

The horizontal maps are injective and finite and the vertical maps are completions.

The prime ideal $\bar{\mathbf{q}} := PR_s[[z_{s+1}, \ldots, z_m]]/(f_1, \ldots, f_{m-s})$ lies over P_s in R_s. By assumption $P_s \subseteq (X)R_s$ and by Theorem 24.11 there is a prime ideal Q_s of R_s such that $P_s \subseteq Q_s \subseteq (X)R_s$, $Q_s \cap V_s = P_s \cap V_s = P_0$, and $\dim(R_s/Q_s) = 2$. There is a prime ideal $\bar{Q}$ in $R_s[[z_{s+1}, \ldots, z_m]]/(f_1, \ldots, f_{m-s})$ lying over Q_s with $\bar{\mathbf{q}} \subseteq \bar{Q}$ by the "going-up theorem" [**103**, Theorem 9.4]. Let Q be the preimage in $\widehat{B} = k[[X, z_1, \ldots, z_m]]$ of $\bar{Q}$. We show the rings and ideals of Theorem 25.1 below.



Then Q has height $n+s-2+m-s = n+m-2$. Moreover, it follows from Diagram 25.1 that Q and P have the same contraction to $V_s[z_{s+1},\dots,z_m]$. This implies that $Q \cap B = (0)$ and completes the proof in the case where $P \not\subseteq X\widehat{B}$.

In the case where $P \subseteq X\widehat{B}$, let $h_1,\dots,h_t \in \widehat{B}$ be a finite set of generators of P, and as above, let $b_{\ell(i)(j)} \in k[[z_1]]$ be the coefficients of the h_ℓ's. Consider the countable set $\Delta = \{b_{\ell(i)(j)}\}$ and the valuation domain $V := k(z_1,\Delta) \cap k[[z_1]]$. Set $P_0 := P \cap V[[X,z_2,\dots,z_m]]$. By Theorem 24.11, there exists a prime ideal Q of $\widehat{B} = k[[X,z_1,\dots,z_m]]$ of height $n+m-2$ such that $P \subset Q$ and $Q \cap V[[X,z_2,\dots,z_m]] = P \cap V[[X,z_2,\dots,z_m]] = P_0$. Therefore $Q \cap B = (0)$. This completes the proof of Theorem 25.1. □

25.2. Weierstrass implications for the ring $C = k[Y]_{(Y)}[[X]]$

As before, k denotes a field, n and m are positive integers, and $X = \{x_1,\dots,x_n\}$ and $Y = \{y_1,\dots,y_m\}$ denote sets of variables over k. Consider the ring

$$C = k[y_1,\dots,y_m]_{(y_1,\dots,y_m)}[[x_1,\dots,x_n]] = k[Y]_{(Y)}[[X]].$$

The completion of C is $\widehat{C} = k[[Y,X]]$.

THEOREM 25.2. *With notation as above, let $Q \in \operatorname{Spec}\widehat{C}$ be maximal with the property that $Q \cap C = (0)$. Then* $\operatorname{ht} Q = n+m-2$.

PROOF. Let $B = k[[X]]\,[Y]_{(X,Y)} \subset C$. If $P \in \operatorname{Spec}\widehat{C} = \operatorname{Spec}\widehat{B}$ and $P \cap C = (0)$, then $P \cap B = (0)$, and so $\operatorname{ht} P \le n+m-2$ by Theorem 25.1. Consider a nonzero prime $P \in \operatorname{Spec}\widehat{C}$ with $P \cap C = (0)$ and $\operatorname{ht} P = r < n+m-2$. If $P \subseteq X\widehat{C}$ then Theorem 24.11 implies the existence of $Q \in \operatorname{Spec}\widehat{C}$ with $\operatorname{ht} Q = n+m-2$ such that $P \subset Q$ and $Q \cap C = (0)$.

Assume that P is not contained in $X\widehat{C}$ and consider the ideal $J := (P,X)\widehat{C}$. Since C is complete in the XC-adic topology, a lemma of Rotthaus implies that if J is primary for the maximal ideal of $\widehat{C}$, then P is extended from C; see [**133**, Lemma 2]. Since we are assuming $P \cap C = (0)$, J is not primary for the maximal ideal of $\widehat{C}$ and we have $\operatorname{ht} J = n+s < n+m$, where $0 < s < m$. Let $W \in \operatorname{Spec}\widehat{C}$ be a minimal prime of J such that $\operatorname{ht} W = n+s$. Let $W_0 = W \cap k[[Y]]$. Then $W = (W_0,X)\widehat{C}$ and W_0 is a prime ideal of $k[[Y]]$ with $\operatorname{ht} W_0 = s$. By Proposition 24.10 applied to $k[[Y]]$ and the prime ideal $W_0 \in \operatorname{Spec} k[[Y]]$, there exists a change of variables $Y \mapsto Z$ with $y_1 \mapsto z_1,\dots,y_m \mapsto z_m$ and elements $f_1,\dots,f_s \in W_0$ so that with $Z_1 = \{z_1,\dots,z_{m-s}\}$, we have

$$\begin{aligned}
&f_1 \in k[[Z_1]]\,[z_{m-s+1},\dots,z_m] && \text{is monic in } z_m\\
&f_2 \in k[[Z_1]]\,[z_{m-s+1},\dots,z_{m-1}] && \text{is monic in } z_{m-1}, \text{ etc}\\
&\quad\vdots\\
&f_s \in k[[Z_1]]\,[z_{m-s+1}] && \text{is monic in } z_{m-s+1}.
\end{aligned}$$

Now $z_1,\dots,z_{m-s},f_1,\dots,f_s$ is a regular sequence in $k[[Z]] = k[[Y]]$. Let $T = \{t_{m-s+1},\dots,t_m\}$ be a set of additional variables and consider the map:

$$\varphi : k[[Z_1,T]] \longrightarrow k[[z_1,\dots,z_m]]$$

defined by $z_i \mapsto z_i$ for all $1 \le i \le m-s$ and $t_{m-i+1} \mapsto f_i$ for all $1 \le i \le s$. The embedding φ is finite (and free) and so is the extension to power series rings in X:

$$\rho : k[[Z_1, T]][[X]] \longrightarrow k[[z_1, \ldots, z_m]][[X]] = \widehat{C}.$$

The contraction $\rho^{-1}(W) \in \operatorname{Spec} k[[Z_1, T, X]]$ of the prime ideal W of $\widehat{C}$ has height $n + s$, since $\operatorname{ht} W = n + s$. Moreover $\rho^{-1}(W)$ contains $(T, X)k[[Z_1, T, X]]$, a prime ideal of height $n + s$. Therefore $\rho^{-1}(W) = (T, X)k[[Z_1, T, X]]$. By construction, $P \subseteq W$ which yields that $\rho^{-1}(P) \subseteq (T, X)k[[Z_1, T, X]]$.

To complete the proof we construct a suitable base ring related to C. Consider the expressions for the f_i's as power series in $z_2, \ldots, z_m$ with coefficients in $k[[z_1]]$:

$$f_j = \sum a_{j(i)} z_2^{i_2} \ldots z_m^{i_m},$$

where $(i) := (i_2, \ldots, i_m), 1 \leq j \leq s, a_{j(i)} \in k[[z_1]]$. Also consider a finite generating system $g_1, \ldots, g_q$ for P and expressions for the g_k, where $1 \leq k \leq q$, as power series in $z_2, \ldots, z_m, x_1, \ldots, x_n$ with coefficients in $k[[z_1]]$:

$$g_k = \sum b_{k(i)(\ell)} z_2^{i_2} \ldots z_m^{i_m} x_1^{\ell_1} \ldots x_n^{\ell_n},$$

where $(i) := (i_2, \ldots, i_m)$, $(\ell) := (\ell_1, \ldots, \ell_n)$, and $b_{k(i)(\ell)} \in k[[z_1]]$. We take the subset $\Delta = \{a_{j(i)}, b_{k(i)(\ell)}\}$ of $k[[z_1]]$ and consider the discrete valuation domain:

$$V := k(z_1, \Delta) \cap k[[z_1]].$$

Since V is countably generated over $k(z_1)$, the field $k((z_1))$ has uncountable transcendence degree over $\mathcal{Q}(V) = k(z_1, \Delta)$. Moreover, by construction the ideal P is extended from $V[[z_2, \ldots, z_m]][[X]]$. Consider the embedding:

$$\psi : V[[z_2, \ldots, z_{m-s}, T]] \longrightarrow V[[z_2, \ldots, z_m]],$$

which is the restriction of φ above, so that $z_i \mapsto z_i$ for all $2 \leq i \leq m - s$ and $t_{m-i+1} \mapsto f_i$ for all i with $1 \leq i \leq s$.

Let σ be the extension of ψ to the power series rings:

$$\sigma : V[[z_2, \ldots, z_{m-s}, T]][[X]] \longrightarrow V[[z_2, \ldots, z_m]][[X]]$$

with $\sigma(x_i) = x_i$ for all i with $1 \leq i \leq n$.

Notice that ρ defined above is the completion $\widehat{\sigma}$ of the map σ, that is, the extension of σ to the completions. Consider the commutative diagram:

$$\begin{array}{ccc} k[[Z_1, T]][[X]] & \xrightarrow{\widehat{\sigma}=\rho} & k[[Z]][[X]] = \widehat{C} \\ \uparrow & & \uparrow \\ V[[z_2, \ldots, z_{m-s}, T]][[X]] & \xrightarrow{\sigma} & V[[z_2, \ldots, z_m]][[X]] \end{array} \tag{25.2.0}$$

where $\widehat{\sigma} = \rho$ is a finite map.

Recall that $\rho^{-1}(W) = (T, X)k[[Z_1, T, X]]$, and so $\rho^{-1}(P) \subseteq (T, X)k[[Z_1, T, X]]$ by Diagram 25.2.0. By Theorem 24.11, there exists a prime ideal Q_0 of the ring $k[[Z_1, T, X]]$ such that $\rho^{-1}(P) \subseteq Q_0$, $\operatorname{ht} Q_0 = n + m - 2$, and

$$Q_0 \cap V[[z_2, \ldots, z_{m-s}, T]][[X]] = \rho^{-1}(P) \cap V[[z_2, \ldots, z_{m-s}, T]][[X]].$$

By the "going-up theorem" [**103**, Theorem 9.4] , there is a prime ideal $Q \in \operatorname{Spec} \widehat{C}$ that lies over Q_0 and contains P. Moreover, Q also has height $n + m - 2$. The commutativity of Diagram 25.2.0 implies that

$$P_1 := P \cap V[[z_2, \ldots, z_{m-s}, T]]\,[[X]] \;\subseteq\; Q_1 := Q \cap V[[z_2, \ldots, z_{m-s}, T]]\,[[X]].$$

Consider the finite homomorphism:

$$\lambda : V[[z_2, \ldots, z_{m-s}]]\,[T]_{(Z_1,T)}[[X]] \longrightarrow V[[z_2, \ldots, z_{m-s}]]\,[z_{m-s+1}, \ldots, z_m]_{(Z)}[[X]]$$

(determined by $t_i \mapsto f_i$ for $1 \le i \le m$) and the commutative diagram:

$$\begin{array}{ccc} V[[z_2, \ldots, z_{m-s}]]\,[[T]]\,[[X]] & \xrightarrow{\;\sigma\;} & V[[z_2, \ldots, z_m]]\,[[X]] \\ \uparrow & & \uparrow \\ V[[z_2, \ldots, z_{m-s}]]\,[T]_{(Z_1,T)}[[X]] & \xrightarrow{\;\lambda\;} & V[[z_2, \ldots, z_{m-s}]]\,[z_{m-s+1}, \ldots, z_m]_{(Z)}[[X]]. \end{array}$$

Since $Q \cap V[[z_2, \ldots, z_{m-s}, T]]\,[[X]] = P \cap V[[z_2, \ldots, z_{m-s}, T]]\,[[X]]$ and since λ is a finite map we conclude that

$$\begin{aligned} Q_1 \cap\; & V[[z_2, \ldots, z_{m-s}]]\,[z_{m-s+1}, \ldots, z_m]_{(Z)}[[X]] \\ & = P_1 \cap V[[z_2, \ldots, z_m]]\,[z_{m-s+1}, \ldots, z_m]_{(Z)}[[X]]. \end{aligned}$$

Since $C \subseteq V[[z_2, \ldots, z_{m-s}]]\,[z_{m-s+1}, \ldots, z_m]_{(Z)}[[X]]$, we obtain that the intersection $Q \cap C = P \cap C = (0)$. This completes the proof of Theorem 25.2. □

Remark 25.3. With B and C as in Sections 25.1 and 25.2, we have

$$B = k[[X]]\,[Y]_{(X,Y)} \hookrightarrow k[Y]_{(Y)}[[X]] = C \quad \text{and} \quad \widehat{B} = k[[X, Y]] = \widehat{C}.$$

Thus for $P \in \operatorname{Spec} k[[X, Y]]$, if $P \cap C = (0)$, then $P \cap B = (0)$. By Theorems 25.1 and 25.2, each prime of $k[[X, Y]]$ maximal in the generic formal fiber of B or C has height $n + m - 2$. Therefore each $P \in \operatorname{Spec} k[[X, Y]]$ maximal with respect to $P \cap C = (0)$ is also maximal with respect to $P \cap B = (0)$. However, if $n + m \ge 3$, the generic fiber of $B \hookrightarrow C$ is nonzero (see Propositions 26.24 and 26.26 of Chapter 26), and so there exist primes of $k[[X, Y]]$ maximal in the generic formal fiber of B that are not in the generic formal fiber of C.

25.3. Weierstrass implications for the localized polynomial ring A

Let n be a positive integer, let $X = \{x_1, \ldots, x_n\}$ be a set of n variables over a field k, and let $A := k[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)} = k[X]_{(X)}$ denote the localized polynomial ring in these n variables over k. Then the completion of A is $\widehat{A} = k[[X]]$.

Theorem 25.4. *For the localized polynomial ring $A = k[X]_{(X)}$ defined above, if Q is an ideal of $\widehat{A}$ maximal with respect to $Q \cap A = (0)$, then Q is a prime ideal of height $n - 1$.*

Proof. It is clear that Q as described in the statement is a prime ideal. Also the assertion holds for $n = 1$. Thus we assume $n \ge 2$. By Proposition 24.17, there exists a nonzero prime $\mathbf{p}$ in $k[[x_1, x_2]]$ such that $\mathbf{p} \cap k[x_1, x_2]_{(x_1,x_2)} = (0)$. It follows that $\mathbf{p}\widehat{A} \cap A = (0)$. Thus the generic formal fiber of A is nonzero.

Let $P \in \operatorname{Spec} \widehat{A}$ be a nonzero prime ideal with $P \cap A = (0)$ and $\operatorname{ht} P = r < n-1$. We construct $Q \in \operatorname{Spec} \widehat{A}$ of height $n - 1$ with $P \subseteq Q$ and $Q \cap A = (0)$. By

Proposition 24.10, there exists a change of variables $x_1 \mapsto z_1, \ldots, x_n \mapsto z_n$ and polynomials

$$\begin{aligned} f_1 &\in k[[z_1, \ldots, z_{n-r}]]\,[z_{n-r+1}, \ldots, z_n] && \text{monic in } z_n \\ f_2 &\in k[[z_1, \ldots, z_{n-r}]]\,[z_{n-r+1}, \ldots, z_{n-1}] && \text{monic in } z_{n-1}, \text{ etc} \\ &\vdots \\ f_r &\in k[[z_1, \ldots, z_{n-r}]]\,[z_{n-r+1}] && \text{monic in } z_{n-r+1}, \end{aligned}$$

so that P is a minimal prime of $(f_1, \ldots, f_r)\widehat{A}$ and P is extended from

$$R := k[[z_1, \ldots, z_{n-r}]]\,[z_{n-r+1}, \ldots, z_n].$$

Let $P_0 := P \cap R$ and extend $f_1, \ldots, f_r$ to a system of generators of P_0, say:

$$P_0 = (f_1, \ldots, f_r, g_1, \ldots, g_s)R.$$

Using an argument similar to that in the proof of Theorem 24.11, write

$$f_j = \sum_{(i)\in\mathbb{N}^{n-1}} a_{j,(i)} z_2^{i_2} \ldots z_n^{i_n} \quad \text{and} \quad g_k = \sum_{(i)\in\mathbb{N}^{n-1}} b_{k,(i)} z_2^{i_2} \ldots z_n^{i_n},$$

where $a_{j,(i)}, b_{k,(i)} \in k[[z_1]]$. Let

$$V_0 := k(z_1, a_{j,(i)}, b_{k,(i)}) \cap k[[z_1]].$$

Then V_0 is a discrete rank-one valuation domain with completion $k[[z_1]]$, and $k((z_1))$ has uncountable transcendence degree over the field of fractions $\mathcal{Q}(V_0)$ of V_0. Let $\gamma_3, \ldots, \gamma_{n-r} \in k[[z_1]]$ be algebraically independent over $\mathcal{Q}(V_0)$ and define

$$\mathbf{q} := (z_3 - \gamma_3 z_2, z_4 - \gamma_4 z_2, \ldots, z_{n-r} - \gamma_{n-r} z_2)k[[z_1, \ldots, z_{n-r}]].$$

We see that $\mathbf{q} \cap V_0[[z_2, \ldots, z_{n-r}]] = (0)$ by an argument similar to that in [**102**] and in Claim 24.12. Let $R_1 := V_0[[z_2, \ldots, z_{n-r}]]\,[z_{n-r+1}, \ldots, z_n]$, let $P_1 := P \cap R_1$ and consider the commutative diagram:

$$\begin{array}{ccc} k[[z_1, \ldots, z_{n-r}]] & \longrightarrow & R/P_0 \\ \uparrow & & \uparrow \\ V_0[[z_2, \ldots, z_{n-r}]] & \longrightarrow & R_1/P_1 \end{array}$$

The horizontal maps are injective finite integral extensions. Let W be a minimal prime of $(\mathbf{q}, P)\widehat{A}$. Then $\operatorname{ht} W = n - 2$ and $\mathbf{q} \cap V_0[[z_2, \ldots, z_{n-r}]] = (0)$ implies that $W \cap R_1 = P_1$. Thus the prime ideal $W \in \operatorname{Spec} \widehat{A}$ satisfies $\operatorname{ht} W = n - 2$, $W \cap A = (0)$ and $P \subseteq W$. Since $f_1, \ldots, f_r \in W$ and since $\widehat{A} = k[[z_1, \ldots, z_n]]$ is the $(f_1, \ldots, f_r)$-adic completion of $k[[z_1, \ldots, z_{n-r}]]\,[z_{n-r+1}, \ldots, z_n]$, the prime ideal W is extended from $k[[z_1, \ldots, z_{n-r}]]\,[z_{n-r+1}, \ldots, z_n]$.

