**1.** Let $G$ be the group of rigid motions in $\mathbb{R}^3$ of a cube. What is the order of $G$? Justify your answer.

**2.** Prove or disprove that the dihedral group $D_{24}$ is isomorphic to the symmetric group $S_4$.

**3.** For the group $A = Z_2 \times Z_4 = \langle a, b \,|\, a^2 = b^4 = 1, ab = ba \rangle$, diagram the lattice of subgroups giving generators for each subgroup.

**4.** Let $\mathbb{F}_3$ denote a field with three elements.

(a) What is the order of the group $SL_2(\mathbb{F}_3)$?

(b) List all the integers that are the order of an element of the group $SL_2(\mathbb{F}_3)$.

**5.** Let $A$ be a nonempty set and let $k$ be a positive integer with $k \leq |A|$. The symmetric group $S_{|A|}$ acts on the set $B$ of all subsets of $A$ of cardinality $k$ by $\sigma\{a_1, \ldots, a_k\} = \{\sigma(a_1), \ldots, \sigma(a_k)\}$.

(a) Describe explicitly how the permutation $\sigma = (1\,2)(3\,4)$ acts on the six 2-element subsets of $\{1, 2, 3, 4\}$.

(b) Describe explicitly how the permutation $\tau = (1\,2\,3\,4)$ acts on the six 2-element subsets of $\{1, 2, 3, 4\}$.

**6.** Diagram the lattice of subgroups of the dihedral group $D_8$ and circle precisely those subgroups that are not normal in $D_8$.

**7.** For $n$ a positive integer, let $Z_n$ denote a cyclic group of order $n$.

(a) What is the order of the group $\mathrm{Aut}(Z_n)$?

(b) Among the groups $\mathrm{Aut}(Z_8)$, $\mathrm{Aut}(Z_{10})$ and $\mathrm{Aut}(Z_{12})$ which are isomorphic and which are not? Justify your answer.

**8.** A subgroup $H$ of a group $G$ is called a *characteristic subgroup* if $\phi(H) = H$ for every automorphism $\phi$ of $G$. Give an example of a group $G$ and a normal subgroup $N$ of $G$ such that $N$ is not a characteristic subgroup of $G$. Explain why in your example $N$ is normal but not characteristic.

**9.** Assume that $G_1$ and $G_2$ are groups and $N_i$ is a normal subgroup of $G_i$, $i = 1, 2$. If $G_1 \cong G_2$ and $N_1 \cong N_2$, prove or disprove that $G_1/N_1$ is isomorphic to $G_2/N_2$.

**10.** Define what is meant by a *group action* of a group $G$ on a set $A$.

**11.** Let $H$ and $K$ be subgroups of a group $G$. If $H \cup K$ is a subgroup of $G$, prove that either $H \subseteq K$ or $K \subseteq H$.

**12.** Let $G$ be the group of rigid motions in $\mathbb{R}^3$ of a regular tetrahedron. What is the order of $G$? Justify your answer.

**13.** Let $\mathbb{F}_2$ denote a field with two elements. Write out the elements of $GL_2(\mathbb{F}_2)$ and indicate the order of each element.

**14.** Diagram the lattice of subgroups of the quaternion group $Q_8$.

**15.** Let $G = \{z \in \mathbb{C} \mid z^n = 1 \text{ for some } n \in \mathbb{Z}^+\}$. Define $\phi : G \to G$ by $\phi(z) = z^5$.

(a) What is the order of $\ker(\phi)$?

(b) Prove or disprove that $\phi$ is surjective.

**16.** Find all subgroups of the cyclic group $Z_{45} = \langle x \rangle$, giving a generator for each. Diagram the lattice of subgroups.

**17.** State the Fundamental Theorem of Arithmetic.

**18.** Find all integers $n$ such that the symmetric group $S_8$ contains an element of order $n$.

**19.** Assume that $G_1$ and $G_2$ are groups and $N_i$ is a normal subgroup of $G_i$, $i = 1, 2$. If $N_1 \cong N_2$ and $G_1/N_1 \cong G_2/N_2$, prove or disprove that $G_1$ is isomorphic to $G_2$.