1. (16 pts)

(a) List representatives for each conjugacy class in the symmetric group S_4 and state the number of elements in each conjugacy class.

(b) List representatives for each conjugacy class in the alternating group A_4 and state the number of elements in each conjugacy class.

(c) Determine the number of elements of order 2 in the symmetric group S_5.

(d) Determine the number of elements of order 2 in the alternating group A_5.

2. (8 pts) Let G be a group having order $2k$, where k is an odd integer. Prove that G has a subgroup of order k.

3. (8 pts) Let H be the cyclic subgroup of the symmetric group S_5 generated by the 5-cycle $(1\,2\,3\,4\,5)$. What is the order of the normalizer N of H in S_5? Justify your answer.

4. (15 pts) Let $\mathbb{F}_3 = \mathbb{Z}/(3)$ denote the finite field with 3 elements.

(a) What is the order of the group $GL_3(\mathbb{F}_3)$ of 3×3 invertible matrices with entries in $\mathbb{F}_3$?

(b) What is the order of the subgroup of $GL_3(\mathbb{F}_3)$ of invertible upper triangular 3×3 matrices?

(c) Does there exist a nonabelian group of order 27 ? Justify your answer.

5. (6 pts) Give an example of a commutative ring R with identity $1 \neq 0$ that has ideals I and J such that $\{ab \mid a \in I, b \in J\}$ is not an ideal of R.

6. (8 pts) Let G be a group of order p^4, where p is prime, and suppose H is a subgroup of G with $|H| = p^2$. Prove or disprove that there must exist a subgroup K of G such that $H \leq K$ and $|K| = p^3$.

7. (10 pts) Let R be a commutative ring with 1. Recall that $a \in R$ is called **nilpotent** if $a^n = 0$ for some $n \in \mathbb{Z}^+$, and a nonzero element $a \in R$ is said to be a **zero divisor** if there exists a nonzero $b \in R$ such that $ab = 0$.

(a) Give an example where R has a zero divisor that is not nilpotent.

(b) Give an example where every zero divisor of R is nilpotent and where R has a nonzero nilpotent element.

8. (15 pts) Let L be the splitting field of the polynomial $x^4 - 2 \in \mathbb{Q}[x]$. Diagram the lattice of subfields of L giving generators for each subfield. Indicate which subfields of L are Galois over $\mathbb{Q}$.

9. (6 pts) Let $\mathbb{Z}$ denote the ring of integers. Diagram the lattice of ideals of the polynomial ring $\mathbb{Z}[x]$ that contain the ideal $(2, x^3 - 1)$. Give generators for each such ideal.

10. (14 pts) Let a be a nonzero nonunit of an integral domain R.

(a) Define "a is irreducible".

(b) Define "a is a prime element".

(c) Prove that if a is a prime element, then a is irreducible.

(d) Give an example of an integral domain R and a nonzero nonunit a of R that is irreducible but not a prime element. Explain why in your example a is irreducible but not a prime element.

11. (6 pts) Let $p > 3$ be a prime integer. Prove that the image of at least one of $2, 3$ or 6 in the field $\mathbb{F}_p$ is a square.

12. (8 pts) Let K/F be a finite algebraic field extension. If $K = F(\alpha)$ for some $\alpha \in K$, prove that there are only finitely many subfields of K that contain F.

13. (8 pts) Let F be an infinite field and let K/F be a finite algebraic field extension. If there are only finitely many subfields of K that contain F, prove that $K = F(\alpha)$ for some $\alpha \in K$

14. (10 pts) Let F be a subfield of the field $\mathbb{C}$ of complex numbers and let $K \subseteq \mathbb{C}$ be an algebraic field extension of F having the property that each nonconstant polynomial in $F[x]$ has at least one root in K. Prove that K is algebraically closed.

15. (10 pts) Let x and y be indeterminates over the field $\mathbb{F}_2$. Explicitly exhibit infinitely many intermediate fields between $K = \mathbb{F}_2(x^2, y^2)$ and $L = \mathbb{F}_2(x, y)$.

16. (9 pts) Let n be a positive integer and d a positive integer that divides n. Suppose $\alpha \in \mathbb{R}$ is a root of the polynomial $x^n - 2 \in \mathbb{Q}[x]$. Prove that there is precisely one subfield F of $\mathbb{Q}(\alpha)$ with $[F : \mathbb{Q}] = d$.

17. (7 pts) Prove that $\mathbb{Q}(\sqrt[3]{2})$ is not a subfield of any cyclotomic field over $\mathbb{Q}$.

18. (10 pts) Let p be a prime number and let F be a field of characteristic p. For a nonzero element $a \in F$, prove that the polynomial $x^p - x + a \in F[x]$ is either irreducible or else factors as a product of distinct linear polynomials in $F[x]$.

19. (12 pts) Let f(x) be an irreducible polynomial of degree $n > 1$ over a field F and let $g(x)$ be a polynomial in $F[x]$ of positive degree.

(i) Prove or disprove that every irreducible factor of the composite polynomial $f(g(x))$ has degree divisible by n.

(ii) Prove or disprove that every irreducible factor of the composite polynomial $g(f(x))$ has degree divisible by n.

20. (14 pts) Let $L/\mathbb{Q}$ be the splitting field of the polynomial $x^5 - 2 \in \mathbb{Q}[x]$. Diagram the lattice of subfields of $L/\mathbb{Q}$. For each subfield, give generators and list its degree over $\mathbb{Q}$. Indicate which of these subfields are Galois over $\mathbb{Q}$.