W. Heinzer Math 553 Practice for the test Oct 3 12 Sept. 07

1. (i) List representatives for each conjugacy class in the symmetric group S_4 and state the number of elements in each conjugacy class.
(ii) List representatives for each conjugacy class in the alternating group A_4 and state the number of elements in each conjugacy class.
(iii) Determine the number of elements of order 2 in the symmetric group S_5.
(iv) Determine the number of elements of order 2 in the alternating group A_5.

2. Let G be a finite group.
(i) What is meant by a composition series for G?
(ii) State the Jordan-Hölder Theorem.

3. Suppose σ is an element of order 2 in the alternating group A_n. Prove or disprove that there exists $\tau \in S_n$ such that $\tau^2 = \sigma$.
4. Let G be a group acting on the nonempty set A. What does it mean for the action of G on A to be transitive?
5. List all positive integers that are the order of an element of the symmetric group S_7.
6. Find all finite groups that have exactly three conjugacy classes.
7. (i) Diagram the lattice of subgroups of the dihedral group D_8.
(ii) How many different composition series exist for the dihedral group D_8?

8. Let G be a group having order $2k$, where k is an odd integer. Prove that G has a subgroup of order k.
9. Let G be a finite group and let $P \in \mathrm{Syl}_p(G)$ be a Sylow p-subgroup of G. If Q is a p-subgroup of G, prove that $Q \cap N_G(P) = Q \cap P$.
10. For n a positive integer, let Z_n denote a cyclic group of order n.
(i) What is the order of the group $\mathrm{Aut}(Z_n)$?
(ii) Are the groups $\mathrm{Aut}(Z_7)$ and $\mathrm{Aut}(Z_9)$ isomorphic? Justify your answer.
(iii) Are the groups $\mathrm{Aut}(Z_8)$ and $\mathrm{Aut}(Z_{10})$ isomorphic? Justify your answer.

11. (i) If H is a subgroup of a group G, what is meant by the normalizer $N_G(H)$ of H in G?
(ii) If H is the cyclic subgroup of S_5 generated by the 5-cycle $(1\ 2\ 3\ 4\ 5)$, what is the order of the normalizer N of H in S_5?

12. Let $\mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$ denote the finite field with 3 elements.

(i) What is the order of the group $GL_3(\mathbb{F}_3)$ of 3×3 invertible matrices with entries in $\mathbb{F}_3$?

(ii) What is the order of the subgroup of $GL_3(\mathbb{F}_3)$ of invertible upper triangular 3×3 matrices?

(iii) Does there exist a nonabelian group of order 27 ? Justify your answer.

13. Let G be a group of order 81 and suppose H is a subgroup of G with $|H| = 9$. Prove or disprove that there must exist a subgroup K of G such that $H \leq K$ and $|K| = 27$.

14. Recall that a subgroup H of a group G is called a *characteristic subgroup* if $\phi(H) = H$ for every automorphism ϕ of G. Give an example of a group G and a normal subgroup N of G such that N is not a characteristic subgroup of G. Explain why in your example N is normal but not characteristic.

15. Suppose G_1 and G_2 are groups and N_i is a normal subgroup of G_i, $i = 1, 2$. If $G_1 \cong G_2$ and $N_1 \cong N_2$, prove or disprove that G_1/N_1 is isomorphic to G_2/N_2.

16. Let a and b be nonzero elements of an integral domain R. If the principal ideals (a) and (b) are equal, prove that $a = ub$ for some unit $u \in R$.

17. Let R be a commutative ring with $1 \neq 0$ and let P be an ideal of R.

(i) Define "P is a prime ideal".

(ii) If P is a prime ideal of R and I and J are ideals of R such that $I \cap J \subseteq P$, prove that either $I \subseteq P$ or $J \subseteq P$.

18. If R is an integral domain, prove that the polynomial ring $R[x]$ has no zero divisors.

19. Prove that a finite integral domain is a field.

20. Let R be a commutative ring with 1.

(i) Define the *characteristic of* R.

(ii) Does there exist a ring having characteristic 4? Justify your answer.

21. Let R be a Boolean ring and let $I = (a, b)$ be an ideal of R generated by elements $a, b \in R$. Prove or disprove that I is a principal ideal.