1. Let $G$ be a finite group with $G \neq 1$.
   (a) What is meant by a composition series for $G$?
   (b) State the Jordan-Hölder theorem.
   (c) What does it mean for $G$ to be simple?
   (d) What does it mean for $G$ to be solvable?
   (e) Give an example of a simple group that is not solvable.

2. Let $P$ be a Sylow 5 subgroup of the alternating group $A_5$.
   (i) What is the order of the normalizer $N_{A_5}(P)$?
   (ii) How many Sylow 5 subgroups does $A_5$ have?

3. Describe all finite groups that have exactly three conjugacy classes.

4. If the finite group $G$ has composition series $1 = N_0 \leq N_1 \leq \cdots \leq N_r = G$ and $1 = M_0 \leq M_1 \leq \cdots \leq M_r = G$,
   prove that $r = 2$.

5. List representatives for each conjugacy class in the alternating group $A_5$ and state the number of elements
   in each conjugacy class.

6. Let $x$ and $y$ be indeterminates over the field $F_2$. Explicitly exhibit infinitely many intermediate fields
   between $K = F_2(x^2, y^2)$ and $L = F_2(x, y)$.

7. Let $p$ be a prime integer and let $G$ be a group of order $p^4$. Suppose $H$ is a subgroup of $G$ with $|H| = p^2$.
   Prove or disprove that there must exist a subgroup $K$ of $G$ such that $H \leq K$ and $|K| = p^3$.

8. Let $G$ be a group with $|G| = 2k$, where $k$ is odd. Prove that $G$ has a subgroup of order $k$.

9. Let $G$ be a finite group and let $P \in \text{Syl}_p(G)$ be a Sylow $p$-subgroup of $G$. If $Q$ is a $p$-subgroup of $G$,
   prove that $Q \cap N_G(P) = Q \cap P$.

10. Assume that $F$ is a field of characteristic zero and that $K/F$ is an algebraic field extension. If each
    nonconstant polynomial in $F[x]$ has at least one root in $K$, prove that $K$ is algebraically closed.

11. Consider the ring $R = Z[x]/(15, x^2 + 1)$.
    (a) How many maximal ideals does the ring $R$ have?
    (b) Give generators for each maximal ideal of the ring $R$.

12. Let $Z$ denote the ring of integers. Diagram the lattice of ideals of the polynomial ring $Z[x]$ that contain
    the ideal $(6, x^3 - 1)$. Give generators for each such ideal.

13. Show that the polynomial
    
    $f_n(x) = (x - 1)(x - 2) \cdots (x - n) - 1$
    
    is irreducible over $Z$ for each integer $n \geq 1$.

14. Show that the polynomial
    
    $g_n(x) = (x - 1)(x - 2) \cdots (x - n) + 1$
    
    is irreducible over $Z$ for each positive integer $n \neq 4$. 

15. Prove that \( \mathbb{Q}(\sqrt[3]{2}) \) is not a subfield of any cyclotomic field over \( \mathbb{Q} \).

16. Suppose \( \alpha \in \mathbb{C} \) is algebraic over \( \mathbb{Q} \).
   (a) Define “\( \alpha \) can be solved for in terms of radicals.”
   (b) For a polynomial \( f(x) \in \mathbb{Q}[x] \), define “\( f(x) \) can be solved by radicals.”

17. Let \( n \) be a positive integer and \( d \) a positive integer that divides \( n \). Suppose \( \alpha \in \mathbb{R} \) is a root of the polynomial \( x^n - 2 \in \mathbb{Q}[x] \). Prove that there is precisely one subfield \( F \) of \( \mathbb{Q}(\alpha) \) with \( [F : \mathbb{Q}] = d \).

18. Let \( K/\mathbb{Q} \) be the splitting field of the polynomial \( x^5 - 1 \in \mathbb{Q}[x] \). Diagram the lattice of subfields of \( K/\mathbb{Q} \). For each subfield, give generators and list its degree over \( \mathbb{Q} \).

19. Let \( L/\mathbb{Q} \) be the splitting field of the polynomial \( x^5 - 2 \in \mathbb{Q}[x] \). Diagram the lattice of subfields of \( L/\mathbb{Q} \). For each subfield, give generators and list its degree over \( \mathbb{Q} \).

20. Let \( G \) be the Galois group of an irreducible polynomial \( f(x) \in \mathbb{Q}[x] \), where \( \deg f = 5 \).
   (a) What integers are possible for the order of \( G \)? Explain your answer.
   (b) If \( G \) contains an element of order 3, what integers are possible for the order of \( G \)? Explain your answer.

21. Suppose \( f(x) \in \mathbb{Q}[x] \) is a monic polynomial of degree \( n \) and \( \alpha_1, \ldots, \alpha_n \in \mathbb{C} \) are the roots of \( f(x) \). Let \( G \) be the Galois group of \( f(x) \) over \( \mathbb{Q} \).
   (a) Prove that \( f(x) \) is irreducible in \( \mathbb{Q}[x] \) if and only if the action of \( G \) on \( \{ \alpha_1, \ldots, \alpha_n \} \) is transitive.
   (b) If the action of \( G \) on \( \{ \alpha_1, \ldots, \alpha_n \} \) is doubly transitive, prove that \( \mathbb{Q} \) is the only proper subfield of \( \mathbb{Q}(\alpha_1) \).

22. Let \( p \) be a prime integer and let \( \mathbb{F}_p \) denote the field with \( p \) elements.
   (a) Prove or disprove that every finite algebraic extension field of \( \mathbb{F}_p \) is Galois.
   (b) Let \( K \) and \( L \) be finite algebraic field extensions of \( \mathbb{F}_p \). If \( [K : \mathbb{F}_p] \leq [L : \mathbb{F}_p] \), does it follow that \( K \) is isomorphic to a subfield of \( L \)? Justify your answer.
   (c) Let \( \overline{\mathbb{F}_p} \) denote the algebraic closure of \( \mathbb{F}_p \). If \( E \) is a subfield of \( \overline{\mathbb{F}_p} \) and \( [E : \mathbb{F}_p] = \infty \), prove or disprove that \( E = \overline{\mathbb{F}_p} \).

23. Let \( G \) be a finite group of order \( pqr \), where \( p > q > r \) are prime.
   (a) If \( G \) fails to have a normal subgroup of order \( p \), determine the number of elements in \( G \) of order \( p \).
   (b) If \( G \) fails to have a normal subgroup of order \( q \), prove that \( G \) has at least \( q^2 \) element of order \( q \).
   (c) Prove that \( G \) has a nontrivial normal subgroup.

24. Let \( K/F \) be a finite algebraic field extension. If \( K = F(\alpha) \) for some \( \alpha \in K \), prove that there are only finitely many subfields of \( K \) that contain \( F \).

25. Let \( F \) be an infinite field and let \( K/F \) be a finite algebraic field extension. If there are only finitely many subfields of \( K \) that contain \( F \), prove that \( K = F(\alpha) \) for some \( \alpha \in K \).
26. Let $L/\mathbb{Q}$ be the Galois closure of the finite algebraic field extension $\mathbb{Q}(\alpha)$ of $\mathbb{Q}$. Let $p$ be a prime that divides the order of $\text{Gal}(L/\mathbb{Q})$. Prove that there exists a subfield $F$ of $L$ such that $[L : F] = p$ and $L = F(\alpha)$. 