We claim that W is actually extended from $k[[z_1, z_2]]\,[z_3, \ldots, z_n]$. To see this, let $g \in W \cap k[[z_1, \ldots, z_{n-r}]]\,[z_{n-r+1}, \ldots, z_n]$ and write:

$$g = \sum_{(i)} a_{(i)} z_{n-r+1}^{i_{n-r+1}} \ldots z_n^{i_n} \in k[[z_1, \ldots, z_{n-r}]]\,[z_{n-r+1}, \ldots, z_n],$$

where the sum is over all $(i) = (i_{n-r}, \ldots, i_n)$ and $a_{(i)} \in k[[z_1, \ldots, z_{n-r}]]$. For all $a_{(i)}$ by Weierstrass, that is, by Theorem 24.7, we can write

$$a_{(i)} = (z_{n-r} - \gamma_{n-r} z_2)h_{(i)} + q_{(i)},$$

where $h_{(i)} \in k[[z_1, \ldots z_{n-r}]]$ and $q_{(i)} \in k[[z_1, \ldots, z_{n-r-1}]]$. If $n - r > 3$, we write

$$q_{(i)} = (z_{n-r-1} - \gamma_{n-r-1} z_2) h'_{(i)} + q'_{(i)},$$

where $h'_{(i)} \in k[[z_1, \dots z_{n-r-1}]]$ and $q'_{(i)} \in k[[z_1, \dots, z_{n-r-2}]]$. In this way we replace a generating set for W in $k[[z_1, \dots, z_{n-r}]]\,[z_{n-r+1}, \dots, z_n]$ by a generating set for W in $k[[z_1, z_2]]\,[z_3, \dots, z_n]$.

In particular, we can replace the elements $f_1, \dots, f_r$ by elements:

$$\begin{aligned}
&h_1 \in k[[z_1, z_2]]\,[z_3, \dots, z_n] && \text{monic in } z_n \\
&h_2 \in k[[z_1, z_2]]\,[z_3, \dots, z_{n-1}] && \text{monic in } z_{n-1}, \text{ etc} \\
&\quad \vdots \\
&h_r \in k[[z_1, z_2]]\,[z_3 \dots, z_{n-r+1}] && \text{monic in } z_{n-r+1}
\end{aligned}$$

and set $h_{r+1} = z_3 - \gamma_3 z_2, \dots, h_{n-2} = z_{n-r} - \gamma_{n-r} z_2$, and then extend to a generating set $h_1, \dots, h_{n+s-2}$ for

$$W_0 = W \cap k[[z_1, z_2]]\,[z_3, \dots, z_n]$$

such that $W_0 \widehat{A} = W$. Consider the coefficients in $k[[z_1]]$ of the h_j:

$$h_j = \sum_{(i)} c_{j(i)} z_2^{i_2} \dots z_n^{i_n}$$

with $c_{j(i)} \in k[[z_1]]$. The set $\{c_{j(i)}\}$ is countable. Define

$$V := \mathcal{Q}(V_0)(\{c_{j(i)}\}) \cap k[[z_1]]$$

Then V is a rank-one discrete valuation domain that is countably generated over $k[z_1]_{(z_1)}$ and W is extended from $V[[z_2]]\,[z_3, \dots, z_n]$.

We may also write each h_i as a polynomial in $z_3, \dots, z_n$ with coefficients in $V[[z_2]]$:

$$h = \sum \omega_{(i)} z_3^{i_3} \dots z_n^{i_n}$$

with $\omega_{(i)} \in V[[z_2]] \subseteq k[[z_1, z_2]]$. By Valabrega's Theorem 4.8, the integral domain

$$D := \mathcal{Q}(V)(z_2, \{\omega_{(i)}\}) \cap k[[z_1, z_2]]$$

is a two-dimensional regular local domain with completion $\widehat{D} = k[[z_1, z_2]]$. Let $W_1 := W \cap D[z_3, \dots, z_n]$. Then $W_1 \widehat{A} = W$. We have shown in Section 24.3 that there exists a prime element $q \in k[[z_1, z_2]]$ with $qk[[z_1, z_2]] \cap D = (0)$. Consider the finite extension

$$D \longrightarrow D[z_3, \dots, z_n]/W_1.$$

Let $Q \in \operatorname{Spec} \widehat{A}$ be a minimal prime of $(q, W)\widehat{A}$. Since $\operatorname{ht} W = n - 2$ and $q \notin W$, $\operatorname{ht} Q = n - 1$. Moreover, $P \subseteq W$ implies $P \subseteq Q$. We claim that

$$Q \cap D[z_3, \dots, z_n] = W_1 \quad \text{and therefore} \quad Q \cap A = (0).$$

To see this consider the commutative diagram:

$$\begin{array}{ccc}
k[[z_1, z_2]] & \longrightarrow & k[[z_1, \dots, z_n]]/W \\
\uparrow & & \uparrow \\
D & \longrightarrow & D[z_3, \dots, z_n]/W_1\,,
\end{array}$$

which has injective finite horizontal maps. Since $qk[[z_1, z_2]] \cap D = (0)$, it follows that $Q \cap D[z_3, \ldots, z_n] = W_1$. This completes the proof of Theorem 25.4. □

25.4. Generic fibers of power series ring extensions

In this section we apply the Weierstrass machinery from Section 24.2 to the generic fiber rings of power series extensions.

THEOREM 25.5. *Let $n \geq 2$ be an integer and let $y, x_1, \ldots, x_n$ be variables over the field k. Let $X = \{x_1, \ldots, x_n\}$ and let R_1 be the formal power series ring $k[[X]]$. Consider the extension $R_1 \hookrightarrow R_1[[y]] = R$. Let $U = R_1 \setminus (0)$. For $P \in \operatorname{Spec} R$ such that $P \cap U = \emptyset$, we have*:

(1) *If $P \not\subseteq XR$, then $\dim R/P = n$ and P is maximal in the generic fiber $U^{-1}R$.*
(2) *If $P \subseteq XR$, then there exists $Q \in \operatorname{Spec} R$ such that $P \subseteq Q$, $\dim R/Q = 2$ and Q is maximal in the generic fiber $U^{-1}R$.*

If $n > 2$ for each prime ideal Q maximal in the generic fiber $U^{-1}R$, we have

$$\dim R/Q = \begin{cases} n & \text{and } R_1 \hookrightarrow R/Q \text{ is finite, or} \\ 2 & \text{and } Q \subset XR. \end{cases}$$

PROOF. Let $P \in \operatorname{Spec} R$ be such that $P \cap U = \emptyset$ or equivalently $P \cap R_1 = (0)$. Then R_1 embeds in R/P. If $\dim(R/P) \leq 1$, then the maximal ideal of R_1 generates an ideal primary for the maximal ideal of R/P. By Theorem 3.9, R/P is finite over R_1, and so $\dim R_1 = \dim(R/P)$, a contradiction. Thus $\dim(R/P) \geq 2$.

If $P \not\subseteq XR$, then there exists a prime element $f \in P$ that contains a term y^s for some positive integer s. By Weierstrass, that is, by Theorem 24.7, it follows that $f = g\epsilon$, where $g \in k[[X]][y]$ is a nonzero monic polynomial in y and ϵ is a unit of R. We have $fR = gR \subseteq P$ is a prime ideal and $R_1 \hookrightarrow R/gR$ is a finite integral extension. Since $P \cap R_1 = (0)$, we must have $gR = P$.

If $P \subseteq XR$ and $\dim(R/P) > 2$, then Theorem 24.11 implies there exists $Q \in \operatorname{Spec} R$ such that $\dim(R/Q) = 2$, $P \subset Q \subset XR$ and $P \cap R_1 = (0) = Q \cap R_1$, and so P is not maximal in the generic fiber. Thus $Q \in \operatorname{Spec} R$ maximal in the generic fiber of $R_1 \hookrightarrow R$ implies that the dimension of $\dim(R/Q)$ is 2, or equivalently that $\operatorname{ht} Q = n - 1$. □

THEOREM 25.6. *Let n and m be positive integers, and let $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$ be sets of independent variables over the field k. Consider the formal power series rings $R_1 = k[[X]]$ and $R = k[[X, Y]]$ and the extension $R_1 \hookrightarrow R_1[[Y]] = R$. Let $U = R_1 \setminus (0)$. Let $Q \in \operatorname{Spec} R$ be maximal with respect to $Q \cap U = \emptyset$. If $n = 1$, then $\dim R/Q = 1$ and $R_1 \hookrightarrow R/Q$ is finite.*

If $n \geq 2$, there are two possibilities:

(1) *$R_1 \hookrightarrow R/Q$ is finite, in which case $\dim R/Q = \dim R_1 = n$, or*
(2) *$\dim R/Q = 2$.*

PROOF. First assume $n = 1$, and let $x = x_1$. Since Q is maximal with respect to $Q \cap U = \emptyset$, for each $P \in \operatorname{Spec} R$ with $Q \subsetneq P$ we have $P \cap U$ is nonempty and therefore $x \in P$. It follows that $\dim R/Q = 1$, for otherwise,

$$Q = \bigcap \{P \mid P \in \operatorname{Spec} R \text{ and } Q \subsetneq P\},$$

which implies $x \in Q$. By Theorem 3.9, $R_1 \hookrightarrow R/Q$ is finite.

It remains to consider the case where $n \geq 2$. We proceed by induction on m. Theorem 25.5 yields the assertion for $m = 1$. Suppose $Q \in \operatorname{Spec} R$ is maximal with respect to $Q \cap U = \emptyset$. As in the proof of Theorem 25.5, we have $\dim R/Q \geq 2$. If $Q \subseteq (X, y_1, \ldots, y_{m-1})R$, then by Theorem 24.11 with $R_0 = k[y_m]_{(y_m)}[[X, y_1, \ldots, y_{m-1}]]$, there exists $Q' \in \operatorname{Spec} R$ with $Q \subseteq Q'$, $\dim R/Q' = 2$, and $Q \cap R_0 = Q' \cap R_0$. Since $R_1 \subseteq R_0$, we have $Q' \cap U = \emptyset$. Since Q is maximal with respect to $Q \cap U = \emptyset$, we have $Q = Q'$, and so $\dim R/Q = 2$.

Otherwise, if $Q \not\subseteq (X, y_1, \ldots, y_{m-1})R$, then there exists a prime element $f \in Q$ that contains a term y_m^s for some positive integer s. Let $R_2 = k[[X, y_1, \ldots, y_{m-1}]]$. By Weierstrass, it follows that $f = g\epsilon$, where $g \in R_2[y_m]$ is a nonzero monic polynomial in y_m and ϵ is a unit of R. We have $fR = gR \subseteq Q$ is a prime ideal and $R_2 \hookrightarrow R/gR$ is a finite integral extension. Thus $R_2/(Q \cap R_2) \hookrightarrow R/Q$ is an integral extension. It follows that $Q \cap R_2$ is maximal in R_2 with respect to being disjoint from U. By induction $\dim R_2/(Q \cap R_2)$ is either n or 2. Since R/Q is integral over $R_2/(Q \cap R_2)$, $\dim R/Q$ is either n or 2. □

Remark 25.7. In the notation of Theorem 24.3, Theorem 25.6 proves the second part of the theorem, since $\dim R = n + m$. Thus if $n = 1$, $\operatorname{ht} Q = m$. If $n \geq 2$, the two cases are (i) $\operatorname{ht} Q = m$ and (ii) $\operatorname{ht} Q = n + m - 2$, as in (a) and (b) of Theorem 24.3, part 4.

Using the TGF terminology of Definition 24.6, we have the following corollary to Theorem 25.6.

Corollary 25.8. *With the notation of Theorem 25.6, assume $P \in \operatorname{Spec} R$ is such that $R_1 \hookrightarrow R/P =: S$ is a TGF extension. Then $\dim S = \dim R_1 = n$ or $\dim S = 2$.*

25.5. Formal fibers of prime ideals in polynomial rings

In this section we present a generalization of Theorem 25.4 above. We also discuss in this section related results concerning generic formal fibers of certain extensions of mixed polynomial-power series rings.

We were inspired to revisit and generalize Theorem 25.4 by Youngsu Kim. His interest in formal fibers and the material in [**75**] inspired us to consider the second question below.

Questions 25.9. For $n \in \mathbb{N}$, let $x_1, \ldots, x_n$ be indeterminates over a field k and let $R = k[x_1, \ldots, x_n]_{(x_1, \ldots, x_n)}$ denote the localized polynomial ring with maximal ideal $\mathbf{m} = (x_1, \ldots, x_n)R$. Let $\widehat{R}$ be the $\mathbf{m}$-adic completion of R.

(1) For $P \in \operatorname{Spec} R$, what is the dimension of the generic formal fiber ring $\operatorname{Gff}(R/P)$?
(2) What heights are possible for maximal ideals of the ring $\operatorname{Gff}(R/P)$?

In connection with Question 25.9.1, for $P \in \operatorname{Spec} R$, the ring R/P is essentially finitely generated over a field and $\dim(\operatorname{Gff}(R/P)) = n - 1 - \operatorname{ht} P$ by a result of Matumura [**102**, Theorem 2 and Corollary, p. 263].

As a sharpening of Matsumura's result and of Theorem 25.4, we prove Theorem 25.10; see also Theorem 25.15. Thus the answer to Question 25.9.2 is that the height of *every* maximal ideal of $\operatorname{Gff}(R/P)$ is $n - 1 - \operatorname{ht} P$.

THEOREM 25.10. *Let S be a local domain essentially finitely generated over a field; thus $S = k[s_1, \ldots, s_r]_{\mathbf{p}}$, where k is a field, $r \in \mathbb{N}$, the elements s_i are in S and $\mathbf{p}$ is a prime ideal of the finitely generated k-algebra $k[s_1, \ldots, s_r]$. Let $\mathbf{n} := \mathbf{p}S$ and let $\widehat{S}$ denote the $\mathbf{n}$-adic completion of S. Then every maximal ideal of* Gff(S) *has height* $\dim S - 1$. *Equivalently, if $Q \in \operatorname{Spec} \widehat{S}$ is maximal with respect to $Q \cap S = (0)$, then* $\operatorname{ht} Q = \dim S - 1$.

This result is relabeled as Theorem 25.15 and the proof is given in Section 25.6.

We make the following observations concerning injective local maps of Noetherian local rings:

DISCUSSION 25.11. Let $\phi : (R, \mathbf{m}) \hookrightarrow (S, \mathbf{n})$ be an injective local map of the Noetherian local ring $(R, \mathbf{m})$ into a Noetherian local ring $(S, \mathbf{n})$. Let $\widehat{R} = \varprojlim_n R/\mathbf{m}^n$ denote the $\mathbf{m}$-adic completion of R and let $\widehat{S} = \varprojlim_n S/\mathbf{n}^n$ denote the $\mathbf{n}$-adic completion of S. For each $n \in \mathbb{N}$, we have $\mathbf{m}^n \subseteq \mathbf{n}^n \cap R$. Hence there exists a map

$$\phi_n : R/\mathbf{m}^n \;\to\; R/(\mathbf{n}^n \cap R) \;\hookrightarrow\; S/\mathbf{n}^n, \qquad \text{for each } \; n \in \mathbb{N}.$$

The family of maps $\{\phi_n\}_{n\in\mathbb{N}}$ determines a unique map $\widehat{\phi} : \widehat{R} \to \widehat{S}$.

Since $\mathbf{m}^n \subseteq \mathbf{n}^n \cap R$, the $\mathbf{m}$-adic topology on R is the subspace topology from S if and only if for each positive integer n there exists a positive integer s_n such that $\mathbf{n}^{s_n} \cap R \subseteq \mathbf{m}^n$. Since $R/\mathbf{m}^n$ is Artinian, the descending chain of ideals $\{\mathbf{m}^n + (\mathbf{n}^s \cap R)\}_{s\in\mathbb{N}}$ stabilizes. The ideal $\mathbf{m}^n$ is closed in the $\mathbf{m}$-adic topology, and it is closed in the subspace topology if and only if $\bigcap_{s\in\mathbb{N}}(\mathbf{m}^n + (\mathbf{n}^s \cap R)) = \mathbf{m}^n$. Hence $\mathbf{m}^n$ is closed in the subspace topology if and only if there exists a positive integer s_n such that $\mathbf{n}^{s_n} \cap R \subseteq \mathbf{m}^n$.

Thus the subspace topology from S is the same as the $\mathbf{m}$-adic topology on R if and only if $\widehat{\varphi}$ is injective.

25.6. Gff(R) and Gff(S) for S an extension domain of R

Theorem 25.12 is useful in considering properties of generic formal fiber rings.

THEOREM 25.12. *Let $\phi : (R, \mathbf{m}) \hookrightarrow (S, \mathbf{n})$ be an injective local map of Noetherian local integral domains. Consider the following properties*:

(1) $\mathbf{m}S$ *is $\mathbf{n}$-primary, and $S/\mathbf{n}$ is finite algebraic over $R/\mathbf{m}$.*
(2) $R \hookrightarrow S$ *is a TGF-extension and* $\dim R = \dim S$*; see Definition 24.6.*
(3) R *is analytically irreducible.*
(4) R *is analytically normal and S is universally catenary.*
(5) *All maximal ideals of* Gff(R) *have the same height.*

If items 1, 2 and 3 hold, then $\dim \operatorname{Gff}(R) = \dim \operatorname{Gff}(S)$. *If, in addition, items 4 and 5 hold, then the maximal ideals of* Gff(S) *all have height* $h = \dim \operatorname{Gff}(R)$.

PROOF. Let $\widehat{R}$ and $\widehat{S}$ denote the $\mathbf{m}$-adic completion of R and $\mathbf{n}$-adic completion of S respectively, and let $\widehat{\phi} : \widehat{R} \to \widehat{S}$ be the natural extension of φ as defined above.

Consider the commutative diagram

$$\begin{array}{ccc} \widehat{R} & \xrightarrow{\widehat{\phi}} & \widehat{S} \\ \uparrow & & \uparrow \\ R & \xrightarrow{\phi} & S\,, \end{array} \tag{25.12.a}$$

where the vertical maps are the natural inclusion maps to the completion. Assume items 1, 2 and 3 hold. Item 1 implies that $\widehat{S}$ is a finite $\widehat{R}$-module with respect to the map $\widehat{\phi}$ by [**103**, Theorem 8.4]. By item 2, we have $\dim \widehat{R} = \dim R = \dim S = \dim \widehat{S}$. Item 3 says that $\widehat{R}$ is an integral domain. It follows that the map $\widehat{\phi} : \widehat{R} \hookrightarrow \widehat{S}$ is injective. Let $Q \in \operatorname{Spec} \widehat{S}$ and let $P = Q \cap \widehat{R}$. Since $R \hookrightarrow S$ is a TGF-extension, by item 2, commutativity of Diagram 25.12.a implies that

$$Q \cap S = (0) \iff P \cap R = (0).$$

Therefore $\widehat{\phi}$ induces an injective finite map $\operatorname{Gff}(R) \hookrightarrow \operatorname{Gff}(S)$. We conclude that $\dim \operatorname{Gff}(R) = \dim \operatorname{Gff}(S)$.

