1. (8 pts) Let $p$ be a prime number and let $\mathbb{F}_p$ denote the field with $p$ elements. For a nonzero element $a \in \mathbb{F}_p$, prove that the polynomial $x^p - x + a$ is irreducible and separable over $\mathbb{F}_p$.

2. (12 pts) Let $f(x)$ be an irreducible polynomial of degree $n > 1$ over a field $F$ and let $g(x)$ be a polynomial in $F[x]$ of positive degree.

(i) Prove or disprove that every irreducible factor of the composite polynomial $g(f(x))$ has degree divisible by $n$.

(ii) Prove or disprove that every irreducible factor of the composite polynomial $f(g(x))$ has degree divisible by $n$. 
3. (18 pts) Let $\mathbb{Z}$ denote the ring of integers and let $x$ be an indeterminate over $\mathbb{Z}$.

(i) Is every ideal of $\mathbb{Z}[x]/(x^2 - 1)$ principal? Justify your answer.

(ii) Is every ideal of $\mathbb{Z}[x]/(x^2 - x)$ principal? Justify your answer.

(iii) Is every ideal of $\mathbb{Z}[x]/(x^2)$ principal? Justify your answer.
4. (6 pts) Give an example of a field $F$ having characteristic $p > 0$ and an irreducible polynomial $g(x) \in F[x]$ that has a multiple root.

5. (14 pts) Let $R$ be an integral domain.

   (i) Define “$N : R \rightarrow \mathbb{Z}^+ \cup \{0\}$ is a Dedekind-Hasse norm.”

   (ii) Prove that if $R$ has a Dedekind-Hasse norm, then $R$ is a principal ideal domain.

   (iii) Prove that if $R$ is a principal ideal domain, then $R$ has a Dedekind-Hasse norm.
6. (15 pts) Let $L$ be the splitting field of the polynomial $x^4 - 2 \in \mathbb{Q}[x]$. Diagram the lattice of subfields of $L$ giving generators for each subfield. Indicate which subfields of $L$ are Galois over $\mathbb{Q}$.

7. (5 pts) Let $\mathbb{Z}$ denote the ring of integers. Diagram the lattice of ideals of the polynomial ring $\mathbb{Z}[x]$ that contain the ideal $(3, x^3 - 1)$. Give generators for each such ideal.
8. (6 pts) Prove that an irreducible monic polynomial \( f(x) \in \mathbb{Q}[x] \) does not have a multiple root.

9. (16 pts) Let \( \zeta \in \mathbb{C} \) be a primitive 9-th root of unity.

   (i) What is \([\mathbb{Q}(\zeta) : \mathbb{Q}]\)?

   (ii) List the distinct conjugates of \( \zeta + \zeta^{-1} \) over \( \mathbb{Q} \).

   (iii) What is the group \( \text{Aut}(\mathbb{Q}(\zeta + \zeta^{-1})/\mathbb{Q}) \)? Is \( \mathbb{Q}(\zeta + \zeta^{-1}) \) Galois over \( \mathbb{Q} \)?

   (iv) Diagram the lattice of subfields of \( \mathbb{Q}(\zeta) \) giving generators for each.