Math 553 **Test 2** **November 28, 2007** **W. Heinzer**

1. (8 pts) Let p be a prime number and let $\mathbb{F}_p$ denote the field with p elements. For a nonzero element $a \in \mathbb{F}_p$, prove that the polynomial $x^p - x + a$ is irreducible and separable over $\mathbb{F}_p$.

2. (12 pts) Let f(x) be an irreducible polynomial of degree $n > 1$ over a field F and let $g(x)$ be a polynomial in $F[x]$ of positive degree.

(i) Prove or disprove that every irreducible factor of the composite polynomial $g(f(x))$ has degree divisible by n.

(ii) Prove or disprove that every irreducible factor of the composite polynomial $f(g(x))$ has degree divisible by n.

3. (18 pts) Let $\mathbb{Z}$ denote the ring of integers and let x be an indeterminate over $\mathbb{Z}$.

(i) Is every ideal of $\mathbb{Z}[x]/(x^2 - 1)$ principal? Justify your answer.

(ii) Is every ideal of $\mathbb{Z}[x]/(x^2 - x)$ principal? Justify your answer.

(iii) Is every ideal of $\mathbb{Z}[x]/(x^2)$ principal? Justify your answer.

4. (6 pts) Give an example of a field F having characteristic $p > 0$ and an irreducible polynomial $g(x) \in F[x]$ that has a multiple root.

5. (14 pts) Let R be an integral domain.

(i) Define "$N : R \to \mathbb{Z}^+ \cup \{0\}$ is a Dedekind-Hasse norm."

(ii) Prove that if R has a Dedekind-Hasse norm, then R is a principal ideal domain

(iii) Prove that if R is a principal ideal domain, then R has a Dedekind-Hasse norm.

6. (15 pts) Let L be the splitting field of the polynomial $x^4 - 2 \in \mathbb{Q}[x]$. Diagram the lattice of subfields of L giving generators for each subfield. Indicate which subfields of L are Galois over $\mathbb{Q}$.

7. (5 pts) Let $\mathbb{Z}$ denote the ring of integers. Diagram the lattice of ideals of the polynomial ring $\mathbb{Z}[x]$ that contain the ideal $(3, x^3 - 1)$. Give generators for each such ideal.

8. (6 pts) Prove that an irreducible monic polynomial $f(x) \in \mathbb{Q}[x]$ does not have a multiple root.

9. (16 pts) Let $\zeta \in \mathbb{C}$ be a primitive 9-th root of unity.

(i) What is $[\mathbb{Q}(\zeta) : \mathbb{Q}]$?

(ii) List the distinct conjugates of $\zeta + \zeta^{-1}$ over $\mathbb{Q}$.

(iii) What is the group $\mathrm{Aut}(\mathbb{Q}(\zeta + \zeta^{-1})/\mathbb{Q})$? Is $\mathbb{Q}(\zeta + \zeta^{-1})$ Galois over $\mathbb{Q}$?

(iv) Diagram the lattice of subfields of $\mathbb{Q}(\zeta)$ giving generators for each.