Assume in addition that items 4 and 5 hold, and let $h = \dim \operatorname{Gff}(R)$. The assumption that S is universally catenary implies that $\dim(\widehat{S}/\mathbf{q}) = \dim S$ for each minimal prime $\mathbf{q}$ of $\widehat{S}$; see [**103**, Theorem 31.7]. Since $\frac{\widehat{R}}{\mathbf{q} \cap \widehat{R}} \hookrightarrow \frac{\widehat{S}}{\mathbf{q}}$ is an integral extension, we have $\mathbf{q} \cap \widehat{R} = (0)$. The assumption that $\widehat{R}$ is a normal domain implies that the going-down theorem holds for $\widehat{R} \hookrightarrow \widehat{S}/\mathbf{q}$ by [**103**, Theorem 9.4(ii)]. Therefore for each $Q \in \operatorname{Spec} \widehat{S}$ we have $\operatorname{ht} Q = \operatorname{ht} P$, where $P = Q \cap \widehat{R}$. Hence if $\operatorname{ht} P = h$ for each $P \in \operatorname{Spec} \widehat{R}$ that is maximal with respect to $P \cap R = (0)$, then $\operatorname{ht} Q = h$ for each $Q \in \operatorname{Spec} \widehat{S}$ that is maximal with respect to $Q \cap S = (0)$. This completes the proof of Theorem 25.12. □

Remark 25.13. We would like to thank Rodney Sharp and Roger Wiegand for their interest in Theorem 25.12. The hypotheses of Theorem 25.12 do not necessarily imply that S is a finite R-module, or even that S is essentially finitely generated over R. If $\phi : (R, \mathbf{m}) \hookrightarrow (T, \mathbf{n})$ is an extension of rank one discrete valuation rings (DVR's) such that $T/\mathbf{n}$ is finite algebraic over $R/\mathbf{m}$, then for every field F that contains R and is contained in the field of fractions of $\widehat{T}$, the ring $S := \widehat{T} \cap F$ is a DVR such that the extension $R \hookrightarrow S$ satisfies the hypotheses of Theorem 25.12.

As a specific example where S is essentially finite over R, but not a finite R-module, let $R = \mathbb{Z}_{5\mathbb{Z}}$, the integers localized at the prime ideal generated by 5, and let A be the integral closure of $\mathbb{Z}_{5\mathbb{Z}}$ in $\mathbb{Q}[i]$. Then A has two maximal ideals lying over $5R$, namely $(1 + 2i)A$ and $(1 - 2i)A$. Let $S = A_{(1+2i)A}$. Then the extension $R \hookrightarrow S$ satisfies the hypotheses of Theorem 25.12. Since S properly contains A, and every element in the field of fractions of A that is integral over R is contained in A, it follows that S is not finitely generated as an R-module. In Remark 25.19, we describe examples in higher dimension where S is not a finite R-module.

Discussion 25.14. As in the statement of Theorem 25.10, let $S = k[z_1, \ldots, z_r]_{\mathbf{p}}$ be a local domain essentially finitely generated over a field k. We observe that S is a localization at a maximal ideal of an integral domain that is a finitely generated algebra over an extension field F of k.

To see this, let $A = k[x_1, \ldots, x_r]$ be a polynomial ring in r variables over k, and let Q denote the kernel of the k-algebra homomorphism of A onto $k[z_1, \ldots, z_r]$ defined by mapping $x_i \mapsto z_i$ for each i with $1 \leq i \leq r$. Using permutability of localization and residue class formation, there exists a prime ideal $N \supset Q$ of A such that $S = A_N/QA_N$. A version of Noether normalization as in [**101**, Theorem 24 (14.F) page 89] states that, if $\operatorname{ht} N = s$, then there exist elements $y_1, \ldots, y_r$ in A such that A is integral over $B = k[y_1, \ldots, y_r]$ and $N \cap B = (y_1, \ldots, y_s)B$. It follows that $y_1, \ldots, y_r$ are algebraically independent over k and A is a finitely generated B-module. Let F denote the field $k(y_{s+1}, \ldots, y_r)$, and let U denote the multiplicatively closed set $k[y_{s+1}, \ldots, y_r] \setminus (0)$. Then $U^{-1}B$ is the polynomial ring $F[y_1, \ldots, y_s]$, and $U^{-1}A := C$ is a finitely generated $U^{-1}B$-module. Moreover NC is a prime ideal of C such that

$$NC \cap U^{-1}B = (y_1, \ldots, y_s)U^{-1}B = (y_1, \ldots, y_s)F[y_1, \ldots, y_s]$$

is a maximal ideal of $U^{-1}B$, and $(y_1, \ldots, y_s)C$ is primary for the maximal ideal of C. Hence NC is a maximal ideal of C and $S = C_{NC}/QC_{NC}$ is a localization of the finitely generated F-algebra $D := C/QC$ at the maximal ideal NC/QC.

Therefore S is a localization of an integral domain D at a maximal ideal of D and D is a finitely generated algebra over an extension field F of k.

THEOREM 25.15. *Let $A = k[s_1, \ldots, s_r]$ be an integral domain that is a finitely generated algebra over a field k, let N be a maximal ideal of A, and let $Q \subset N$ be a prime ideal of A. Set $S = A_N/QA_N$ and $\mathbf{n} = NS$. If $\dim S = d$, then every maximal ideal of the generic formal fiber ring* Gff(S) *has height $d - 1$.*

PROOF. Choose $x_1, \ldots, x_d$ in $\mathbf{n}$ such that $x_1, \ldots, x_d$ are algebraically independent over k and $(x_1, \ldots, x_d)S$ is $\mathbf{n}$-primary. Set $R = k[x_1, \ldots, x_d]_{(x_1, \ldots, x_d)}$, a localized polynomial ring over k, and let $\mathbf{m} = (x_1, \ldots, x_d)R$. To prove Theorem 25.15, it suffices to show that the inclusion map $\phi : R \hookrightarrow S$ satisfies items 1 - 5 of Theorem 25.12. By construction ϕ is an injective local homomorphism and $\mathbf{m}S$ is $\mathbf{n}$-primary. Also $R/\mathbf{m} = k$ and $S/\mathbf{n} = A/N$ is a field that is a finitely generated k-algebra and hence a finite algebraic extension field of k; see [**103**, Theorem 5.2]. Therefore item 1 holds. Since $\dim S = d = \dim A/Q$, the field of fractions of S has transcendence degree d over the field k. Therefore S is algebraic over R. It follows that $R \hookrightarrow S$ is a TGF extension. Thus item 2 holds. Since R is a regular local ring, R is analytically irreducible and analytically normal. Since S is essentially finitely generated over a field, S is universally catenary. Therefore items 3 and 4 hold. Since R is a localized polynomial ring in d variables, Theorem 25.4 implies that every maximal ideal of Gff(R) has height $d - 1$. By Theorem 25.12, every maximal ideal of Gff(S) has height $d - 1$. □

Other results on generic formal fibers

Theorems 25.1 and 25.3 give descriptions of the generic formal fiber ring of mixed polynomial-power series rings. We use Theorems 25.12, 25.1 and 25.3 to deduce Theorem 25.16.

THEOREM 25.16. *Let R be either $k[[X]][Y]_{(X,Y)}$ or $k[Y]_{(Y)}[[X]]$, where m and n are positive integers and $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$ are sets of independent variables over a field k. Let $\mathbf{m}$ denote the maximal ideal $(X, Y)R$ of R. Let $(S, \mathbf{n})$ be a Noetherian local integral domain containing R such that:*

(1) *The injection* $\varphi : (R, \mathbf{m}) \hookrightarrow (S, \mathbf{n})$ *is a local map.*
(2) $\mathbf{m}S$ *is* $\mathbf{n}$*-primary, and* $S/\mathbf{n}$ *is finite algebraic over* $R/\mathbf{m}$*.*
(3) $R \hookrightarrow S$ *is a TGF-extension and* $\dim R = \dim S$*.*
(4) S *is universally catenary.*

Then every maximal ideal of the generic formal fiber ring $\mathrm{Gff}(S)$ *has height* $n+m-2$*. Equivalently, if* P *is a prime ideal of* $\widehat{S}$ *maximal with respect to* $P \cap S = (0)$*, then* $\mathrm{ht}(P) = n + m - 2$*.*

PROOF. We check that the conditions 1–5 of Theorem 25.12 are satisfied for R and S and the injection φ. Since the completion of R is $k[[X, Y]]$, R is analytically normal, and so also analytically irreducible. Items 1–4 of Theorem 25.16 ensure that the rest of conditions 1–4 of Theorem 25.12 hold. By Theorems 25.1 and 25.3, every maximal ideal of $\mathrm{Gff}(R)$ has height $n+m-2$, and so condition 5 of Theorem 25.12 holds. Thus we have every maximal ideal of $\mathrm{Gff}(S)$ has height $n + m - 2$ by Theorem 25.12. □

REMARK 25.17. Let k, X, Y, and R be as in Theorem 25.16. Let A be a finite integral extension domain of R and let S be the localization of A at a maximal ideal. As observed in the proof of Theorem 25.16, R is a local analytically normal integral domain. Since S is a localization of a finitely generated R-algebra and R is universally catenary, it follows that S is universally catenary. We also have that conditions 1–3 of Theorem 25.16 hold. Thus the extension $R \hookrightarrow S$ satisfies the hypotheses of Theorem 25.16. Hence every maximal ideal of $\mathrm{Gff}(S)$ has height $n + m - 2$.

Example 25.18 is an application of Theorem 25.16 and Remark 25.17.

EXAMPLE 25.18. Let k, X, Y, and R be as in Theorem 25.16. Let K denote the field of fractions of R, and let L be a finite algebraic extension field of K. Let A be the integral closure of R in L, and let S be a localization of A at a maximal ideal. The ring R is a Nagata ring by a result of Marot; see [**98**, Prop.3.5]. Therefore A is a finite integral extension of R and the conditions of Remark 25.17 apply to show that every maximal ideal of $\mathrm{Gff}(S)$ has height $n + m - 2$.

REMARK 25.19. With notation as in Example 25.18, since the sets X and Y are nonempty, the field K is a simple transcendental extension of a subfield. It follows that the regular local ring R is not Henselian; see the book of Berger, Kiehl, Kunz and Nastoid [**16**, Satz 2.3.11, p. 60] and the paper of Schmidt [**139**]. Hence there exists a finite algebraic field extension L/K such that the integral closure A of R in L has more than one maximal ideal. It follows that the localization S of A at any one of these maximal ideals is not a finite R-module, and gives an example $R \hookrightarrow S$ that satisfies the hypotheses of Theorem 25.12.

Exercise

(1) Let x and y be indeterminates over a field k. Let $R = k[[x]][y]$ and let $\tau \in k[[y]]$ be such that y and τ are algebraically independent over k. Then we have the embedding $R = k[[x]][y] \hookrightarrow k[[x, y]]$. For $P := (x - \tau)k[[x, y]]$, prove that
   (a) $P \cap k[x, y] = (0)$, but
   (b) $P \cap R \neq (0)$.

   Suggestion: For item b, apply Theorem 3.9.

CHAPTER 26

Mixed polynomial-power series rings and relations among their spectra

We are interested in the following sequence of two-dimensional nested mixed polynomial-power series rings:

$$A := k[x,y] \hookrightarrow B := k[[y]]\,[x] \hookrightarrow C := k[x]\,[[y]] \hookrightarrow E := k[x,1/x]\,[[y]], \tag{26.0.1}$$

where k is a field and x and y are indeterminates over k. [1] That is, A is the usual polynomial ring in the two variables x and y over k, the ring B is all polynomials in the variable x with coefficients in the power series ring $k[[y]]$, the ring C is all power series in the variable y over the polynomial ring $k[x]$, and E is power series in the variable y over the ring $k[x,1/x]$. In Sequence 26.0.1 all the maps are flat; see Propositions 2.31.4 and 3.2.2. We also consider Sequence 26.0.2 consisting of embeddings between the rings C and E of Sequence 26.0.1:

$$C \hookrightarrow D_1 := k[x]\,[[y/x]] \hookrightarrow \cdots \hookrightarrow D_n := k[x]\,[[y/x^n]] \hookrightarrow \cdots \hookrightarrow E. \tag{26.0.2}$$

With regard to Sequence 26.0.2, for n a positive integer, the map $C \hookrightarrow D_n$ is not flat, since $\mathrm{ht}(xD_n \cap C) = 2$ but $\mathrm{ht}(xD_n) = 1$; see Proposition 2.31.10. The map $D_n \hookrightarrow E$ is a localization followed by an ideal-adic completion of a Noetherian ring and therefore is flat. We discuss the spectra of the rings in Sequences 26.0.1 and 26.0.1, and we consider the maps induced on the spectra by the inclusion maps on the rings. For example, we determine whether there exist nonzero primes of one of the larger rings that intersect a smaller ring in zero.

26.1. Two motivations

We were led to consider these rings by questions that came up in two contexts.

The first motivation is a question about formal schemes that is discussed in the introduction to the paper [**10**] by Alonzo-Tarrio, Jeremias-Lopez and Lipman:

Question 26.1. If a map between Noetherian formal schemes can be factored as a closed immersion followed by an open immersion, can this map also be factored as an open immersion followed by a closed immersion? [2]

Brian Conrad observed that an example to show the answer to Question 26.1 is "No" can be constructed for every triple $(R, x, \mathbf{p})$ that satisfies the following three conditions; see [**10**]:

[1] The material in this chapter is adapted from our article [**76**] dedicated to Robert Gilmer, an outstanding algebraist, scholar and teacher.

[2] See Scheme Terminology 26.3 for a brief explanation of this terminology.

(26.1.1) R is an ideal-adic domain, that is, R is a Noetherian domain that is separated and complete with respect to the powers of a proper ideal I.

(26.1.2) x is a nonzero element of R such that the completion of $R[1/x]$ with respect to the powers of $IR[1/x]$, denoted $S := R_{\{x\}}$, is an integral domain.

(26.1.3) $\mathbf{p}$ is a nonzero prime ideal of S that intersects R in (0).

The following example of such a triple $(R, x, \mathbf{p})$ is described in [**10**]:

EXAMPLE 26.2. Let w, x, y, z be indeterminates over a field k. Let

$$R := k[w, x, z]\,[[y]] \quad \text{and} \quad S := k[w, x, 1/x, z]\,[[y]].$$

Notice that R is complete with respect to yR and S is complete with respect to yS. An indirect proof that there exist nonzero primes p of S for which $p \cap R = (0)$ is given in the paper [**10**] of Lipman, Alonzo-Tarrio and Jeremias-Lopez, using a result of Heinzer and Rotthaus [**59**, Theorem 1.12, p. 364]. A direct proof is given in [**76**, Proposition 4.9]. In Proposition 26.31 below we give a direct proof of a more general result due to Dumitrescu [**34**, Corollary 4].

In Scheme Terminology 26.3 we explain some of the terminology of formal schemes necessary for understanding Question 26.1; more details may be found in [**52**]. In Remark 26.4 we explain why a triple satisfying (26.1.1) to (26.1.3) yields examples that answer Question 26.1.

SCHEME TERMINOLOGY 26.3. Let R be a Noetherian integral domain and let K be its field of fractions. Let X denote the topological space $\operatorname{Spec} R$ with the Zariski topology defined in Section 2.1. We form a *sheaf*, denoted $\mathcal{O}$, on X by associating, to each open set U of X, the ring

$$\mathcal{O}(U) \;=\; \bigcap_{x \,\in\, U} R_{\mathfrak{p}_x},$$

where $\mathfrak{p}_x$ is the prime associated to the point $x \in U$; see [**142**, p. 235 and Theorem 1, p. 238]. For each pair $U \subseteq V$ of open subsets of X, there exists a natural inclusion map $\rho_U^V : \mathcal{O}(V) \hookrightarrow \mathcal{O}(U)$. The "ringed space" $(X, \mathcal{O})$ is identified with $\operatorname{Spec} R$ and is called an *affine scheme*; see [**142**, p. 242-3], [**52**, Definition I.10.1.2, p. 402]. Assume that $R = R^*$ is complete with respect to the I-adic topology, where I is a nonzero proper ideal of R (see Definition 3.1). Then the ringed space $(X, \mathcal{O})$ is denoted Spf (R) and is called the *formal spectrum* of R. It is also called a *Noetherian formal adic affine scheme*; see [**52**, I.10.1.7, p. 403]. An *immersion* is a morphism $f : Y \to X$ of schemes that factors as an isomorphism to a subscheme Z of X followed by a canonical injection $Z \to X$; see [**52**, (I.4.2.1)].

REMARK 26.4. Assume, in addition to R being a Noetherian integral domain complete with respect to the I-adic topology, that x is a nonzero element of R, that S is the completion of $R[1/x]$ with respect to the powers of $IR[1/x]$, and that $\mathbf{p}$ is a prime ideal of S such that the triple $(R, x, \mathbf{p})$ satisfies the three conditions 26.1.1 to 26.1.3.

The composition of the maps $R \to S \to S/\mathbf{p}$ determines a map on formal spectra Spf $(S/\mathbf{p}) \to$ Spf $(S) \to$ Spf (R) that is a closed immersion followed by an open immersion. This is because a surjection such as $S \to S/\mathbf{p}$ of adic rings gives rise to a closed immersion Spf $(S/\mathbf{p}) \to$ Spf (S) while a localization, such as that of R with respect to the powers of x, followed by the completion of $R[1/x]$

with respect to the powers of $IR[1/x]$ to obtain S gives rise to an open immersion Spf $(S) \to$ Spf (R) [**52**, I.10.14.4].

The map Spf $(S/\mathbf{p}) \to$ Spf (R) cannot be factored as an open immersion followed by a closed one. This is because a closed immersion into Spf (R) corresponds to a surjective map of adic rings $R \to R/J$, where J is an ideal of R [**52**, page 441]. Thus if the map Spf $(S/\mathbf{p}) \to$ Spf (R) factored as an open immersion followed by a closed one, we would have R-algebra homomorphisms from $R \to R/J \to S/\mathbf{p}$, where Spf $(S/\mathbf{p}) \to$ Spf (R/J) is an open immersion. Since $\mathbf{p} \cap R = (0)$, we must have $J = (0)$. This implies Spf $(S/\mathbf{p}) \to$ Spf (R) is an open immersion, that is, the composite map Spf $(S/\mathbf{p}) \to$ Spf $(S) \to$ Spf (R), is an open immersion. But also Spf $(S) \to$ Spf (R) is an open immersion. It follows that Spf $(S/\mathbf{p}) \to$ Spf (S) is both open and closed. Since S is an integral domain this implies Spf $(S/\mathbf{p}) \cong$ Spf (S). Since $\mathbf{p}$ is nonzero, this is a contradiction. Thus Example 26.2 shows that the answer to Question 26.1 is "No".

The second motivation for the material in this chapter comes from Question 24.4 of Melvin Hochster and Yongwei Yao "Can one describe or somehow classify the local maps $R \hookrightarrow S$ of complete local domains R and S such that every nonzero prime ideal of S has nonzero intersection with R?" The following example is a local map of the type described in the Hochster-Yao question.

EXAMPLE 26.5. Let x and y be indeterminates over a field k and consider the extension $R := k[[x, y]] \hookrightarrow S := k[[x]]\,[[y/x]]$.

