1. (16 pts)

(a) List representatives for each conjugacy class in the symmetric group $S_4$ and state the number of elements in each conjugacy class.

(b) List representatives for each conjugacy class in the alternating group $A_4$ and state the number of elements in each conjugacy class.

(c) Determine the number of elements of order 2 in the symmetric group $S_5$.

(d) Determine the number of elements of order 2 in the alternating group $A_5$.
2. (8 pts) Let $G$ be a group having order $2k$, where $k$ is an odd integer. Prove that $G$ has a subgroup of order $k$.

3. (8 pts) Let $H$ be the cyclic subgroup of the symmetric group $S_5$ generated by the 5-cycle $(12345)$. What is the order of the normalizer $N$ of $H$ in $S_5$? Justify your answer.
4. (15 pts) Let $\mathbb{F}_3 = \mathbb{Z}/(3)$ denote the finite field with 3 elements.

(a) What is the order of the group $GL_3(\mathbb{F}_3)$ of $3 \times 3$ invertible matrices with entries in $\mathbb{F}_3$?

(b) What is the order of the subgroup of $GL_3(\mathbb{F}_3)$ of invertible upper triangular $3 \times 3$ matrices?

(c) Does there exist a nonabelian group of order 27? Justify your answer.

5. (6 pts) Give an example of a commutative ring $R$ with identity $1 \neq 0$ that has ideals $I$ and $J$ such that

$$\{ab \mid a \in I, b \in J\}$$

is not an ideal of $R$. 
6. (8 pts) Let $G$ be a group of order $p^4$, where $p$ is prime, and suppose $H$ is a subgroup of $G$ with $|H| = p^2$. Prove or disprove that there must exist a subgroup $K$ of $G$ such that $H \leq K$ and $|K| = p^3$.

7. (10 pts) Let $R$ be a commutative ring with 1. Recall that $a \in R$ is called \textbf{nilpotent} if $a^n = 0$ for some $n \in \mathbb{Z}^+$, and a nonzero element $a \in R$ is said to be a \textbf{zero divisor} if there exists a nonzero $b \in R$ such that $ab = 0$.

(a) Give an example where $R$ has a zero divisor that is not nilpotent.

(b) Give an example where every zero divisor of $R$ is nilpotent and where $R$ has a nonzero nilpotent element.
8. (15 pts) Let $L$ be the splitting field of the polynomial $x^4 - 2 \in \mathbb{Q}[x]$. Diagram the lattice of subfields of $L$ giving generators for each subfield. Indicate which subfields of $L$ are Galois over $\mathbb{Q}$.

9. (6 pts) Let $\mathbb{Z}$ denote the ring of integers. Diagram the lattice of ideals of the polynomial ring $\mathbb{Z}[x]$ that contain the ideal $(2, x^3 - 1)$. Give generators for each such ideal.
10. (14 pts) Let $a$ be a nonzero nonunit of an integral domain $R$.

(a) Define “$a$ is irreducible”.

(b) Define “$a$ is a prime element”.

(c) Prove that if $a$ is a prime element, then $a$ is irreducible.

(d) Give an example of an integral domain $R$ and a nonzero nonunit $a$ of $R$ that is irreducible but not a prime element. Explain why in your example $a$ is irreducible but not a prime element.

11. (6 pts) Let $p > 3$ be a prime integer. Prove that the image of at least one of $2, 3$ or $6$ in the field $\mathbb{F}_p$ is a square.
12. (8 pts) Let $K/F$ be a finite algebraic field extension. If $K = F(\alpha)$ for some $\alpha \in K$, prove that there are only finitely many subfields of $K$ that contain $F$.

13. (8 pts) Let $F$ be an infinite field and let $K/F$ be a finite algebraic field extension. If there are only finitely many subfields of $K$ that contain $F$, prove that $K = F(\alpha)$ for some $\alpha \in K$. 

14. (10 pts) Let $F$ be a subfield of the field $\mathbb{C}$ of complex numbers and let $K \subseteq \mathbb{C}$ be an algebraic field extension of $F$ having the property that each nonconstant polynomial in $F[x]$ has at least one root in $K$. Prove that $K$ is algebraically closed.

15. (10 pts) Let $x$ and $y$ be indeterminates over the field $\mathbb{F}_2$. Explicitly exhibit infinitely many intermediate fields between $K = \mathbb{F}_2(x^2, y^2)$ and $L = \mathbb{F}_2(x, y)$. 
16. (9 pts) Let \( n \) be a positive integer and \( d \) a positive integer that divides \( n \). Suppose \( \alpha \in \mathbb{R} \) is a root of the polynomial \( x^n - 2 \in \mathbb{Q}[x] \). Prove that there is precisely one subfield \( F \) of \( \mathbb{Q}(\alpha) \) with \( [F : \mathbb{Q}] = d \).

17. (7 pts) Prove that \( \mathbb{Q}(\sqrt[3]{2}) \) is not a subfield of any cyclotomic field over \( \mathbb{Q} \).
18. (10 pts) Let $p$ be a prime number and let $F$ be a field of characteristic $p$. For a nonzero element $a \in F$, prove that the polynomial $x^p - x + a \in F[x]$ is either irreducible or else factors as a product of distinct linear polynomials in $F[x]$.

19. (12 pts) Let $f(x)$ be an irreducible polynomial of degree $n > 1$ over a field $F$ and let $g(x)$ be a polynomial in $F[x]$ of positive degree.

(i) Prove or disprove that every irreducible factor of the composite polynomial $f(g(x))$ has degree divisible by $n$.

(ii) Prove or disprove that every irreducible factor of the composite polynomial $g(f(x))$ has degree divisible by $n$. 


20. (14 pts) Let $L/Q$ be the splitting field of the polynomial $x^5 - 2 \in \mathbb{Q}[x]$. Diagram the lattice of subfields of $L/Q$. For each subfield, give generators and list its degree over $\mathbb{Q}$. Indicate which of these subfields are Galois over $\mathbb{Q}$. 