1. Let G be a finite group with $G \neq 1$.

(a) What is meant by a composition series for G?

(b) State the Jordan-Hölder theorem.

(c) What does it mean for G to be simple?

(d) What does it mean for G to be solvable?

(e) Give an example of a simple group that is not solvable.

2. Let P be a Sylow 5 subgroup of the alternating group A_5.
(i) What is the order of the normalizer $N_{A_5}(P)$?

(ii) How many Sylow 5 subgroups does A_5 have?

3. Describe all finite groups that have exactly three conjugacy classes.

4. If the finite group G has composition series $1 = N_0 \leq N_1 \leq N_2 = G$ and $1 = M_0 \leq M_1 \leq \cdots \leq M_r = G$, prove that $r = 2$.

5. List representatives for each conjugacy class in the alternating group A_5 and state the number of elements in each conjugacy class.

6. Let x and y be indeterminates over the field $\mathbb{F}_2$. Explicitly exhibit infinitely many intermediate fields between $K = \mathbb{F}_2(x^2, y^2)$ and $L = \mathbb{F}_2(x, y)$.

7. Let p be a prime integer and let G be a group of order p^4. Suppose H is a subgroup of G with $|H| = p^2$. Prove or disprove that there must exist a subgroup K of G such that $H \leq K$ and $|K| = p^3$.

8. Let G be a group with $|G| = 2k$, where k is odd. Prove that G has a subgroup of order k.

9. Let G be a finite group and let $P \in \mathrm{Syl}_p(G)$ be a Sylow p-subgroup of G. If Q is a p-subgroup of G, prove that $Q \cap N_G(P) = Q \cap P$.

10. Assume that F is a field of characteristic zero and that K/F is an algebraic field extension. If each nonconstant polynomial in $F[x]$ has at least one root in K, prove that K is algebraically closed.

11. Consider the ring $R = \mathbb{Z}[x]/(15,\ x^2 + 1)$.

(a) How many maximal ideals does the ring R have?

(b) Give generators for each maximal ideal of the ring R.

12. Let $\mathbb{Z}$ denote the ring of integers. Diagram the lattice of ideals of the polynomial ring $\mathbb{Z}[x]$ that contain the ideal $(6,\ x^3 - 1)$. Give generators for each such ideal.

13. Show that the polynomial

$$f_n(x) = (x-1)(x-2)\cdots(x-n) - 1$$

is irreducible over $\mathbb{Z}$ for each integer $n \geq 1$.

14. Show that the polynomial

$$g_n(x) = (x-1)(x-2)\cdots(x-n) + 1$$

is irreducible over $\mathbb{Z}$ for each positive integer $n \neq 4$.

15. Prove that $\mathbb{Q}(\sqrt[3]{2})$ is not a subfield of any cyclotomic field over $\mathbb{Q}$.

16. Suppose $\alpha \in \mathbb{C}$ is algebraic over $\mathbb{Q}$.

(a) Define "α *can be solved for in terms of radicals.*"

(b) For a polynomial $f(x) \in \mathbb{Q}[x]$, define "$f(x)$ *can be solved by radicals.*"

17. Let n be a positive integer and d a positive integer that divides n. Suppose $\alpha \in \mathbb{R}$ is a root of the polynomial $x^n - 2 \in \mathbb{Q}[x]$. Prove that there is precisely one subfield F of $\mathbb{Q}(\alpha)$ with $[F : \mathbb{Q}] = d$.

18. Let $K/\mathbb{Q}$ be the splitting field of the polynomial $x^5 - 1 \in \mathbb{Q}[x]$. Diagram the lattice of subfields of $K/\mathbb{Q}$. For each subfield, give generators and list its degree over $\mathbb{Q}$.

19. Let $L/\mathbb{Q}$ be the splitting field of the polynomial $x^5 - 2 \in \mathbb{Q}[x]$. Diagram the lattice of subfields of $L/\mathbb{Q}$. For each subfield, give generators and list its degree over $\mathbb{Q}$.

20. Let G be the Galois group of an irreducible polynomial $f(x) \in \mathbb{Q}[x]$, where $\deg f = 5$.

(a) What integers are possible for the order of G? Explain your answer.

(b) If G contains an element of order 3, what integers are possible for the order of G? Explain your answer.

21. Suppose $f(x) \in \mathbb{Q}[x]$ is a monic polynomial of degree n and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ are the roots of $f(x)$. Let G be the Galois group of $f(x)$ over $\mathbb{Q}$.

(a) Prove that $f(x)$ is irreducible in $\mathbb{Q}[x]$ if and only if the action of G on $\{\alpha_1, \ldots, \alpha_n\}$ is transitive.

(b) If the action of G on $\{\alpha_1, \ldots, \alpha_n\}$ is doubly transitive, prove that $\mathbb{Q}$ is the only proper subfield of $\mathbb{Q}(\alpha_1)$.

22. Let p be a prime integer and let $\mathbb{F}_p$ denote the field with p elements.

(a) Prove or disprove that every finite algebraic extension field of $\mathbb{F}_p$ is Galois.

(b) Let K and L be finite algebraic field extensions of $\mathbb{F}_p$. If $[K : \mathbb{F}_p] \leq [L : \mathbb{F}_p]$, does it follow that K is isomorphic to a subfield of L? Justify your answer.

(c) Let $\overline{\mathbb{F}_p}$ denote the algebraic closure of $\mathbb{F}_p$. If E is a subfield of $\overline{\mathbb{F}_p}$ and $[E : \mathbb{F}_p] = \infty$, prove or disprove that $E = \overline{\mathbb{F}_p}$.

23. Let G be a finite group of order pqr, where $p > q > r$ are prime.

(a) If G fails to have a normal subgroup of order p, determine the number of elements in G of order p.

(b) If G fails to have a normal subgroup of order q, prove that G has at least q^2 element of order q.

(c) Prove that G has a nontrivial normal subgroup.

24. Let K/F be a finite algebraic field extension. If $K = F(\alpha)$ for some $\alpha \in K$, prove that there are only finitely many subfields of K that contain F.

25. Let F be an infinite field and let K/F be a finite algebraic field extension. If there are only finitely many subfields of K that contain F, prove that $K = F(\alpha)$ for some $\alpha \in K$

26. Let $L/\mathbb{Q}$ be the Galois closure of the finite algebraic field extension $\mathbb{Q}(\alpha)$ of $\mathbb{Q}$. Let p be a prime that divides the order of $\mathrm{Gal}(L/\mathbb{Q})$. Prove that there exists a subfield F of L such that $[L : F] = p$ and $L = F(\alpha)$.