To see this extension is TGF—the "trivial generic fiber" condition of Definition 24.6, it suffices to show $P \cap R \neq (0)$ for each $P \in \operatorname{Spec} S$ with $\operatorname{ht} P = 1$. This is clear if $x \in P$, while if $x \notin P$, then $k[[x]] \cap P = (0)$, and so $k[[x]] \hookrightarrow R/(P \cap R) \hookrightarrow S/P$ and S/P is finite over $k[[x]]$. Therefore $\dim R/(P \cap R) = 1$, and so $P \cap R \neq (0)$.

Definition 24.6 is related to Question 26.1. If a ring R and a nonzero element x of R satisfies conditions 26.1.1 and 26.1.2, then condition 26.1.3 simply says that the extension $R \hookrightarrow R_{\{x\}}$ is *not* TGF.

In some correspondence to Lipman regarding Question 26.1, Conrad asked: "Is there a nonzero prime ideal of $E := k[x, 1/x]\,[[y]]$ that intersects $C = k[x]\,[[y]]$ in zero?" If there were such a prime ideal $\mathbf{p}$, then the triple $(C, x, \mathbf{p})$ would satisfy conditions 26.1.1 to 26.1.3. This would yield a two-dimensional example to show the answer to Question 26.1 is "No". Thus one can ask:

QUESTION 26.6. Let x and y be indeterminates over a field k. Is the extension $C := k[x]\,[[y]] \hookrightarrow E := k[x, 1/x]\,[[y]]$ TGF?

We show in Proposition 26.12.2 below that the answer to Question 26.6 is "Yes"; thus the triple $(C, x, \mathbf{p})$ does not satisfy condition 26.1.3, although it does satisfy conditions 26.1.1 and 26.1.2. This is part of our analysis of the prime spectra of A, B, C, D_n and E, and the maps induced on these spectra by the inclusion maps on the rings.

REMARKS 26.7. (1) The extension $k[[x, y]] \hookrightarrow k[[x, y/x]]$ is, up to isomorphism, the same as the extension $k[[x, xy]] \hookrightarrow k[[x, y]]$.

(2) We show in Chapter 27 that the extension $R := k[[x, y, xz]] \hookrightarrow S := k[[x, y, z]]$ is not TGF. We also give more information about TGF extensions of local rings there.

(3) Takehiko Yasuda gives additional information on the TGF property in [**161**]. In particular, he shows that

$$\mathbb{C}[x,y][[z]] \hookrightarrow \mathbb{C}[x,x^{-1},y][[z]]$$

is not TGF, where $\mathbb{C}$ is the field of complex numbers [**161**, Theorem 2.7].

26.2. Trivial generic fiber (TGF) extensions and prime spectra

We record in Proposition 26.8 several basic facts about TGF extensions. We omit the proofs since they are straightforward.

PROPOSITION 26.8. *Let $R \hookrightarrow S$ and $S \hookrightarrow T$ be injective maps where R, S and T are integral domains.*

1. *If $R \hookrightarrow S$ and $S \hookrightarrow T$ are TGF extensions, then so is the composite map $R \hookrightarrow T$. Equivalently if the composite map $R \hookrightarrow T$ is not TGF, then at least one of the extensions $R \hookrightarrow S$ or $S \hookrightarrow T$ is not TGF.*
2. *If $R \hookrightarrow T$ is TGF, then $S \hookrightarrow T$ is TGF.*
3. *If the map $\operatorname{Spec} T \to \operatorname{Spec} S$ is surjective and $R \hookrightarrow T$ is TGF, then $R \hookrightarrow S$ is TGF.*

We use the following remark about prime ideals in a formal power series ring.

REMARKS 26.9. Let R be a commutative ring and let $R[[y]]$ denote the formal power series ring in the variable y over R. Then

1. Each maximal ideal of $R[[y]]$ is of the form $(\mathbf{m}, y)R[[y]]$ where $\mathbf{m}$ is a maximal ideal of R. Thus y is in every maximal ideal of $R[[y]]$.
2. If R is Noetherian with $\dim R[[y]] = n$ and $x_1, \ldots, x_m$ are independent indeterminates over $R[[y]]$, then y is in every height $n+m$ maximal ideal of the polynomial ring $R[[y]][x_1, \ldots, x_m]$.

PROOF. Item 1 follows from [**117**, Theorem 15.1]. For item 2, let $\mathbf{m}$ be a maximal ideal of $R[[y]][x_1, \ldots, x_m]$ with $\operatorname{ht}(\mathbf{m}) = n+m$. By [**85**, Theorem 39], $\operatorname{ht}(\mathbf{m} \cap R[[y]]) = n$; thus $\mathbf{m} \cap R[[y]]$ is maximal in $R[[y]]$, and so, by item 1, $y \in \mathbf{m}$. □

PROPOSITION 26.10. *Let n be a positive integer, let R be an n-dimensional Noetherian domain, let y be an indeterminate over R, and let $\mathbf{q}$ be a prime ideal of height n in the power series ring $R[[y]]$. If $y \notin \mathbf{q}$, then $\mathbf{q}$ is contained in a unique maximal ideal of $R[[y]]$.*

PROOF. Since $R[[y]]$ has dimension $n+1$ and $y \notin \mathbf{q}$, the ring $S := R[[y]]/\mathbf{q}$ has dimension one. Moreover, S is complete with respect to the yS-adic topology [**103**, Theorem 8.7] and every maximal ideal of S is a minimal prime of the principal ideal yS. Hence S is a complete semilocal ring. Since S is also an integral domain, it must be local by [**103**, Theorem 8.15]. Therefore $\mathbf{q}$ is contained in a unique maximal ideal of $R[[y]]$. □

In Section 26.3 we use the following corollary to Proposition 26.10.

COROLLARY 26.11. *Let R be a one-dimensional Noetherian domain and let $\mathbf{q}$ be a height-one prime ideal of the power series ring $R[[y]]$. If $\mathbf{q} \neq yR[[y]]$, then $\mathbf{q}$ is contained in a unique maximal ideal of $R[[y]]$.*

PROPOSITION 26.12. *Consider the nested mixed polynomial-power series rings:*

$$\begin{aligned} A := k[x,y] \hookrightarrow B := k[[y]][x] \hookrightarrow C := k[x][[y]] \\ \hookrightarrow D_1 := k[x][[y/x]] \hookrightarrow D_2 := k[x][[y/x^2]] \hookrightarrow \cdots \\ \hookrightarrow D_n := k[x][[y/x^n]] \hookrightarrow \cdots \hookrightarrow E := k[x,1/x][[y]], \end{aligned}$$

where k is a field and x and y are indeterminates over k. Then

1. *If $S \in \{B, C, D_1, D_2, \cdots, D_n, \cdots, E\}$, then $A \hookrightarrow S$ is not TGF.*
2. *If $\{R, S\} \subset \{B, C, D_1, D_2, \cdots, D_n \cdots, E\}$ are such that $R \subseteq S$, then $R \hookrightarrow S$ is TGF.*
3. *Each of the proper associated maps on spectra fails to be surjective.*

PROOF. For item 1, let $\sigma(y) \in yk[[y]]$ be such that $\sigma(y)$ and y are algebraically independent over k. Then $(x - \sigma(y))S \cap A = (0)$, and so $A \hookrightarrow S$ is not TGF.

For item 2, observe that every maximal ideal of C, D_n or E is of height two with residue field finite algebraic over k. To show $R \hookrightarrow S$ is TGF, it suffices to show $\mathbf{q} \cap R \neq (0)$ for each height-one prime ideal $\mathbf{q}$ of S. This is clear if $y \in \mathbf{q}$. If $y \notin \mathbf{q}$, then $k[[y]] \cap \mathbf{q} = (0)$, and so $k[[y]] \hookrightarrow R/(\mathbf{q} \cap R) \hookrightarrow S/q$ are injections. By Corollary 26.11, $S/\mathbf{q}$ is a one-dimensional local domain. Since the residue field of $S/\mathbf{q}$ is finite algebraic over k, it follows that $S/\mathbf{q}$ is finite over $k[[y]]$. Therefore $S/\mathbf{q}$ is integral over $R/(\mathbf{q} \cap R)$. Hence $\dim(R/(\mathbf{q} \cap R) = 1$ and so $\mathbf{q} \cap R \neq (0)$.

For item 3, observe that xD_n is a prime ideal of D_n and x is a unit of E. Thus $\operatorname{Spec} E \to \operatorname{Spec} D_n$ is not surjective. Now, considering $C = D_0$ and $n > 0$, we have $xD_n \cap D_{n-1} = (x, y/x^{n-1})D_{n-1}$. Therefore xD_{n-1} is not in the image of the map $\operatorname{Spec} D_n \to \operatorname{Spec} D_{n-1}$. The map from $\operatorname{Spec} C \to \operatorname{Spec} B$ is not onto, because $(1+xy)$ is a prime ideal of B, but $1+xy$ is a unit in C. Similarly $\operatorname{Spec} B \to \operatorname{Spec} A$ is not onto, because $(1 + y)$ is a prime ideal of A, but $1 + y$ is a unit in B. This completes the proof. □

QUESTION AND REMARKS 26.13. Which of the Spec maps of Proposition 26.12 are one-to-one and which are finite-to-one?

1. For $S \in \{B, C, D_1, D_2, \cdots, D_n, \cdots, E\}$, the generic fiber ring of the map $A \hookrightarrow S$ has infinitely many prime ideals and has dimension one. Every height-two maximal ideal of S contracts in A to a maximal ideal. Every maximal ideal of S containing y has height two. Also $yS \cap A = yA$ and the map $\operatorname{Spec} S/yS \to \operatorname{Spec} A/yA$ is one-to-one.
2. Suppose $R \hookrightarrow S$ is as in Proposition 26.12.2. Each height-two prime of S contracts in R to a height-two maximal ideal of R. Each height-one prime of R is the contraction of at most finitely many prime ideals of S and all of these prime ideals have height one. If $R \hookrightarrow S$ is flat, which is true if $S \in \{B, C, E\}$, then "going-down" holds for $R \hookrightarrow S$, and so, for P a height-one prime of S, we have $\operatorname{ht}(P \cap R) \leq 1$.
3. As mentioned in [**80**, Remark 1.5], C/P is Henselian for every nonzero prime ideal P of C other than yC.

26.3. Spectra for two-dimensional mixed polynomial-power series rings

Let x and y be indeterminates over a field k. We consider the prime spectra, as partially ordered sets, of the mixed polynomial-power series rings A, B, C,

$D_1, D_2, \cdots, D_n, \cdots$ and E as given in Sequences 26.0.1 and 26.0.2 at the beginning of this chapter.

Even for k a countable field there are at least two non-order-isomorphic partially ordered sets that can be the prime spectrum of the polynomial ring $A := k[x, y]$. Let $\mathbb{Q}$ be the field of rational numbers, let F be a field contained in the algebraic closure of a finite field and let $\mathbb{Z}$ denote the ring of integers. Then, by [**157**] and [**158**], $\operatorname{Spec} \mathbb{Q}[x, y] \ncong \operatorname{Spec} F[x, y] \cong \operatorname{Spec} \mathbb{Z}[y]$.

The prime spectra of the rings B, C, $D_1, \cdots, D_n, \cdots$, and E of Sequences 26.0.1 and 26.0.2 are simpler since they involve power series in y. Remark 26.9.2 implies that y is in every maximal ideal of height two of each of these rings.

The partially ordered set $\operatorname{Spec} B = \operatorname{Spec}(k[[y]][x])$ is similar to a prime ideal space studied by Heinzer and S. Wiegand in the countable case in [**80**] and then generalized by Shah to other cardinalities in [**143**]. The ring $k[[y]]$ is uncountable, even if k is countable. It follows that $\operatorname{Spec} B$ is also uncountable. The partially ordered set $\operatorname{Spec} B$ can be described uniquely up to isomorphism by the axioms of [**143**] (similar to the CHP axioms of [**80**]), since $k[[y]]$ is Henselian and has cardinality at least equal to c, the cardinality of the real numbers $\mathbb{R}$. [3]

Theorem 26.14 characterizes $U := \operatorname{Spec} B$, for the ring B of Sequence 26.01, as a *Henselian affine* partially ordered set (where the "$\leq$" relation is "set containment").

THEOREM 26.14. [**80**, Theorem 2.7] [**143**, Theorem 2.4] *Let $B = k[[y]][x]$ be as in Sequence 26.0.1, where k is a field, the cardinality of the set of maximal ideals of $k[x]$ is α and the cardinality of $k[[y]]$ is β. Then the partially ordered set $U := \operatorname{Spec} B$ under containment is called Henselian affine of type (β, α) and is characterized as a partially ordered set by the following axioms:*

(1) $|U| = \beta$.
(2) *U has a unique minimal element.*
(3) $\dim(U) = 2$ *and* $|\{$ *height-two elements of* $U\}| = \alpha$.
(4) *There exists a special height-one element $u \in U$ such that u is less than every height-two element of U, namely $u = (x)$. If $|\max(R)| > 1$, then the special element is unique.*
(5) *Every nonspecial height-one element of U is less than at most one height-two element.*
(6) *Every height-two element $t \in U$ is greater than exactly β height-one elements such that t is the unique height-two element above each. If $t_1, t_2 \in U$ are distinct height-two elements, then the special element from (4) is the unique height-one element less than both.*
(7) *There are exactly β height-one elements that are maximal.*

REMARK 26.15. (1) The axioms of Theorem 26.14 are redundant. We feel this redundancy helps in understanding the relationships among the prime ideals.

(2) The theorem applies to the spectrum of B by defining the unique minimal element to be the ideal (0) of B and the special height-one element to be the prime ideal yB. Every height-two maximal ideal $\mathbf{m}$ of B has nonzero intersection with $k[[y]]$. Thus $\mathbf{m}/yB$ is principal and so $\mathbf{m} = (y, f(x))$, for some monic irreducible

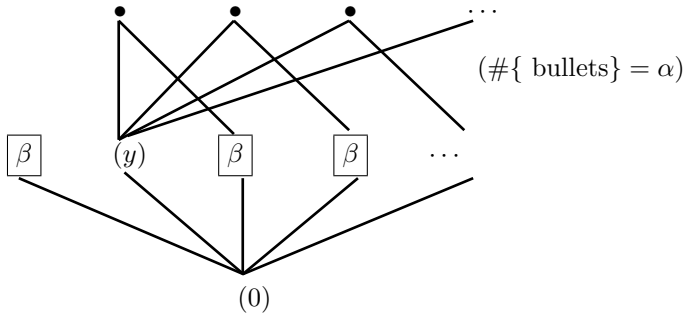
[3] Kearnes and Oman observe in [**87**] that some cardinality arguments are incomplete in the paper [**143**]. R. Wiegand and S. Wiegand show that Shah's results are correct in [**160**]. In Remarks 26.15.2 we give proofs of some items of Theorem 26.14.

polynomial $f(x)$ of $k[x]$. Consider $\{f(x) + ay \,|\, a \in k[[y]]\}$. This set has cardinality β and each $f(x) + ay$ is contained in a nonempty finite set of height-one primes contained in $\mathbf{m}$. If $\mathbf{p}$ is a height-one prime contained in $\mathbf{m}$ with $\mathbf{p} \neq yB$, then $\mathbf{p} \cap k[[y]] = (0)$, and so $\mathbf{p}k((y))[x]$ is generated by a monic polynomial in $k((y))[x]$. But for $a, b \in k[[y]]$ with $a \neq b$, we have $(f(x) + ay, f(x) + by)k((y))[x] = k((y))[x]$. Therefore no height-one prime contained in $\mathbf{m}$ contains both $f(x)+ay$ and $f(x)+by$. Since B is Noetherian and $|B| = \beta$ is an infinite cardinal, we conclude that the cardinality of the set of height-one prime ideals contained in $\mathbf{m}$ is β. Examples of height-one maximals are $(1 + xyf(x, y)\,)$, for various $f(x, y) \in k[[y]]\,[x]$. The set of height-one maximal ideals of B also has cardinality β.

(3) We observe that $\alpha = |k| \cdot \aleph_0$ and $\beta = |k|^{\aleph_0}$ in Theorem 26.14, where $\aleph_0 = |\mathbb{N}|$. The proof of this is Exercise 26.1.

(4) The axioms given in Theorem 26.14 *characterize* $\operatorname{Spec} B$ in the sense that every two partially ordered sets satisfying these axioms are order-isomorphic.

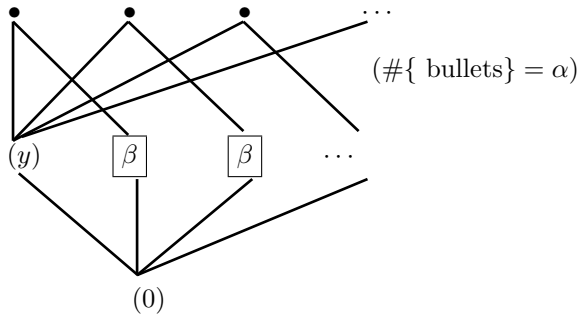
The picture of $\operatorname{Spec} B$ is shown below:



$\operatorname{Spec} k[[y]]\,[x]$

In the diagram, β is the cardinality of $k[[y]]$, and α is the cardinality of the set of maximal ideals of $k[x]$ (and also the cardinality of the set of maximal ideals of $k[[y]]\,[x]\,$); the boxed β means there are cardinality β height-one primes in that position with respect to the partial ordering.

Next we consider $\operatorname{Spec} R[[y]]$, for R a Noetherian one-dimensional domain. We show in Theorem 26.18 below that $\operatorname{Spec} R[[y]]$ has the following picture:



$$\mathrm{Spec}(R[[y]])$$

Here α is the cardinality of the set of maximal ideals of R (and also the cardinality of the set of maximal ideals of $R[[y]]$ by Remark 26.9.1) and β is the cardinality (uncountable) of $R[[y]]$; the boxed β (one for each maximal ideal of R) means that there are exactly β prime ideals in that position.

We give the following lemma and add some more arguments in order to justify the cardinalities that occur in the spectra of power series rings more precisely.

LEMMA 26.16. [**160**, Lemma 4.2] *Let T be a Noetherian domain, y an indeterminate and I a proper ideal of T. Let $\delta = |T|$ and $\gamma = |T/I|$. Then $\delta \leq \gamma^{\aleph_0}$, and $|T[[y]]| = \delta^{\aleph_0} = \gamma^{\aleph_0}$.*

PROOF. The first equality holds by Exercise 26.1. That $\delta^{\aleph_0} \geq \gamma^{\aleph_0}$ follows from $\gamma \leq \delta$. For the reverse inequality, $\bigcap_{n\geq 1} I^n = 0$ by the Krull Intersection Theorem [**103**, Theorem 8.10 (ii)]. Therefore there is a monomorphism

$$T \hookrightarrow \prod_{n\geq 1} T/I^n. \tag{26.16.0}$$

Now T/I^n has a finite filtration with factors I^{r-1}/I^r for each r with $1 \leq r \leq n$. Since I^{r-1}/I^r is a finitely generated (T/I)-module, $|I^{r-1}/I^r| \leq \gamma^{\aleph_0}$. Therefore $|T/I^n| \leq (\gamma^{\aleph_0})^n = \gamma^{\aleph_0}$, for each n. Thus $\delta \leq (\gamma^{\aleph_0})^{\aleph_0} = \gamma^{(\aleph_0^2)} = \gamma^{\aleph_0}$ by Equation 26.16.0. Finally, $\delta^{\aleph_0} \leq (\gamma^{\aleph_0})^{\aleph_0} = \gamma^{\aleph_0}$, and so $\delta^{\aleph_0} = \gamma^{\aleph_0}$. □

The following remarks, observed in the article [**160**] of R. Wiegand and S. Wiegand, are helpful for establishing the cardinaliies in Theorem 26.18.

