W. Heinzer Math 553 Test 1 3 October 07

(8) 1. Let G be a finite group.

(i) What is meant by a composition series for G?

(ii) State the Jordan-Hölder Theorem.

(9) 2. (i) Diagram the lattice of subgroups of the dihedral group D_8.

(ii) How many different composition series exist for the dihedral group D_8?

(8) 3. Let G be a finite group and let $P \in \mathrm{Syl}_p(G)$ be a Sylow p-subgroup of G. If Q is a p-subgroup of G, prove that $Q \cap N_G(P) = Q \cap P$.

(5) 4. List all positive integers that are the order of an element of the symmetric group S_8.

(8) 5. For n a positive integer, let Z_n denote a cyclic group of order n.

(i) Are the groups $\mathrm{Aut}(Z_7)$ and $\mathrm{Aut}(Z_9)$ isomorphic? Justify your answer.

(ii) Are the groups $\mathrm{Aut}(Z_8)$ and $\mathrm{Aut}(Z_{10})$ isomorphic? Justify your answer.

(8) 6. Let G be a group of order 18. Prove that G has a subgroup of order 9.

(10 7. Let H be the cyclic subgroup of S_4 generated by the 4-cycle $(1\ 2\ 3\ 4)$.

(i) What is the order of the normalizer N of H in S_4?

(ii) Give generators for the group N.

(7) 8. Assume that G_1 and G_2 are groups and that N_i is a normal subgroup of G_i, $i = 1, 2$. If $G_1 \cong G_2$ and $G_1/N_1 \cong G_2/N_2$, prove or disprove that N_1 is isomorphic to N_2.

(7) 9. Recall that a subgroup H of a group G is called a *characteristic subgroup* if $\phi(H) = H$ for every automorphism ϕ of G. Give an example of a group G and a normal subgroup N of G such that N is not a characteristic subgroup of G. Explain why in your example N is normal but not characteristic.

(6) 10. Give an example of a commutative ring R with identity $1 \neq 0$ that has ideals I and J such that $\{ab \mid a \in I, b \in J\}$ is not an ideal of R.

(6) 11. Diagram the lattice of ideals of the ring $\mathbb{Z}/(30)$.

(6) 12. Diagram the lattice of ideals of the ring $\mathbb{Z}/(36)$.

(6) 13. Let R be a commutative ring with 1.

(i) Define the *characteristic of* R.

(ii) Does there exist a ring having characteristic 4? Justify your answer.

(6) 14. If R is an integral domain, prove that the polynomial ring $R[x]$ has no zero divisors.