1. (i) List representatives for each conjugacy class in the symmetric group $S_4$ and state the number of elements in each conjugacy class.

(ii) List representatives for each conjugacy class in the alternating group $A_4$ and state the number of elements in each conjugacy class.

(iii) Determine the number of elements of order 2 in the symmetric group $S_5$.

(iv) Determine the number of elements of order 2 in the alternating group $A_5$.

2. Let $G$ be a finite group.

(i) What is meant by a composition series for $G$?

(ii) State the Jordan-Hölder Theorem.

3. Suppose $\sigma$ is an element of order 2 in the alternating group $A_n$. Prove or disprove that there exists $\tau \in S_n$ such that $\tau^2 = \sigma$.

4. Let $G$ be a group acting on the nonempty set $A$. What does it mean for the action of $G$ on $A$ to be transitive?

5. List all positive integers that are the order of an element of the symmetric group $S_7$.

6. Find all finite groups that have exactly three conjugacy classes.

7. (i) Diagram the lattice of subgroups of the dihedral group $D_8$.

(ii) How many different composition series exist for the dihedral group $D_8$?

8. Let $G$ be a group having order $2k$, where $k$ is an odd integer. Prove that $G$ has a subgroup of order $k$.

9. Let $G$ be a finite group and let $P \in \text{Syl}_p(G)$ be a Sylow $p$-subgroup of $G$.
   If $Q$ is a $p$-subgroup of $G$, prove that $Q \cap N_G(P) = Q \cap P$.

10. For $n$ a positive integer, let $Z_n$ denote a cyclic group of order $n$.

   (i) What is the order of the group $\text{Aut}(Z_n)$?

   (ii) Are the groups $\text{Aut}(Z_7)$ and $\text{Aut}(Z_9)$ isomorphic? Justify your answer.

   (iii) Are the groups $\text{Aut}(Z_8)$ and $\text{Aut}(Z_{10})$ isomorphic? Justify your answer.

11. (i) If $H$ is a subgroup of a group $G$, what is meant by the normalizer $N_G(H)$ of $H$ in $G$?

   (ii) If $H$ is the cyclic subgroup of $S_5$ generated by the 5-cycle $(1 \ 2 \ 3 \ 4 \ 5)$, what is the order of the normalizer $N$ of $H$ in $S_5$?
12. Let $\mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$ denote the finite field with 3 elements.
   (i) What is the order of the group $GL_3(\mathbb{F}_3)$ of $3 \times 3$ invertible matrices with entries in $\mathbb{F}_3$?
   (ii) What is the order of the subgroup of $GL_3(\mathbb{F}_3)$ of invertible upper triangular $3 \times 3$ matrices?
   (iii) Does there exist a nonabelian group of order 27? Justify your answer.

13. Let $G$ be a group of order 81 and suppose $H$ is a subgroup of $G$ with $|H| = 9$. Prove or disprove that there must exist a subgroup $K$ of $G$ such that $H \leq K$ and $|K| = 27$.

14. Recall that a subgroup $H$ of a group $G$ is called a characteristic subgroup if $\phi(H) = H$ for every automorphism $\phi$ of $G$. Give an example of a group $G$ and a normal subgroup $N$ of $G$ such that $N$ is not a characteristic subgroup of $G$. Explain why in your example $N$ is normal but not characteristic.

15. Suppose $G_1$ and $G_2$ are groups and $N_i$ is a normal subgroup of $G_i$, $i = 1, 2$.
   If $G_1 \cong G_2$ and $N_1 \cong N_2$, prove or disprove that $G_1/N_1$ is isomorphic to $G_2/N_2$.

16. Let $a$ and $b$ be nonzero elements of an integral domain $R$. If the principal ideals $(a)$ and $(b)$ are equal, prove that $a = ub$ for some unit $u \in R$.

17. Let $R$ be a commutative ring with $1 \neq 0$ and let $P$ be an ideal of $R$.
   (i) Define “$P$ is a prime ideal”.
   (ii) If $P$ is a prime ideal of $R$ and $I$ and $J$ are ideals of $R$ such that $I \cap J \subseteq P$, prove that either $I \subseteq P$ or $J \subseteq P$.

18. If $R$ is an integral domain, prove that the polynomial ring $R[x]$ has no zero divisors.

19. Prove that a finite integral domain is a field.

20. Let $R$ be a commutative ring with 1.
   (i) Define the characteristic of $R$.
   (ii) Does there exist a ring having characteristic 4? Justify your answer.

21. Let $R$ be a Boolean ring and let $I = (a, b)$ be an ideal of $R$ generated by elements $a, b \in R$. Prove or disprove that $I$ is a principal ideal.