REMARKS 26.17. Let $\aleph_0$ denote the cardinality of the set of natural numbers. Suppose that T is a commutative ring of cardinality δ, that $\mathbf{m}$ is a maximal ideal of T and that γ is the cardinality of $T/\mathbf{m}$. Then:

(1) The cardinality of $T[[y]]$ is $\delta^{\aleph_0}$, by Lemma 26.16 and Exercise 26.1. If T is Noetherian, then $T[[y]]$ is Noetherian, and so every prime ideal of $T[[y]]$ is finitely generated. Since the cardinality of the finite subsets of $T[[y]]$ is $\delta^{\aleph_0}$, it follows that $T[[y]]$ has at most $\delta^{\aleph_0}$ prime ideals.

(2) If T is Noetherian, then there are at least $\gamma^{\aleph_0}$ distinct height-one prime ideals (other than $(y)T[[y]]$) of $T[[y]]$ contained in $(\mathbf{m}, y)T[[y]]$. To see this, choose a set $C = \{c_i \,|\, i \in I\}$ of elements of T so that $\{c_i + \mathbf{m} \,|\, i \in I\}$ gives the distinct coset representatives for $T/\mathbf{m}$. Thus there are γ elements of C, and for $c_i, c_j \in C$ with $c_i \neq c_j$, we have $c_i - c_j \notin \mathbf{m}$. Now also let $a \in \mathbf{m}, a \neq 0$. Consider the set

$$G = \{a + \sum_{n\in\mathbb{N}} d_n y^n \,|\, d_n \in C\, \forall n \in \mathbb{N}\}.$$

Each of the elements of G is in $(\mathbf{m}, y)T[[y]] \setminus yT[[y]]$ and hence is contained in a height-one prime contained in $(\mathbf{m}, y)T[[y]]$ distinct from $yT[[y]]$.

Moreover, $|G| = |C|^{\aleph_0} = \gamma^{\aleph_0}$. Let P be a height-one prime ideal of $T[[y]]$ contained in $(\mathbf{m}, y)T[[y]]$ but such that $y \notin P$. If two distinct elements of G, say $f = a + \sum_{n\in\mathbb{N}} d_n y^n$ and $g = a + \sum_{n\in\mathbb{N}} e_n y^n$, with the $d_n, e_n \in C$, are both in P, then so is their difference; that is

$$f - g = \sum_{n\in\mathbb{N}} d_n y^n - \sum_{n\in\mathbb{N}} e_n y^n = \sum_{n\in\mathbb{N}} (d_n - e_n) y^n \in P.$$

Now let t be the smallest power of y so that $d_t \neq e_t$. Then $(f-g)/y^t \in P$, since P is prime and $y \notin P$, but the constant term, $d_t - e_t \notin \mathbf{m}$, which contradicts the fact that $P \subseteq (\mathbf{m}, y)T[[y]]$. Thus there must be at least $|C|^{\aleph_0} = \gamma^{\aleph_0}$ distinct height-one primes contained in $(\mathbf{m}, y)T[[y]]$.

(3) If T is Noetherian, then there are exactly $\gamma^{\aleph_0} = \delta^{\aleph_0}$ distinct height-one prime ideals (other than $yT[[y]]$) of $T[[y]]$ contained in $(\mathbf{m}, y)T[[y]]$. This follows from (1) and (2) and Lemma 26.16.

THEOREM 26.18. [**76**] [**160**] *Let R be a one-dimensional Noetherian domain with cardinality δ, let $\beta = \delta^{\aleph_0}$ and let α be the cardinality of the set of maximal ideals of R, where α may be finite. Let $U = \operatorname{Spec} R[[y]]$, where y is an indeterminate over R. Then U as a partially ordered set (where the "$\leq$" relation is "set containment") satisfies the following axioms:*

(1) $|U| = \beta$.
(2) *U has a unique minimal element, namely* (0).
(3) $\dim(U) = 2$ *and* $|\{$ *height-two elements of* $U\}| = \alpha$.
(4) *There exists a special height-one element $u \in U$ such that u is less than every height-two element of U, namely $u = (y)$. If $|\max(R)| > 1$, then the special element is unique.*
(5) *Every nonspecial height-one element of U is less than exactly one height-two element.*
(6) *Every height-two element $t \in U$ is greater than exactly β height-one elements that are less than only t. If $t_1, t_2 \in U$ are distinct height-two elements, then the special element from (4) is the unique height-one element less than both.*
(7) *There are no height-one maximal elements in U. Every maximal element has height two.*

The set U is characterized as a partially ordered set by the axioms 1-7. Every partially ordered set satisfying the axioms 1-7 is isomorphic to every other such partially ordered set.

PROOF. Item 1 follows from Remarks 26.17.1 and 26.17.3. Item 2 and the first part of item 3 are clear. The second part of item 3 follows immediately from Remark 26.9.1.

For items 4 and 5, suppose that P is a height-one prime of $R[[y]]$. If $P = yR[[y]]$, then P is contained in each maximal ideal of $R[[y]]$ by Remark 26.9.1, and so $yR[[y]]$ is the special element. If $y \notin P$, then, by Corollary 26.11, P is contained in a unique maximal ideal of $R[[y]]$.

For item 6, use Remarks 26.17.2 and 26.17.3.

All partially ordered sets satisfying the axioms of Theorem 26.14 are order-isomorphic, and the partially ordered set U of the present theorem satisfies the same axioms as in Theorem 26.14 except axiom (7) that involves height-one maximals. Since U has no height-one maximals, an order-isomorphism between two partially ordered sets as in Theorem 26.18 can be deduced by adding on height-one maximals and then deleting them. □

COROLLARY 26.19. *In the terminology of Sequences 26.0.1 and 26.0.2 at the beginning of this chapter, we have* $\operatorname{Spec} C \cong \operatorname{Spec} D_n \cong \operatorname{Spec} E$, *but* $\operatorname{Spec} B \ncong \operatorname{Spec} C$.

PROOF. The rings C, D_n, and E are all formal power series rings in one variable over a one-dimensional Noetherian domain R, where R is either $k[x]$ or $k[x, 1/x]$. Thus the domain R satisfies the hypotheses of Theorem 26.18. Also the number of maximal ideals is the same for C, D_n, and E, because in each case, it is the same as the number of maximal ideals of R which is $|k[x]| = |k| \cdot \aleph_0$.

Thus in the picture of $R[[y]]$ shown above, for $R[[y]] = C, D_n$ or E, we have $\alpha = |k| \cdot \aleph_0$ and $\beta = |R[[y]]| = |R|^{\aleph_0}$, and so the spectra are isomorphic. The spectrum of B is not isomorphic to that of C, however, because B contains height-one maximal ideals, such as that generated by $1 + xy$, whereas C has no height-one maximal ideals. □

REMARK 26.20. As mentioned at the beginning of this section, it is shown in [**157**] and [**158**] that $\operatorname{Spec} \mathbb{Q}[x, y] \ncong \operatorname{Spec} F[x, y] \cong \operatorname{Spec} \mathbb{Z}[y]$, where F is a field contained in the algebraic closure of a finite field. Corollary 26.21 shows that the spectra of power series extensions in y behave differently in that $\operatorname{Spec} \mathbb{Z}[[y]] \cong \operatorname{Spec} \mathbb{Q}[x][[y]] \cong \operatorname{Spec} F[x][[y]]$.

COROLLARY 26.21. *If $\mathbb{Z}$ is the ring of integers, $\mathbb{Q}$ is the rational numbers, F is a field contained in the algebraic closure of a finite field, and $\mathbb{R}$ is the real numbers, then*

$$\operatorname{Spec} \mathbb{Z}[[y]] \cong \operatorname{Spec} \mathbb{Q}[x][[y]] \cong \operatorname{Spec} F[x][[y]] \ncong \operatorname{Spec} \mathbb{R}[x][[y]].$$

PROOF. The rings $\mathbb{Z}, \mathbb{Q}[x]$ and $F[x]$ are all countable with countably infinitely many maximal ideals. Thus if $R = \mathbb{Z}, \mathbb{Q}[x]$ or $F[x]$, then R satisfies the hypotheses of Theorem 26.18 with the cardinality conditions of parts (b) and (c). On the other hand, $\mathbb{R}[x]$ has uncountably many maximal ideals; thus $\mathbb{R}[x][[y]]$ also has uncountably many maximal ideals. □

26.4. Higher dimensional mixed polynomial-power series rings

In analogy to Sequence 26.0.1, we display several embeddings involving three variables.

(26.4.0.1)

$$k[x, y, z] \overset{\alpha}{\hookrightarrow} k[[z]][x, y] \overset{\beta}{\hookrightarrow} k[x][[z]][y] \overset{\gamma}{\hookrightarrow} k[x, y][[z]] \overset{\delta}{\hookrightarrow} k[x][[y, z]],$$

$$k[[z]][x, y] \overset{\epsilon}{\hookrightarrow} k[[y, z]][x] \overset{\zeta}{\hookrightarrow} k[x][[y, z]] \overset{\eta}{\hookrightarrow} k[[x, y, z]],$$

where k is a field and x, y and z are indeterminates over k.

REMARKS 26.22.

(1) By Proposition 26.12.2 every nonzero prime ideal of $C = k[x][[y]]$ has nonzero intersection with $B = k[[y]][x]$. In three or more variables, however, the analogous statements fail. We show below that the maps $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta$ in Sequence 26.4.0.1 fail to be TGF. Thus, by Proposition 26.8.2, no proper inclusion in Sequence 26.4.0.1 is TGF. The dimensions of the generic fiber rings of the maps in the diagram are either one or two.

(2) For those rings in Sequence 26.4.0.1 of form $R = S[[z]]$ (ending in a power series variable) where S is a ring, such as $R = k[x, y][[z]]$, we have some information concerning the prime spectra. By Proposition 26.10 every height-two prime ideal not containing z is contained in a unique maximal ideal. By [**117**, Theorem 15.1] the maximal ideals of $S[[z]]$ are of the

form $(\mathbf{m}, z)S[[z]]$, where $\mathbf{m}$ is a maximal ideal of S, and thus the maximal ideals of $S[[z]]$ are in one-to-one correspondence with the maximal ideals of S. As in section 26.3, using Remarks 26.9, we see that maximal ideals of $\operatorname{Spec} k[[z]][x, y]$ can have height two or three, that (z) is contained in every height-three prime ideal, and that every height-two prime ideal not containing (z) is contained in a unique maximal ideal.

(3) It follows by arguments analogous to that in Proposition 26.12.1, that α, δ, ϵ are not TGF. For α, let $\sigma(z) \in zk[[z]]$ be transcendental over $k(z)$; then $(x-\sigma)k[[z]][x, y] \cap k[x, y, z] = (0)$. For δ and ϵ: let $\sigma(y) \in yk[[y]]$ be transcendental over $k(y)$; then $(x-\sigma)k[x][[z, y]] \cap k[x][[z]][y] = (0)$, and $(x-\sigma)k[[y, z]][x] \cap k[[z]][x, y] = (0)$.

(4) By Main Theorem 24.3.4.a of Chapters 24 and 26 (proved in Theorem 25.2), η is not TGF and the dimension of the generic fiber ring of η is one.

In order to show in Proposition 26.24 below that the map β is not TGF, we first observe:

PROPOSITION 26.23. *The element $\sigma = \sum_{n=1}^{\infty}(xz)^{n!} \in k[x][[z]]$ is transcendental over $k[[z]][x]$.*

PROOF. Consider an expression

$$Z := a_\ell\sigma^\ell + a_{\ell-1}\sigma^{\ell-1} + \cdots + a_1\sigma + a_0,$$

where the $a_i \in k[[z]][x]$ and $a_\ell \neq 0$. Let m be an integer greater than $\ell + 1$ and greater than $\deg_x a_i$ for each i such that $0 \leq i \leq \ell$ and $a_i \neq 0$. Regard each $a_i\sigma^i$ as a power series in x with coefficients in $k[[z]]$.

For each i with $0 \leq i \leq \ell$, we have $i(m!) < (m+1)!$. It follows that the coefficient of $x^{i(m!)}$ in σ^i is nonzero, and the coefficient of x^j in σ^i is zero for every j with $i(m!) < j < (m+1)!$. Thus if $a_i \neq 0$ and $j = i(m!)+\deg_x a_i$, then the coefficient of x^j in $a_i\sigma^i$ is nonzero, while for j such that $i(m!) + \deg_x a_i < j < (m+1)!$, the coefficient of x^j in $a_i\sigma^i$ is zero. By our choice of m, for each i such that $0 \leq i < \ell$ and $a_i \neq 0$, we have

$$(m+1)! > \ell(m!) + \deg_x a_\ell \geq i(m!) + m! > i(m!) + \deg_x a_i.$$

Thus in Z, regarded as a power series in x with coefficients in $k[[z]]$, the coefficient of x^j is nonzero for $j = \ell(m!) + \deg_x a_\ell$. Therefore $Z \neq 0$. We conclude that σ is transcendental over $k[[z]][x]$. □

PROPOSITION 26.24. *In Sequence 26.4.0.1, $k[[z]][x, y] \overset{\beta}{\hookrightarrow} k[x][[z]][y]$ is not a TGF-extension.*

PROOF. Fix an element $\sigma \in k[x][[z]]$ that is transcendental over $k[[z]][x]$. We define $\pi : k[x][[z]][y] \to k[x][[z]]$ to be the identity map on $k[x][[z]]$ and $\pi(y) = \sigma z$. Let $\mathbf{q} = \ker \pi$. Then $y - \sigma z \in \mathbf{q}$. If $h \in \mathbf{q} \cap (k[[z]][x, y])$, then

$$h = \sum_{j=0}^{s}\sum_{i=0}^{t}(\sum_{\ell\in\mathbb{N}} a_{ij\ell}z^\ell)x^iy^j, \text{ for some } s, t \in \mathbb{N} \text{ and } a_{ij\ell} \in k, \text{ and so}$$

$$0 = \pi(h) = \sum_{j=0}^{s}\sum_{i=0}^{t}(\sum_{\ell\in\mathbb{N}} a_{ij\ell}z^\ell)x^i(\sigma z)^j = \sum_{j=0}^{s}\sum_{i=0}^{t}(\sum_{\ell\in\mathbb{N}} a_{ij\ell}z^{\ell+j})x^i\sigma^j.$$

Since σ is transcendental over $k[[z]][x]$, we have that x and σ are algebraically independent over $k((z))$. Thus each of the $a_{ij\ell} = 0$. Therefore $\mathbf{q} \cap (k[[z]][x, y]) = (0)$, and so the embedding β is not TGF. □

The concept of "analytic independence" is useful in several arguments below.

DEFINITION AND REMARKS 26.25. Let I be an ideal of an integral domain A. Assume that A is complete and Hausdorff in the I-adic topology. Let B be a subring of A, let $a_1, \ldots, a_n \in I$ and let $v_1, \ldots, v_n$ be indeterminates over A. We say $a_1, \ldots, a_n$ are *analytically independent* over B if the B-algebra homomorphism $\varphi : B[[v_1, \ldots, v_n]] \to A$, where $\varphi(v_i) = a_i$ for each i, is injective.

(1) This definition of "analytically independent" is given in the book of Zariski and Samuel [**165**, page 258]. This use of the term applies to the work of Abhankar and Moh [**11**], and of Dumitrescu [**34**]. However, this definition does not agree with the use of the term "analytically independent" in Matsumura [**103**, page 107] and Swanson and Huneke [**149**, page 175].

(2) If, for example, a and b are elements of I, then we have power series expressions $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} c_{ij} a^i b^j$, where each $c_{ij} \in B$. If a and b are analytically independent over B, then the expression above is unique. Thus $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} c_{ij} a^i b^j = 0$ implies that every $c_{ij} = 0$.

PROPOSITION 26.26. *In Sequence 26.4.0.1, the extensions*

$$k[[y, z]][x] \overset{\zeta}{\hookrightarrow} k[x][[y, z]] \quad \text{and} \quad k[x][[z]][y] \overset{\gamma}{\hookrightarrow} k[x, y][[z]]$$

are not TGF.

PROOF. For ζ, let $t = xy$ and let $\sigma \in k[[t]]$ be algebraically independent over $k(t)$. Define $\pi : k[x][[y, z]] \to k[x][[y]]$ as follows. For

$$f := \sum_{\ell=0}^{\infty} \sum_{m+n=\ell} f_{mn}(x) y^m z^n \in k[x][[y, z]],$$

where $f_{mn}(x) \in k[x]$, define

$$\pi(f) := \sum_{\ell=0}^{\infty} \sum_{m+n=\ell} f_{mn}(x) y^m (\sigma y)^n \in k[x][[y]].$$

In particular, $\pi(z) = \sigma y$. Let $\mathbf{p} := \ker \pi$. Then $z - \sigma y \in \mathbf{p}$, and so $\mathbf{p} \neq (0)$. Let $h \in \mathbf{p} \cap k[[y, z]][x]$. We show $h = 0$. Now h is a polynomial with coefficients in $k[[y, z]]$, and we define $g \in k[[y, z]][t]$, by, if $a_i(y, z) \in k[[y, z]]$ and

$$h := \sum_{i=0}^{r} a_i(y, z) x^i, \text{ then set } g := y^r h = \sum_{i=0}^{r} (\sum_{\ell=0}^{\infty} \sum_{m+n=\ell} b_{imn} y^m z^n) t^i.$$

The coefficients of g are in $k[[y, z]]$, since $y^r x^i = y^{r-i} t^i$. Thus

$$0 = \pi(g) = \sum_{i=0}^{r} (\sum_{\ell=0}^{\infty} \sum_{m+n=\ell} b_{imn} y^m (\sigma y)^n) t^i = \sum_{i=0}^{r} (\sum_{\ell=0}^{\infty} \sum_{m+n=\ell} b_{imn} \sigma^n y^\ell) t^i$$
$$= \sum_{\ell=0}^{\infty} (\sum_{m+n=\ell} (\sum_{i=0}^{r} b_{imn} t^i) \sigma^n) y^\ell.$$

The elements $t = xy$ and y of $k[x][[y,z]]$ are *analytically independent* over k in the sense of Definition 26.25; hence the coefficient of each y^ℓ (in $k[[t]]$) is 0. Since σ and t are algebraically independent over k, the coefficient of each σ^n is 0. It follows that each $b_{imn} = 0$, that $g = 0$ and hence that $h = 0$. Thus $\mathbf{p} \cap k[[y,z]][x] = (0)$, and so the extension ζ is not TGF.

