1. Let R be a commutative ring with $1 \neq 0$ and let P be an ideal of R.

(a) Define "P is a prime ideal".

(b) If P is a prime ideal of R and I and J are ideals of R such that $I \cap J \subseteq P$, prove that $I \subseteq P$ or $J \subseteq P$.

(c) Define "P is a maximal ideal".

(d) If R is a unique factorization domain and P is a nonzero prime ideal of R is P a maximal ideal? Justify your answer.

2. Let $\mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$ denote the field with 3 elements. Diagram the lattice of ideals of the polynomial ring $R = \mathbb{F}_3[x]$ that contain $x^3 - 1$.

The following are two variants of a question.

3. Prove or disprove that the rings $\frac{\mathbb{Z}}{(15)}$ and $\frac{\mathbb{Z}}{(3)} \times \frac{\mathbb{Z}}{(5)}$ are isomorphic.

[In this case, you should not just quote a theorem.]

4. Are the rings $\frac{\mathbb{Z}}{(15)}$ and $\frac{\mathbb{Z}}{(3)} \times \frac{\mathbb{Z}}{(5)}$ isomorphic? Justify your answer.

[In this case, a justification of your answer could be to state a relevant theorem from Dummit and Foote.]

5. Let $R = \mathbb{Z}[i]$ denote the ring of Gaussian integers.

(a) Diagram the lattice of ideals of $R = \mathbb{Z}[i]$ that contain $5R$.

(b) Diagram the lattice of ideals of $R = \mathbb{Z}[i]$ that contain $15R$.

(c) Factor $3 + 4i$ into irreducible elements in $\mathbb{Z}[i]$.

(d) What is the cardinality of the ring $\frac{\mathbb{Z}[i]}{(3+4i)}$?

6. Let R be an integral domain.

(a) Define "$N : R \to \mathbb{Z}^+ \cup \{0\}$ is a Dedekind-Hasse norm".

(b) Prove that if R has a Dedekind-Hasse norm, then R is a principal ideal domain.

(c) Prove that if R is a principal ideal domain, then R has a Dedekind-Hasse norm.

7. Let R be a principal ideal domain. Prove that a nonzero prime ideal of R is a maximal ideal.

8. Let R be an integral domain.

(a) Define "R is a Euclidean domain".

(b) Prove that a Euclidean domain is a principal ideal domain.

(c) Prove that the ring $\mathbb{Z}[i]$ of Gaussian integers is a Euclidean domain.

9. Prove that a finite integral domain is a field.

10. Let R be a commutative ring with 1. If I and J are ideals of R such that $I + J = R$, prove that $IJ = I \cap J$.

11. Let $\mathbb{Z}$ denote the ring of integers and consider the ring $R = \mathbb{Z} \times \mathbb{Z}$ with respect to componentwise addition and multiplication.

(a) Prove or disprove that R is an integral domain.

(b) True or false: Every ideal of R is a principal ideal. Justify your answer.

12. Let a is a nonzero nonunit of an integral domain R.

(a) Define "a is an irreducible element".

(b) Define "a is a prime element".

(c) Prove that if a is a prime element, then a is irreducible.

13. Give an example of an integral domain R and an irreducible element $r \in R$ that is not prime. Explain why the element r you give is irreducible and is not prime.

14. If R is an integral domain, prove or disprove that the polynomial ring $R[x]$ is an integral domain.

15. Let F be a field and let $p(x) \in F[x]$ be a nonconstant monic polynomial. Describe all the ideals of the quotient ring $\frac{F[x]}{(p(x))}$ in terms of the factorization of $p(x)$ in $F[x]$.

16. Let $n > 1$ be a positive integer and let p be a prime integer. Let $\varphi : \frac{\mathbb{Z}}{(pn)} \to \frac{\mathbb{Z}}{(n)}$ be the natural surjective ring projection.

(a) If p divides n, prove or disprove that u is a unit in $\frac{\mathbb{Z}}{(pn)}$ if $\varphi(u)$ is a unit in $\frac{\mathbb{Z}}{(n)}$.

(b) If p does not divide n, prove or disprove that u is a unit in $\frac{\mathbb{Z}}{(pn)}$ if $\varphi(u)$ is a unit in $\frac{\mathbb{Z}}{(n)}$.