To see that γ is not TGF, we switch variables in the proof for ζ, so that $t = yz$. Again choose $\sigma \in k[[t]]$ to be algebraically independent over $k(t)$. Define $\psi : k[x,y][[z]] \to k[y][[z]]$ by $\psi(x) = \sigma z$ and ψ is the identity on $k[y][[z]]$. Then ψ can be extended to $\pi : k[y][[x,z]] \to k[y][[z]]$, which is similar to the π in the proof above. As above, set $\mathbf{p} := \ker \pi$; then $\mathbf{p} \cap k[[x,z]][y] = (0)$. Thus $\mathbf{p} \cap k[x][[z]][y] = (0)$ and γ is not TGF. □

PROPOSITION 26.27. *Let D be an integral domain and let x and t be indeterminates over D. Then $\sigma = \sum_{n=1}^{\infty} t^{n!} \in D[[x,t]]$ is algebraically independent over $D[[x,xt]]$.*

PROOF. Suppose that σ is algebraically dependent over $D[[x,xt]]$. Then there exists an equation

$$\gamma_\ell \sigma^\ell + \cdots + \gamma_i \sigma^i + \cdots + \gamma_1 \sigma + \gamma_0 = 0,$$

where each $\gamma_i \in D[[x,xt]]$, ℓ is a positive integer and $\gamma_\ell \neq 0$. This implies

$$\gamma := \gamma_\ell \sigma^\ell + \cdots + \gamma_i \sigma^i + \cdots + \gamma_1 \sigma = -\gamma_0$$

is an element of $D[[x,xt]]$. We obtain a contradiction by showing that $\gamma \notin D[[x,xt]]$.

For each i with $1 \leq i \leq \ell$, write

$$\gamma_i := \sum_{j=0}^{\infty} f_{ij}(x)(xt)^j \in D[[x,xt]],$$

where each $f_{ij}(x) \in D[[x]]$. Since $\gamma_\ell \neq 0$, there exists j such that $f_{\ell j}(x) \neq 0$. Let a_ℓ be the smallest such j, and let m_ℓ be the order of $f_{\ell a_\ell}(x)$, that is, $f_{\ell a_\ell}(x) = x^{m_\ell} g_\ell(x)$, where $g_\ell(0) \neq 0$. Let n be a positive integer such that

$$n \geq 2 + \max\{\ell, m_\ell, a_\ell\}.$$

Since $\ell < n$ and $1 \leq i \leq \ell$, we have

$$\begin{aligned} \sigma^i = (t + \ldots + t^{n!} + \ldots)^i &= t(t + \ldots + t^{(n-1)!} + t^{n!} + \ldots)^{(i-1)} \\ &\quad + \ldots + t^{(n-1)!}(t + \ldots + t^{(n-1)!} + t^{n!} + \ldots)^{(i-1)} \\ &\quad + t^{n!}(t + \ldots + t^{(n-1)!} + t^{n!} + \ldots)^{(i-1)} + \\ &\quad + t^{(n+1)!}(t + t^2 + \ldots)^{(i-1)} + \ldots \\ &= \delta_i(t) + c_i t^{i(n!)} + t^{(n+1)!}\tau_i(t), \end{aligned} \tag{26.27.0}$$

where $c_i = 1$ is a nonzero element of D, $\delta_i(t)$ is a polynomial in $D[t]$ of degree at most $(i-1)n! + (n-1)!$ and $\tau_i(t) \in D[[t]]$, for each i. We use the following two claims to complete the proof. □

CLAIM 26.28. *The coefficient of $t^{\ell(n!)+a_\ell}$ in $\sigma^\ell \gamma_\ell = \sigma^\ell(\sum_{j=a_\ell}^{\infty} f_{\ell j}(x)(xt)^j)$ as a power series in $D[[x]]$ has order $m_\ell + a_\ell$, and hence, in particular, is nonzero.*

PROOF. By the choice of n, $(n+1)! = (n+1)n! > \ell(n!) + n > \ell(n!) + a_\ell$. Hence, by the expression for σ^ℓ given in Equation 26.27.0, we see that all of the terms in $\sigma^\ell\gamma_\ell$ of the form $bt^{\ell(n!)+a_\ell}$, for some $b \in D[[x]]$, appear in the product

$$(\, t^{\ell(n!)} + \delta_\ell(t)\,)(x^{m_\ell} g_\ell(x)(xt)^{a_\ell} + \sum_{j=1+a_\ell}^{\ell(n!)+a_\ell} f_{\ell j}(x)(xt)^j).$$

One of the terms of the form $bt^{\ell(n!)+a_\ell}$ in this product is

$$(x^{m_\ell+a_\ell} g_\ell(x))t^{\ell(n!)+a_\ell} = (x^{m_\ell+a_\ell}(g_\ell(0) + xh_\ell(x)))t^{\ell(n!)+a_\ell},$$

where we write $g_\ell(x) = g_\ell(0) + xh_\ell(x)$ with $h_\ell(x) \in D[[x]]$. Since $g_\ell(0)$ is a nonzero element of D, $x^{m_\ell+a_\ell} g_\ell(x) \in D[[x]]$ has order $m_\ell + a_\ell$. The other terms in the product $\sigma^\ell\gamma_\ell$ that have the form $bt^{\ell(n!)+a_\ell}$, for some $b \in D[[x]]$, are in the product

$$(\delta_\ell(t))(\sum_{j=1+a_\ell}^{\ell(n!)+a_\ell} f_{\ell j}(x)(xt)^j) = \sum_{j=1+a_\ell}^{\ell(n!)+a_\ell} f_{\ell j}(x)(xt)^j \delta_\ell(t).$$

Since $\deg_t \delta_\ell \leq (\ell-1)n! + (n-1)!$ and since, for each j with $f_{\ell j}(x) \neq 0$, we have $\deg_t f_{\ell j}(x)(xt)^j = j$, we see that each term $f_{\ell j}(x)(xt)^j\delta_\ell(t)$ has degree in t less than or equal to $j + (\ell-1)n! + (n-1)!$. Thus each nonzero term in this product of the form $bt^{\ell(n!)+a_\ell}$ has $\ell(n!) + a_\ell \leq j + (\ell-1)n! + (n-1)!$. That is,

$$\begin{aligned} j &\geq \ell(n!) + a_\ell - (\ell-1)(n!) - (n-1)! = a_\ell + n! - (n-1)! \\ &= a_\ell + (n-1)!(n-1) > a_\ell + m_\ell, \end{aligned}$$

since $n-1 > m_\ell$. Moreover, for j such that $f_{\ell j}(x) \neq 0$, the order in x of $f_{\ell j}(x)(xt)^j$ is bigger than or equal to j. Thus the coefficient of $t^{\ell(n!)+a_\ell}$ in $\sigma^\ell\gamma_\ell$ as a power series in x has order $m_\ell + a_\ell$, as desired for Claim 26.28. □

CLAIM 26.29. *For $i < \ell$, the coefficient of $t^{\ell(n!)+a_\ell}$ in $\sigma^i\gamma_i$ as a power series in $D[[x]]$ is either zero or has order greater than $m_\ell + a_\ell$.*

PROOF. As in the proof of Claim 26.28, all of the terms in $\sigma^i\gamma_i$ of the form $bt^{\ell(n!)+a_\ell}$, for some $b \in D[[x]]$, appear in the product

$$(\delta_i + t^{i(n!)})(\sum_{j=0}^{\ell(n!)+a_\ell} f_{ij}(x)(xt)^j) = \sum_{j=0}^{\ell(n!)+a_\ell} f_{ij}(x)(xt)^j(\delta_i + t^{i(n!)}).$$

Since $\deg_t(\delta_i + t^{i(n!)}) = i(n!)$, each term in $f_{ij}(x)(xt)^j(\delta_i + t^{i(n!)})$ has degree in t at most $j + i(n!)$. Thus each term in this product of the form $bt^{\ell(n!)+a_\ell}$, for some nonzero $b \in D[[x]]$, has

$$j \geq \ell(n!) + a_\ell - i(n!) \geq n! + a_\ell > m_\ell + a_\ell.$$

Thus $\operatorname{ord}_x b \geq j > m_\ell + a_\ell$. This completes the proof of Claim 26.29. Hence $\gamma \notin D[[x, xt]]$. This completes the proof of Proposition 26.27. □

QUESTION AND REMARKS 26.30. (1) As we show in Proposition 26.12, the embeddings from Equation 1 involving two mixed polynomial-power series rings of dimension two over a field k with inverted elements are TGF. In the article [**76**] we ask whether this is true in the three-dimensional case. For example, is the embedding θ below TGF?

$$k[x, y]\,[[z]] \overset{\theta}{\hookrightarrow} k[x, y, 1/x]\,[[z]]$$

Yasuda shows the answer for this example is "No" in [**161**]. Dumitrescu establishes the answer is "No" in more generality; see Theorem 26.31.

(2) For the four-dimensional case, as observed in the discussion of Question 26.1, it follows from a result of Heinzer and Rotthaus [**59**, Theorem 1.12, p. 364] that the extension $k[x,y,u]\,[[z]] \hookrightarrow k[x,y,u,1/x]\,[[z]]$ is not TGF. Theorem 26.31 yields a direct proof of this fact.

We close this chapter with a result of Dumitrescu that shows many extensions involving only one power series variable are not TGF.

THEOREM 26.31. [**34**, Cor. 4 and Prop. 3] *Let D be an integral domain and let x, y, z be indeterminates over D. For every subring B of $D[[x,y]]$ that contains $D[x,y]$, the extension $B[[z]] \hookrightarrow B[1/x][[z]]$ is not TGF.*

PROOF. Let K be the field of fractions of D and let $\theta(z) \in D[[z]]$ be algebraically independent over $K(z)$.

CLAIM 26.32. *The elements xz and $x\theta(z) \in K[[x,z]]$ are analytically independent over $K[[x]]$, and $x\theta(z)$ is analytically independent over $K[[x,xz]]$.*

Proof of Claim: Let v and w be indeterminates over $K[[x]]$ and consider the $K[[x]]$-algebra homomorphism $\varphi : K[[x,v,w]] \to K[[x,z]]$ where $\varphi(v) = xz$ and $\varphi(w) = x\theta$. Let $g \in K[[x,v,w]]$ and write $g = \sum_{n\geq 0} g_n(x,v,w)$, where g_n is a form of degree n in x, v, w with coefficients in K. Thus $\varphi(g) = \sum_{n\geq 0} \varphi(g_n)$ and

$$g_n \;=\; \sum_{i+j+k=n} c_{ijk} x^i v^j w^k \;\;\Longrightarrow\;\; \varphi(g_n) \;=\; \sum_{i+j+k=n} c_{ijk} x^n z^j \theta^k.$$

If $\varphi(g) = 0$, then $\varphi(g_n) = 0$ for each n. Since z and θ are algebraically independent over K, each $c_{ijk} = 0$. Thus φ is injective. By Definition 26.25, Claim 26.32 holds.

It follows from Claim 26.32 that $\lambda := x\theta(z/x) \in zD[x,1/x][[z]]$ is analytically independent over $D[[x,z]]$.

Consider the $D[[x]][1/x][[z]]$-algebra homomorphism

$$\pi : D[[x]]\,[\frac{1}{x}]\,[[z,y]] \;\to\; D[[x]]\,[\frac{1}{x}]\,[[z]], \quad \text{where} \quad \pi(y) \;=\; \lambda.$$

Let $\mathbf{p} = \ker \pi$. Then $y - \lambda$ is a nonzero element of $\mathbf{p} \;\cap D[x,y][1/x][[z]]$, and so $0 \neq \mathbf{p} \;\cap B[1/x][[z]]$. We show $\mathbf{p} \cap D[[x,y,z]] = (0)$, and so also $\mathbf{p} \;\cap B\,[[z]] = (0)$.

The restriction of π to $D[[x,z,y]]$ is injective because λ is analytically independent over $D[[x,z]]$. Therefore $\mathbf{p} \cap D[[x,z,y]] = (0)$. This completes the proof of Theorem 26.31. □

Exercises

(1) Let k be a field and let $\aleph_0 = |\mathbb{N}|$. Prove that $\alpha = |k| \cdot \aleph_0$ and $\beta = |k|^{\aleph_0}$ in Theorem 26.14.
Suggestion: Notice that every polynomial of the form $x - a$, for $a \in k$, generates a maximal ideal of $k[x]$ and also that $|k[x]| = |k| \cdot \aleph_0$, since $k[x]$ can be considered as an infinite union of polynomials of each finite degree.

(2) Let y denote an indeterminate over the ring of integers $\mathbb{Z}$, and let $A = \mathbb{Z}[[y]]$.
   (a) Prove that every maximal ideal of A has height two.
   (b) Describe and make a diagram of the partially ordered set Spec A.

(c) Let $B = A[\frac{1}{y+2}]$. Describe the partially ordered set $\operatorname{Spec} B$. Prove that B has maximal ideals of height one, and deduce that $\operatorname{Spec} B$ is not order-isomorphic to $\operatorname{Spec} A$.

(d) Let $C = A[\frac{y}{2}]$. Describe the partially ordered set $\operatorname{Spec} C$. Prove that C has precisely two nonmaximal prime ideal of height one that are an intersection of maximal ideals, while each of A and B has precisely one nonmaximal prime ideal of height one that is an intersection of maximal ideals. Deduce that $\operatorname{Spec} C$ is not order-isomorphic to either $\operatorname{Spec} A$ or $\operatorname{Spec} B$.

CHAPTER 27

Extensions of local domains with trivial generic fiber

We consider injective local maps from a local domain R to a local domain S such that the generic fiber of the inclusion map $R \hookrightarrow S$ is trivial, that is $P \cap R \neq (0)$ for every nonzero prime ideal P of S. [1] We recall that S is said to be a *trivial generic fiber extension of* R, or more briefly, a *TGF extension*, if each nonzero ideal of S has a nonzero intersection with R, or equivalently, if each nonzero element of S has a nonzero multiple in R. We present in this chapter examples of injective local maps involving power series that are or fail to be TGF extensions.

Let $R \hookrightarrow S$ be an injective map of integral domains. Since ideals of S maximal with respect to not meeting the multiplicative system of nonzero elements of R are prime ideals, S is a TGF extension of R if and only if $P \cap R \neq (0)$ for each nonzero prime ideal P of S. Another condition equivalent to S is a TGF extension of R is that $U^{-1}S$ is a field, where $U = R \setminus (0)$.

Our work in this chapter is motivated by Question 24.4 asked by Melvin Hochster and Yongwei Yao. In this connection we use the following definition.

DEFINITION 27.1. Let $(R, \mathbf{m}) \hookrightarrow (S, \mathbf{n})$ be an injective local homomorphism of complete Noetherian local domains; thus $\mathbf{n} \cap R = \mathbf{m}$. We say that S is a *TGF-complete extension* of R if S is a TGF extension of R.

By Remark 24.5, in the equicharacteristic zero case such extensions arise as a composition

$$R = K[[x_1, ..., x_n]] \hookrightarrow T = L[[x_1, ..., x_n, y_1, ..., y_m]] \to T/P = S, \tag{27.1.0}$$

where K is a subfield of L, the x_i, y_j are formal indeterminates, and P is a prime ideal of T maximal with respect to being disjoint from the image of $R \setminus \{0\}$.

We discuss several topics and questions related to Question 24.4. Previous work related to this question concerning homomorphisms of formal power series rings appears in articles of Matsumura, Rotthaus, Abhyankar, Moh [**134**],[**3**] [**11**], among others.

In particular, if R and S contain a field and are both complete regular local rings of dimension two, then the map from R to S factors as a composition of maps $R_i := k[[x, xy]] \hookrightarrow S_i := k[[x, y]]$, where x and y are variables over a field k, see [**11**, Section 3].

[1] The material in this chapter is adapted from our paper [**77**] dedicated to Phil Griffith in honor of his contributions to commutative algebra.

27.1. General remarks about TGF extensions

We consider in Proposition 27.2 relatively easy TGF-complete extensions where the base ring has dimension one.

PROPOSITION 27.2. *Let $(R, \mathbf{m})$ be a complete one-dimensional local domain. Assume that $(S, \mathbf{n})$ is a TGF-complete extension of R. Then*

(1) $\dim(S) = 1$ *and* $\mathbf{m}S$ *is* $\mathbf{n}$*-primary.*
(2) *If $[S/\mathbf{n} : R/\mathbf{m}] < \infty$, then S is a finite integral extension of R.*

Thus, if $R \hookrightarrow S$ is a TGF extension with finite residue extension and $\dim S \geq 2$, then $\dim R \geq 2$.

PROOF. By Krull's Altitude Theorem 2.17, $\mathbf{n}$ is the union of the height-one primes of S. If $\dim S > 1$, then S has infinitely many height-one primes. Each nonzero element of $\mathbf{n}$ is contained in only finitely many of these height-one primes. If $\dim S > 1$, then the intersection of the height-one primes of S is zero. Since $\dim R = 1$ and $R \hookrightarrow S$ is TGF, every nonzero prime of S contains $\mathbf{m}$. Thus $\dim S = 1$ and $\mathbf{m}S$ is $\mathbf{n}$-primary. Moreover, if $[S/\mathbf{n} : R/\mathbf{m}] < \infty$, then S is finite over R by Theorem 3.9. □

REMARKS 27.3. (1) Notice that there exist TGF-complete extensions of R that have an arbitrarily large extension of residue field. For example, if k is a subfield of a field F and x is an indeterminate over F, then $R := k[[x]] \subseteq S := F[[x]]$ is a TGF-complete extension.

(2) Let $(R, \mathbf{m}) \hookrightarrow (T, \mathbf{q})$ be an injective local homomorphism of complete local domains. For $P \in \operatorname{Spec} T$, $S := T/P$ is a TGF-complete extension of R if and only if P is an ideal of T maximal with respect to the property that $P \cap R = (0)$.

REMARKS 27.4. Let $X = \{x_1 \ldots, x_n\}$, $Y = \{y_1 \ldots, y_m\}$ and $Z = \{z_1 \ldots, z_r\}$ be algebraically independent finite sets of indeterminates over a field k, where $n \geq 2$, $m, r \geq 1$. Set $R := k[[X]]$ and let P be a prime ideal of $k[[X, Y, Z]]$ that is maximal with respect to $P \cap R = (0)$. Then we have the inclusions:

$$R := k[[X]] \overset{\sigma}{\hookrightarrow} S := k[[X, Y]]/(P \cap k[[X, Y]]) \overset{\tau}{\hookrightarrow} T := k[[X, Y, Z]]/P.$$

By Remark 27.3.2, $\tau \cdot \sigma$ is a TGF extension. By Proposition 26.8.3, $S \hookrightarrow T$ is TGF.

(1) If the map $\operatorname{Spec} T \to \operatorname{Spec} S$ is surjective, then $\sigma : R \hookrightarrow S$ is TGF by Proposition 26.8.2.
(2) If $R \hookrightarrow T$ is finite, then $R \hookrightarrow S$ is also finite, and so $\sigma : R \hookrightarrow S$ is TGF.
(3) If $R \hookrightarrow T$ is not finite, then $\dim T = 2$ by Theorem 25.6.
(4) If $P \cap k[[X, Y]] = 0$, then $S = R[[Y]]$ and $R \hookrightarrow S$ is not TGF. (We show in Example 27.16 that this can occur.)

REMARKS AND QUESTION 27.5. (1) With notation as in Remarks 27.4 and with $Y = \{y\}$, a singleton set, it is always true that $\operatorname{ht}(P \cap R[[y]]) \leq n - 1$. (See Theorem 25.5.) Moreover, if $\operatorname{ht}(P \cap R[[y]]) = n - 1$, then $R \hookrightarrow S$ is TGF. Thus if $n = 2$ and $P \cap R[[y]] \neq 0$, then $R \hookrightarrow S$ is TGF.

(2) With notation as in (1) and $n = 3$, it can happen that $P \cap k[[X, y]] \neq (0)$ and $R \hookrightarrow R[[y]]/(P \cap R[[y]])$ is not a TGF extension. To construct an example of such a prime ideal P, we proceed as follows: Since $\dim(k[[X, y]]) = 4$, there exists

a prime ideal Q of $k[[X,y]]$ with $\operatorname{ht} Q = 2$ and $Q \cap k[[X]] = (0)$, see Theorem 25.5. Let $\mathbf{p} \subset Q$ be a prime ideal with $\operatorname{ht} \mathbf{p} = 1$. Since $\mathbf{p} \subsetneq Q$ and $Q \cap k[[X]] = (0)$, the extension $k[[X]] \hookrightarrow k[[X,y]]/\mathbf{p}$ is not a TGF extension. In particular, it is not finite. Let $P \in \operatorname{Spec}(k[[X,y,Z]])$ be maximal with respect to $P \cap k[[X,y]] = \mathbf{p}$. By Corollary 27.10 below, $\dim(k[[X,y,Z]]/P) = 2$. Hence P is maximal in the generic fiber over $k[[X]]$.

(3) If $(R,\mathbf{m}) \hookrightarrow (S,\mathbf{n})$ is a TGF-complete extension with $S/\mathbf{n}$ finite algebraic over $R/\mathbf{m}$, can the transcendence degree of S over R be finite but nonzero?

(4) If $(R,\mathbf{m}) \hookrightarrow (S,\mathbf{n})$ is a TGF-complete extension as in (3) with R equicharacteristic and $\dim R \geq 2$, then by Corollary 27.9 below it follows that either S is a finite integral extension of R or $\dim S = 2$.

PROPOSITION 27.6. *Let $A \hookrightarrow B$ be a TGF extension, where B is a Noetherian integral domain. For each $Q \in \operatorname{Spec} B$, we have $\operatorname{ht} Q \leq \operatorname{ht}(Q \cap A)$. In particular, $\dim A \geq \dim B$.*

PROOF. If $\operatorname{ht} Q = 1$, it is clear that $\operatorname{ht} Q \leq \operatorname{ht}(Q \cap A)$ since $Q \cap A \neq (0)$. Let $\operatorname{ht} Q = n \geq 2$, and assume by induction that $\operatorname{ht} Q' \leq \operatorname{ht}(Q' \cap A)$ for each $Q' \in \operatorname{Spec} B$ with $\operatorname{ht} Q' \leq n-1$. Since B is Noetherian,

$$(0) = \bigcap \{Q' \mid Q' \subset Q \quad \text{and} \quad \operatorname{ht} Q' = n-1\}.$$

Hence there exists $Q' \subset Q$ with $\operatorname{ht} Q' = n-1$ and $Q' \cap A \subsetneq Q \cap A$. We have $n-1 \leq \operatorname{ht}(Q' \cap A) < \operatorname{ht}(Q \cap A)$, and so $\operatorname{ht}(Q \cap A) \geq n$. □

27.2. TGF-complete extensions with finite residue field extension

SETTING 27.7. Let $n \geq 2$ be an integer, let $X = \{x_1, \ldots, x_n\}$ be a set of independent variables over the field k and let $R = k[[X]]$ be the formal power series ring in n variables over the field k.

THEOREM 27.8. *Let $R = k[[X]]$ be as in Setting 27.7. Assume that $R \hookrightarrow S$ is a TGF-complete extension, where $(S,\mathbf{n})$ is a complete Noetherian local domain and $S/\mathbf{n}$ is finite algebraic over k. Then either $\dim S = n$ and S is a finite integral extension of R or $\dim S = 2$.*

PROOF. It is clear that if S is a finite integral extension of R, then $\dim S = n$. Assume S is not a finite integral extension of R. Let $b_1, \ldots, b_m \in \mathbf{n}$ be such that $\mathbf{n} = (b_1, \ldots, b_m)S$, and let $Y = \{y_1, \ldots, y_m\}$ be a set of independent variables over R. Since S is complete the R-algebra homomorphism $\varphi : T := R[[Y]] \to S$ such that $\varphi(y_i) = b_i$ for each i with $1 \leq i \leq m$ is well defined. Let $Q = \ker \varphi$. We have

$$R \hookrightarrow T/Q \hookrightarrow S.$$

By Theorem 3.9 S is a finite module over T/Q. Hence $\dim S = \dim(T/Q)$ and the map $\operatorname{Spec} S \to \operatorname{Spec}(T/Q)$ is surjective, and so by Proposition 26.8(3) $R \hookrightarrow T/Q$ is TGF. By Corollary 25.8, $\dim(T/Q) = 2$, and so $\dim S = 2$. □

COROLLARY 27.9. *Let $(A,\mathbf{m})$ and $(S,\mathbf{n})$ be complete equicharacteristic Noetherian local domains with $\dim A = n \geq 2$ and suppose that $A \hookrightarrow S$ is a local injective homomorphism and that the residue field $S/\mathbf{n}$ is finite algebraic over the residue field $A/\mathbf{m} := k$. If $A \hookrightarrow S$ is a TGF-complete extension, then either $\dim S = n$ and S is a finite integral extension of A or $\dim S = 2$.*

PROOF. By [**103**, Theorem 29.4(3)], A is a finite integral extension of $R = k[[X]]$, where X is as in Setting 27.7. We have $R \hookrightarrow A \hookrightarrow S$. By Proposition 26.8(1), $R \hookrightarrow S$ is TGF. By Theorem 27.8, either $\dim S = n$ and S is a finite integral extension of A or $\dim S = 2$. □

For example, if $R = k[[x_1, \dots, x_4]]$ and $S = k[[y_1, y_2, y_3]]$, then every k-algebra embedding $R \hookrightarrow S$ fails to be TGF.

COROLLARY 27.10. *Let $R = k[[X]]$ be as in Setting 27.7. Let $Y = \{y_1, \dots, y_m\}$ be a set of m independent variables over R and let $S = R[[Y]]$. If $P \in \operatorname{Spec} R$ is such that $\dim(R/P) \geq 2$ and $Q \in \operatorname{Spec} S$ is maximal with respect to $Q \cap R = P$, then either*

(i) $\dim(S/Q) = 2$, *or*
(ii) $R/P \hookrightarrow S/Q$ *is a finite integral extension (and so* $\dim(R/P) = \dim(S/Q)$*).*

PROOF. Let $A := R/P \hookrightarrow S/Q =: B$, and apply Corollary 27.9. □

GENERAL EXAMPLE 27.11. It is known that, for each positive integer n, the power series ring $R = k[[x_1, \dots, x_n]]$ in n variables over a field k can be embedded into a power series ring in two variables over k. The construction is based on the fact that the power series ring $k[[z]]$ in the single variable z contains an infinite set of algebraically independent elements over k. Let $\{f_i\}_{i=1}^{\infty} \subset k[[z]]$ with $f_1 \neq 0$ and $\{f_i\}_{i=2}^{\infty}$ algebraically independent over $k(f_1)$. Let $(S := k[[z, w]], \mathbf{n} := (z, w))$ be the formal power series ring in the two variables z, w. Fix a positive integer n and consider the subring $R_n := k[[f_1 w, \dots, f_n w]]$ of S with maximal ideal $\mathbf{m}_n = (f_1 w, \dots, f_n w)$. Let $x_1, \dots, x_n$ be new indeterminates over k and define a k-algebra homomorphism $\varphi : k[[x_1, \dots, x_n]] \to R_n$ by setting $\varphi(x_i) = f_i w$ for $i = 1, \dots, n$.

CLAIM 27.12. *(see* [**165**, pp. 219-220]*) φ is an isomorphism.*

PROOF. Suppose $g = \sum_{m=0}^{\infty} g_m$, where g_m is a form of degree m in $k[x_1, \dots, x_n]$. Then

$$\varphi(g) = \sum_{m=0}^{\infty} \varphi(g_m) \quad \text{and} \quad \varphi(g_m) = g_m(f_1 w, \dots, f_n w) = w^m g_m(f_1, \dots, f_n),$$

where $g_m(f_1, \dots, f_n) \in k[[z]]$. If $\varphi(g) = 0$, then $g_m(f_1, \dots, f_n) = 0$ for each m. Thus

$$0 = g_m(f_1, \dots, f_n) = \sum_{i_1 + \cdots + i_n = m} a_{i_1, \dots, i_n} f_1^{i_1} \cdots f_n^{i_n},$$

where the $a_{i_1, \dots, i_n} \in k$ and the i_j are nonnegative integers. Our hypothesis on the f_j implies that each of the $a_{i_1, \dots, i_n} = 0$, and so $g_m = 0$ for each m. □

PROPOSITION 27.13. *With notation as in Example 27.11, for each integer $n \geq 2$, the extension $(R_n, \mathbf{m}_n) \hookrightarrow (S, \mathbf{n})$ is nonfinite TGF-complete with trivial residue extension. Moreover $\operatorname{ht}(P \cap R_n) \geq n - 1$, for each nonzero prime $P \in \operatorname{Spec} S$.*

PROOF. We have $k = R_n/\mathbf{m}_n = S/\mathbf{n}$, so the residue field of S is a trivial extension of that of R_n. Since $\mathbf{m}_n S$ is not $\mathbf{n}$-primary, S is not finite over R_n. If $P \cap R_n = \mathbf{m}_n$, then $\operatorname{ht}(P \cap R_n) = n \geq n - 1$. Since $\dim S = 2$, if $\mathbf{m}_n$ is not contained in P, then $\operatorname{ht} P = 1$, S/P is a one-dimensional local domain, and $\mathbf{m}_n(S/P)$ is primary for the maximal ideal $\mathbf{n}/P$ of S/P. It follows that $R_n/(P \cap R_n) \hookrightarrow S/P$ is a finite integral extension by Theorem 3.9. Therefore $\dim(R_n/(P \cap R_n)) = 1$. Since R_n is catenary and $\dim R_n = n$, $\operatorname{ht}(P \cap R_n) = n - 1$. □

COROLLARY 27.14. *Let X and $R = k[[X]]$ be as in Setting 27.7. Then there exists an infinite properly ascending chain of two-dimensional TGF-complete extensions $R =: S_0 \hookrightarrow S_1 \hookrightarrow S_2 \hookrightarrow \cdots$ such that each S_i has the same residue field as R and S_{i+1} is a nonfinite TGF-complete extension of S_i for each i.*

PROOF. Example 27.11 and Proposition 27.13 imply that R can be identified with a proper subring of the power series ring in two variables so that $k[[y_1, y_2]]$ is a TGF-complete extension of R and the extension is not finite. Now Example 27.11 and Proposition 27.13 can be applied again, to $k[[y_1, y_2]]$, and so on. □

EXAMPLE 27.15. A particular case of Example 27.11.

For $R := k[[x, y]]$, the extension ring $S := k[[x, y/x]]$ has infinite transcendence over R by Sheldon's work; see [**147**]. The method used in [**147**] to prove that S has infinite transcendence degree over R is by constructing power series in y/x with 'special large gaps'. Since $k[[x]]$ is contained in R, it follows that S is a TGF-complete extension of R. To show this, it suffices to show $P \cap R \neq (0)$ for each $P \in \operatorname{Spec} S$ with $\operatorname{ht} P = 1$. This is clear if $x \in P$, while if $x \notin P$, then $k[[x]] \cap P = (0)$, and so $k[[x]] \hookrightarrow R/(P \cap R) \hookrightarrow S/P$ and S/P is finite over $k[[x]]$. Therefore $\dim(R/(P \cap R)) = 1$, and so $P \cap R \neq (0)$.

Notice that the extension $k[[x, y]] \hookrightarrow k[[x, y/x]]$ is, up to isomorphism, the same as the extension $k[[x, xy]] \hookrightarrow k[[x, y]]$.

In Example 27.16 we show the situation of Remark 27.4.4 does occur.

EXAMPLE 27.16. Let k, $X = \{x_1, x_2\}$, $Y = \{y\}$, $Z = \{z\}$ and $R = k[[x_1, x_2]]$ be as in Remarks 27.4. Let $f_1, f_2 \in k[[z]]$ be algebraically independent over k. Let P denote the ideal of $k[[x_1, x_2, y, z]]$ generated by $(x_1 - f_1 y, x_2 - f_2 y)$, Then P is the kernel of the k-algebra homomorphism $\theta : k[[x_1, x_2, y, z]] \to k[[y, z]]$ obtained by defining $\theta(x_1) = f_1 y$, $\theta(x_2) = f_2 y$, $\theta(y) = y$ and $\theta(z) = z$. In the notation of Remark 27.4,

$$T = k[[x_1, x_2, y, z]]/P \cong k[[y, z]].$$

Let $\varphi := \theta|_R$ and $\tau := \theta|_{R[[y]]}$. The proof of Claim 27.12 shows that φ and τ are embeddings. Hence $P \cap R[[y]] = (0)$. By Proposition 27.13, φ and τ are TGF. We have

$$R \overset{\sigma}{\hookrightarrow} S = \frac{R[[y]]}{P \cap R[[y]]} = R[[y]] \overset{\tau}{\hookrightarrow} \frac{R[[y, z]]}{P} \cong k[[y, z]],$$

where $\sigma : R \hookrightarrow S$ is the inclusion map. Also $\tau \cdot \sigma = \varphi$ is TGF as in Remark 27.4.4. Since $yS \cap R = (0)$, $\sigma : R \hookrightarrow S$ is not TGF.

QUESTIONS 27.17. (1) If $\varphi : R \hookrightarrow S$ is a TGF-complete nonfinite extension with finite residue field extension, is it always true that φ can be extended to a TGF-complete nonfinite extension $R[[y]] \hookrightarrow S$?

(2) Suppose that $R \hookrightarrow S$ is a TGF-complete extension and y is an indeterminate over S. It is natural to ask: Does $R[[y]] \hookrightarrow S[[y]]$ have the TGF property? Computing with elements, one may ask if for $s \in S \setminus R$, does $y + s$ have a multiple in $R[[y]]$? There is a $t \in S$ with $ts \in R$, but is there a $t' \in S$ with both $t't$ and $t'ts \in R$?

(3) A related question is whether the given $R \hookrightarrow S$ is extendable to an injective local homomorphism $\varphi : R[[y]] \hookrightarrow S$. For example, with k a field, $k[[x_1]][y]_{(x_1,y)} \hookrightarrow k[y][[x_1]]_{(x_1,y)}$ is TGF. Can we extend to $k[[x_1]][y][[x_2]]_{(x_1,x_2,y)} \hookrightarrow k[y][[x_1]]_{(x_1,y)}$, say by $x_2 \to \sum_{n=0}^{\infty}(yx)^n$, which is still local injective?

We show in Proposition 27.18 that the answer to (27.17.2) is 'no' if the answer to (27.17.3) is "yes", that is, the given $R \hookrightarrow S$ is extendable to an injective local homomorphism $R[[y]] \hookrightarrow S$. In Example 27.19 we present an example where this occurs.

PROPOSITION 27.18. *Let $R \hookrightarrow S$ be a TGF-complete extension and let y be an indeterminate over S. If $R \hookrightarrow S$ is extendable to an injective local homomorphism $\varphi : R[[y]] \hookrightarrow S$, then $R[[y]] \hookrightarrow S[[y]]$ is not TGF.*

PROOF. Let $a := \varphi(y)$ and consider the ideal $Q = (y-a)S[[y]]$. The canonical map $S[[y]] \to S[[y]]/Q = S$ extends φ. Thus $Q \cap R[[y]] = (0)$ and $R[[y]] \hookrightarrow S[[y]]$ is not TGF. □

EXAMPLE 27.19. Let $R := R_n = k[[f_1 w, \dots, f_n w]] \hookrightarrow S := k[[z, w]]$ be as in Example 27.11 with $n \geq 2$. Define the extension $\varphi : R[[y]] \hookrightarrow S$ by setting $\varphi(y) = f_{n+1} w \in S$. By Proposition 27.13, $\varphi : R[[y]] \hookrightarrow S$ is TGF-complete. Thus by Proposition 27.18, $R[[y]] \hookrightarrow S[[y]]$ is not TGF.

REMARK AND QUESTIONS 27.20. Let $(R, \mathbf{m}) \hookrightarrow (S, \mathbf{n})$ be a TGF-complete extension. Assume that $[S/\mathbf{n} : R/\mathbf{m}] < \infty$ and that S is not finite over R. By Theorem 3.9, $\mathbf{m}S$ is not $\mathbf{n}$-primary. Thus $\dim S > \operatorname{ht}(\mathbf{m}S)$. Therefore $\dim S > 1$, and so by Proposition 27.2, $\dim R > 1$.

(1) If $(R, \mathbf{m})$ is equicharacteristic, then by Corollary 27.9, $\dim S = 2$. Is it true in general that $\dim S = 2$?
(2) Is it possible to have $\dim S - \operatorname{ht}(\mathbf{m}S) > 1$?

EXAMPLES 27.21. (1) Let $R := k[[x, xy, z]] \hookrightarrow S := k[[x, y, z]]$. We show this is not a TGF extension. By (27.15), $\varphi : k[[x, xy]] \hookrightarrow k[[x, y]]$ is TGF-complete. By Proposition 27.18, it suffices to extend φ to an injective local homomorphism of $k[[x, xy, z]]$ to $k[[x, y]]$. Let $f \in k[[x]]$ be such that x and f are algebraically independent over k, and so $(1, x, f)$ is not a solution to any nonzero homogeneous form over k. As in (27.8) and (27.11), the extension of φ obtained by mapping $z \to fy$ is an injective local homomorphism.

(2) The extension $R = k[[x, xy, xz]] \hookrightarrow S = k[[x, y, z]]$ is not a TGF-complete extension, since $R = k[[x, xy, xz]] \hookrightarrow k[[x, xy, z]] \hookrightarrow S = k[[x, y, z]]$ is a composition of two extensions that are not TGF by part (1). Now apply Proposition 26.8.

27.3. The case of transcendental residue extension

In this section we address but do not fully resolve the following question.

QUESTION 27.22. If $(S, \mathbf{n})$ is a TGF-complete extension of $(R, \mathbf{m})$ and if $S/\mathbf{n}$ is transcendental over $R/\mathbf{m}$ does it follow that $\dim S \leq 1$?

In Proposition 27.23 we prove every complete Noetherian local domain of positive dimension has a one-dimensional TGF-complete extension.

PROPOSITION 27.23. *Let $(R, \mathbf{m})$ be a Noetherian local domain of positive dimension.*

(1) *There exists a one-dimensional complete Noetherian local domain $(S, \mathbf{n})$ that is a TGF extension of R.*

(2) *If R is complete, there exists a one-dimensional TGF-complete extension of R.*

PROOF. By Chevalley's Theorem 2.29 there exists a discrete rank-one valuation domain (DVR) $(S, \mathbf{n})$ that dominates R. The $\mathbf{n}$-adic completion $\widehat{S}$ of S is a DVR that dominates R. Moreover, if $(S, \mathbf{n})$ is a one-dimensional Noetherian local domain that dominates a Noetherian local domain $(R, \mathbf{m})$ of positive dimension, then it is obvious that S is a TGF extension of R, and so if R and S are also complete, then S is a TGF-complete extension of R. □

SETTING 27.24. Let $n \geq 2$ be an integer, let $X = \{x_1, \ldots, x_n\}$ be a set of independent variables over the field k and let $R = k[[X]]$ be the formal power series ring in n variables over the field k. Let z, w, t, v be independent variables over R.

PROPOSITION 27.25. *Let notation be as in Setting 27.24.*

(1) *There exists a TGF embedding $\theta : k[[z, w]] \to k(t)[[v]]$ defined by $\theta(z) = tv$ and $\theta(w) = v$.*
(2) *Moreover, the composition $\psi = \theta \circ \varphi$ of θ with $\varphi : R \to k[[z, w]]$ given in General Example 27.11 is also TGF.*

PROOF. Suppose $f \in \ker \theta$. Write $f = \sum_{n=0}^{\infty} f_n(z, w)$, where f_n is a homogeneous form of degree n with coefficients in k. We have

$$0 = \theta(f) = \sum_{n=0}^{\infty} f_n(tv, v) = \sum_{n=0}^{\infty} v^n f_n(t, 1)\,.$$

This implies $f_n(t, 1) = 0$ for each n. Since t is algebraically independent over k, we have $f_n(z, w) = 0$ for each n. Thus $f = 0$ and θ is an embedding. Since θ is a local homomorphism and $\dim(k(t)[[v]]) = 1$, it is clear that θ is TGF. The second statement is clear since a local embedding or a local domain into a one-dimensional local domain is TGF. □

As a consequence of Proposition 27.25, we prove:

COROLLARY 27.26. *Let $R = k[[X]]$ be as above and let $A = k(t)[[X]]$. There exists a prime ideal $P \in \operatorname{Spec} A$ in the generic fiber over R with $\operatorname{ht} P = n - 1$. In particular, the inclusion map $R = k[[X]] \hookrightarrow A = k(t)[[X]]$ is not TGF.*

PROOF. Define $\varphi : R \to k[[z, w]] := S$, by

$$\varphi(x_1) = z, \quad \varphi(x_2) = h_2(w)z, \quad \ldots, \quad \varphi(x_n) = h_n(w)z,$$

where $h_2(w), \ldots, h_n(w) \in k[[w]]$ are algebraically independent over k. Also define $\theta : S \to k(t)[[v]] := B$ by $\theta(z) = tv$ and $\theta(w) = v$. Consider the following diagram

$$\begin{array}{ccc} R = k[[X]] & \xrightarrow{\subset} & A = k(t)[[X]] \\ {\scriptstyle\varphi}\big\downarrow & & {\scriptstyle\Psi}\big\downarrow \\ S = k[[z, w]] & \xrightarrow{\theta} & B = k(t)[[v]], \end{array}$$

where $\Psi : A \to B$ is the identity map on $k(t)$ and is defined by

$$\Psi(x_1) = tv, \quad \Psi(x_2) = h_2(v)tv, \quad \ldots, \quad \Psi(x_n) = h_n(v)tv.$$

Notice that $\Psi|_R = \psi = \theta \circ \varphi$. Therefore the diagram is commutative. Let $P = \ker \Psi$. Since Ψ is surjective, $\operatorname{ht} P = n - 1$. Commutativity of the diagram implies that $P \cap R = (0)$. □

DISCUSSION 27.27. Let us describe generators for the prime ideal $P = \ker \Psi$ given in Corollary 27.26. Under the map Ψ, $x_1 \mapsto tv$, and so $\frac{x_1}{t} \mapsto v$. Since also $x_2 \mapsto h_2(v)tv, \ldots, x_n \mapsto h_n(v)tv$, we see that

$$(x_2 - h_2(\frac{x_1}{t})x_1,\, x_3 - h_3(\frac{x_1}{t})x_1, \ldots, x_n - h_n(\frac{x_1}{t})x_1)A \subseteq P$$

(that is, $\Psi(x_2 - h_2(x_1/t)x_1) = h_2(v)tv - h_2(v)tv = 0$ etc.) Since the ideal on the left-hand-side is a prime ideal of height $n-1$, the inclusion is an equality. Thus we have generators for the prime ideal $P = \ker \Psi$ resulting from the definitions of φ and θ given in the corollary.

On the other hand, in Corollary 27.26 if we change the definition of θ and we define $\theta' : k[[z,w]] \to k(t)[[v]]$ by $\theta'(z) = v$ and $\theta'(w) = tv$ (but we keep φ as above), then ψ' defined by $\psi'|_R = \theta' \cdot \varphi$ maps $x_1 \to v$, $x_2 \to h_2(tv)v$, $\ldots$, $x_n \to h_n(tv)v$. In this case

$$(x_2 - h_2(tx_1)x_1,\, x_3 - h_3(tx_1)x_1, \ldots, x_n - h_n(tx_1)x_1)A \subseteq \ker \Psi' = P'.$$

Again the ideal on the left-hand-side is a prime ideal of height $n-1$, and so we have equality. This yields a different prime ideal P'.

In this case one can also see directly for

$$P' = (x_2 - h_2(tx_1)x_1,\, x_3 - h_3(tx_1)x_1, \ldots, x_n - h_n(tx_1)x_1)A$$

that $P' \cap R = (0)$. We have $\Psi : A \to A/P' = k(t)[[v]]$. Suppose $f \in R \cap P'$. We write $f = \sum_{\ell=0}^{\infty} f_\ell(x_1, \ldots, x_n)$, where $f_\ell \in k[x_1, \ldots, x_n]$ is a homogeneous form of degree ℓ. We have

$$0 = \Psi'(f) = \sum_{\ell=0}^{\infty} f_\ell(v, h_2(tv)v, \ldots, h_n(tv)v) = \sum_{\ell=0}^{\infty} v^\ell f_\ell(1, h_2(tv), \ldots, h_n(tv)).$$

This implies $f_\ell(1, h_2(tv), \ldots, h_n(tv)) = 0$ for each ℓ. Since $h_2, \ldots, h_n$ are algebraically independent over k, each of the homogeneous forms $f_\ell(x_1, \ldots, x_n) = 0$. Hence $f = 0$.

QUESTION 27.28. With notation as in Corollary 27.26, does every prime ideal of the ring A maximal in the generic fiber over R have height $n-1$?

THEOREM 27.29. *Let $(A, \mathbf{m}) \hookrightarrow (B, \mathbf{n})$ be an extension of two-dimensional regular local domains. Assume that B dominates A and that $B/\mathbf{n}$ as a field extension of $A/\mathbf{m}$ is not algebraic. Then $A \hookrightarrow B$ is not TGF.*

PROOF. Since $\dim A = \dim B$, the assumption that $B/\mathbf{n}$ is transcendental over $A/\mathbf{m}$ implies that B is not algebraic over A by Cohen's Theorem 2.20 on extensions. If $\mathbf{m}B$ is $\mathbf{n}$-primary, then B is faithfully flat over A [**103**, Theorem 23.1], and a result of Heinzer and Rotthaus, [**59**, Theorem 1.12], implies that $A \hookrightarrow B$ is not TGF in this case.

If $\mathbf{m}B$ is principal, then $\mathbf{m}B = xB$ for some $x \in \mathbf{m}$ since B is local. It follows that $\mathbf{m}/x \subset B$. Localizing $A[\mathbf{m}/x]$ at the prime ideal $\mathbf{n} \cap A[\mathbf{m}/x]$ gives a local quadratic transform $(A_1, \mathbf{m}_1)$ of A, see Definition 14.1. If $\dim A_1 = 1$, then $A_1 \hookrightarrow B$ is not TGF because only finitely many prime ideals of B can contract to the maximal ideal of A_1. Hence $A \hookrightarrow B$ is not TGF if $\dim A_1 = 1$. If $\dim A_1 = 2$, then $(A_1, \mathbf{m}_1)$ is a 2-dimensional regular local domain dominated by $(B, \mathbf{n})$ and the field $A_1/\mathbf{m}_1$ is finite algebraic over $A/\mathbf{m}$, and so $B/\mathbf{n}$ is transcendental over $A_1/\mathbf{m}_1$. Thus we can repeat the above analysis: If $\mathbf{m}_1 B$ is $\mathbf{n}$-primary, then as above $A \hookrightarrow B$

is not TGF. If $\mathbf{m}_1 B$ is principal, we obtain a local quadratic transform $(A_2, \mathbf{m}_2)$ of A_1. If this process does not end after finitely many steps, we have a union $V = \cup_{n=1}^{\infty} A_n$ of an infinite sequence A_n of quadratic transforms of a 2-dimensional regular local domains. By [**2**], the integral domain V is a valuation domain of rank at most 2 contained in B, and so at most finitely many of the height-one primes of B have a nonzero intersection with V. Therefore $V \hookrightarrow B$ is not TGF and hence also $A \hookrightarrow B$ is not TGF.

Thus by possibly replacing A by an iterated local quadratic transform A_n of A, we may assume that $\mathbf{m}B$ is neither $\mathbf{n}$-primary nor principal. Let $\mathbf{m} = (x, y)A$. There exist $f, g, h \in B$ such that $x = gf, y = hf$ and g, h is a regular sequence in B. Hence $(g, h)B$ is $\mathbf{n}$-primary. Let $f = f_1^{e_1} \cdots f_r^{e_r}$, where $f_1 B, \ldots f_r B$ are distinct height-one prime ideals and the e_i are positive integers. Then $f_1 B, \ldots, f_r B$ are precisely the height-one primes of B that contain $\mathbf{m}$.

Let $t \in B$ be such that the image to t in $B/\mathbf{n}$ is transcendental over $A/\mathbf{m}$. Modifying t if necessary by an element of $\mathbf{n}$ we may assume that t is transcendental over A. We have $\mathbf{n} \cap A[t] = \mathbf{m}[t]$. Let $A(t) = A[t]_{\mathbf{m}[t]}$. Notice that $A(t)$ is a 2-dimensional regular local domain with maximal ideal $\mathbf{m}A(t)$ that is dominated by $(B, \mathbf{n})$. We have

$$A \hookrightarrow A[t] \hookrightarrow A(t) \hookrightarrow B.$$

For each positive integer i, let $P_i = (xt^i - y)A(t)$. Since t is transcendental over A, we have $P_i \cap A = (0)$ for each $i \in \mathbb{N}$. Notice that $P_i B = (gft^i - hf)B = f(gt^i - h)B$. If $i \neq j$, the element $t^i - t^j$ is a unit of B. Hence $(gt^i - h, gt^j - h)B = (g, h)B$ is $\mathbf{n}$-primary if $i \neq j$. Therefore a height one prime Q of B contains $gt^i - h$ for at most one integer i. Hence there exists a positive integer n such that if Q is a minimal prime of $(gt^n - h)B$, then $Q \notin \{f_1 B, \ldots, f_r B\}$. It follows that $Q \cap A(t)$ has height one. Since $P_n \subseteq (gt^n - h)B \subseteq Q$, we have $Q \cap A(t) = P_n$. Thus $Q \cap A = (0)$. This completes the proof. □

We have the following immediate corollary to Theorem 27.29.

COROLLARY 27.30. *Let x, y, z, w, t be indeterminates over the field k and let*

$$\varphi : R = k[[x, y]] \hookrightarrow S := k(t)[[z, w]]$$

be an injective local k-algebra homomorphism. Then $\varphi(R) \hookrightarrow S$ is not TGF.

In relation to Question 27.22, Example 27.31 is a TGF extension $A \hookrightarrow B$ that is not complete for which the residue field of B is transcendental over that of A and $\dim B = 2$.

EXAMPLE 27.31. Let $A = k[x, y, z, w]_{(x,y,z,w)}$, where k is a field and $xw = yz$. Thus A is a 3-dimensional normal local domain with maximal ideal $\mathbf{m} := (x, y, z, w)A$ and residue field $A/\mathbf{m} \cong k$. Since $y/x = w/z$, we have

$$C \ := \ A[\frac{y}{x}] \ = \ k[\frac{y}{x}, \ x, \ z]$$

is a polynomial ring in 3 variables over k. Thus $B := C_{(x,z)}$ is a 2-dimensional regular local domain with maximal ideal $\mathbf{n} = (x, z)B$. Notice that $(B, \mathbf{n})$ birationally dominates $(A, \mathbf{m})$. Hence $(A, \mathbf{m}) \hookrightarrow (B, \mathbf{n})$ is a TGF extension. Also $B = k(y/x)[x, z]_{(x,z)}$, and so $k(y/x)$ is a coefficient field for B. The image t of y/x in $B/\mathbf{n}$ is transcendental over k and $B/\mathbf{n} = k(t)$. The completion of A is the normal local domain $\widehat{A} = k[[x, y, z, w]]$, where $xw = yz$. By a form of

Zariski's Subspace Theorem [**3**, (10.6)], $\widehat{A}$ is dominated by $\widehat{B}$ and $\widehat{B}$ is isomorphic to $k(t)[[x,z]]$, where t is transcendental over k. We have $\varphi : \widehat{A} \hookrightarrow \widehat{B}$, where $\varphi(x) = x, \varphi(z) = z, \varphi(y) = tx, \varphi(w) = tz$, and $\varphi(xw) = xtz = \varphi(yz)$.

Exercise

(1) With notation as in Example 27.31, prove that $\widehat{A} \hookrightarrow \widehat{B}$ is not a TGF-complete extension. Equivalently, prove that the inclusion map

$$R := k[[x, z, tx, tz]] \hookrightarrow k(t)[[x, z]] := S$$

is not a TGF extension.

CHAPTER 28

Examples discussed in this book

Here is a list of examples presented in this book, with a brief description of each.

(1) The "simplest" example of a Noetherian local domain A on an algebraic function field L/k of at least two variables that is not essentially finitely generated over its ground field k, i.e., A is not the localization of a finitely generated k-algebra; see Example 4.7.
(2) A two-dimensional regular local domain A that is a nested union of three-dimensional regular local domains that A birationally dominates; see Example 4.9.
(3) A two-dimensional regular local domain A that is a nested union of four-dimensional regular local domains that A birationally dominates; see Example 4.10.
(4) A one-dimensional Noetherian local domain A that is the local coordinate ring of a nodal plane curve singularity; see Example 4.12. The integral closure of A is a homomorphic image of a regular Noetherian domain of dimension two with precisely two maximal ideals.
(5) A two-dimensional regular local domain A that is not Nagata and thus not excellent. The ring A contains a prime element f that factors as a square in the completion $\widehat{A}$ of A, that is, $f = g^2$ for some element $g \in \widehat{A}$; see Example 4.14, Remarks 4.15.2, Proposition 6.13 and Remark 6.14, [**117**, Example 7, pp. 209-211].
(6) A two-dimensional normal Noetherian local domain D that is analytically reducible; see Example 4.14 and Remarks 4.15.1, [**117**, Example 7, pp. 209-211]. Moreover, there exists a two-dimensional regular local domain that birationally dominates D and is not essentially finitely generated over D.
(7) A three-dimensional regular local domain A that is Nagata but not excellent. The formal fibers of A are reduced but not regular; see Examples 4.16 and 6.17 and Remark 4.17, [**131**].
(8) A non-Noetherian three-dimensional local Krull domain $(B, \mathbf{n})$ such that $\mathbf{n}$ is two-generated, the $\mathbf{n}$-adic completion of B is a two-dimensional regular local domain, and B birationally dominates a four-dimensional regular local domain; see Theorem 12.3 and Example 12.6.
(9) Every Noetherian local domain $(A, \mathbf{n})$ having a coefficient field k, and having the property that the field of fractions L of A is finitely generated over k is realizable as an intersection $L \cap \widehat{R}/I$, where R is a Noetherian local domain essentially finitely generated over k with $\mathcal{Q}(R) = L$, and

I is an ideal in the completion $\widehat{R}$ of R such that $P \cap R = (0)$ for each associated prime P of I; see Corollary 4.3.

(10) An example of Inclusion Construction 5.3 where the approximation domain B is equal to the intersection domain A; see Remark 4.19, Local Prototype Example 4.25 and Example 12.19.

(11) A strictly descending chain of one-dimensional analytically ramified Noetherian local domains that birationally dominate a polynomial ring in two variables over a field; see Example 17.18.

(12) A non-excellent DVR obtained by Localized Polynomial Example Theorem 17.28; see Proposition 9.4.

(13) A two-dimensional non-excellent regular local domain obtained by Localized Polynomial Example Theorem 17.28; see Remark 9.5.

(14) For each pair of positive integers r, n, a Noetherian local domain A with $\dim A = r$ and a principal ideal-adic completion A^* of A such that A^* has nilradical with nilpotency index n; see Example 17.30.

(15) A non-universally catenary two-dimensional Noetherian local domain B that birationally dominates a three-dimensional regular local domain. The completion of B has two minimal primes, one of dimension one and one of dimension two. The ring B is not a homomorphic image of a regular local ring; see Example 18.13.

(16) An example of Insider Construction 10.1 where the approximation domain B is equal to the intersection domain A. The domain B is a non-catenary non-Noetherian four-dimensional local UFD that is very close to being Noetherian. The ring B has exactly one prime ideal Q of height three; the ideal Q is not finitely generated; see Examples 6.18 and 16.1.

(17) For every $m, n \in \mathbb{N}$ with $n \geq 4$, an example of Insider Construction 10.1 where the approximation domain B is equal to the intersection domain A, B has dimension n, and B has exactly m prime ideals of height $n-1$. The domain B is a non-catenary non-Noetherian UFD, and every prime ideal of B of height $n-1$ is not finitely generated; see Examples 6.18 and 10.9.

(18) An example of Insider Construction 10.1, where the approximation domain B is properly contained in the intersection domain A, and neither A nor B is Noetherian. The local domain B is a UFD that fails to have Cohen-Macaulay formal fibers; see Example 6.20 and Section 23.4.

From Chapter 27

(19) A general example of a nonfinite TGF-complete embedding of a power series ring $R = k[[x_1, \ldots, x_n]]$ in n variables over a field k into a power series ring in two variables over k; see Example 27.11 and Section 23.4. A particular case is given in Example 27.15¿

(20) An example where $\sigma : R \hookrightarrow S$ is an inclusion map, $\tau : S \hookrightarrow T$ is a TGF-embedding, and $\tau \cdot \sigma = \varphi$ is TGF, but $\sigma : R \hookrightarrow S$ is not TGF.

(21) An example where A is a 3-dimensional normal local domain, B is a 2-dimensional regular local domain, the residue field of B is transcendental over that of A and $(A, \mathbf{m}) \hookrightarrow (B, \mathbf{n})$ is a TGF extension, but $\widehat{A} \hookrightarrow \widehat{B}$ is not TGF-complete.

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