

1. Determine the number of elements of order 3 in the symmetric group S_4 .
2. Determine the number of elements of order 3 in the symmetric group S_5 .
3. For the group $A = Z_2 \times Z_4 = \langle a, b \mid a^2 = b^4 = 1, ab = ba \rangle$, diagram the lattice of subgroups giving generators for each subgroup.
4. Let G be the group of rigid motions in \mathbb{R}^3 of a regular tetrahedron. List the possible orders of elements of G and the number of elements of G having each order.
5. Let Z_{15} be a cyclic group of order 15.
 - (a) What is the order of the group $\text{Aut}(Z_{15})$?
 - (b) Prove or disprove that the group $\text{Aut}(Z_{15})$ is cyclic.
6. For n a positive integer, let $\phi(n)$ denote the Euler ϕ -function. Thus $\phi(n)$ is the number of positive integers $a \leq n$ with a relatively prime to n .
 - (a) If p is a prime number, what is $\phi(p^3)$?
 - (b) Find all positive integers n such that $\phi(n) = \phi(2n)$
 - (c) Assuming that for each positive integer k there exist only finitely many integers n with $\phi(n) = k$, write out a careful proof that $\phi(n)$ tends to infinity as n tends to infinity. Include a definition of $\lim_{n \rightarrow \infty} \phi(n) = \infty$
7. Let \mathbb{F}_5 denote a finite field with 5 elements and let $GL_2(\mathbb{F}_5)$ denote the multiplicative group of 2×2 matrices whose entries come from \mathbb{F}_5 and whose determinant is nonzero. Let $SL_2(\mathbb{F}_5)$ denote the subgroup of $GL_2(\mathbb{F}_5)$ of matrices whose determinant is $1 \in \mathbb{F}_5$.
 - (a) What is the order of $GL_2(\mathbb{F}_5)$?
 - (b) Is $SL_2(\mathbb{F}_5)$ a normal subgroup of $GL_2(\mathbb{F}_5)$? Justify your answer.
 - (c) What is the index of $SL_2(\mathbb{F}_5)$ in $GL_2(\mathbb{F}_5)$?
 - (d) What is the order of $SL_2(\mathbb{F}_5)$?
8. Let \mathbb{F}_3 denote a field with three elements. Prove or disprove that the group $SL_2(\mathbb{F}_3)$ is isomorphic to the symmetric group S_4 .
9. Consider the polynomial ring $R = \mathbb{Z}[x]$.
 - (a) Give generators for each maximal ideal of R that contain the ideal $(6, x^2 - 5)R$?
 - (b) Diagram the lattice of ideals of R that contain the ideal $(6, x^2 - 5)R$.

10. Let R be a commutative ring with 1.
- (a) Define the *characteristic* of R .
 - (b) Does there exist a ring having characteristic 4? Justify your answer.
11. Does there exist a field having exactly 8 elements? If so, describe how to obtain such a field; if not, explain why not.
12. Let R be a commutative ring with 1 and let I and J be ideals of R such that $I + J = R$. Prove or disprove that $IJ = I \cap J$.
13. For which prime integers $p \in \mathbb{Z}$ is p irreducible in the ring of Gaussian integers $\mathbb{Z}[i]$? Explain.
14. For n a positive integer, prove that the polynomial

$$f(x) = (x - 1)(x - 2) \cdots (x - n) - 1$$

is irreducible in $\mathbb{Z}[x]$.

15. Let a and b be relatively prime positive integers.
- (a) Prove that every integer n has the form $n = ax + by$, where $x, y \in \mathbb{Z}$ and $0 \leq x < b$.
 - (b) What is the largest integer n that cannot be written in the form $ax + by$, where x and y are both nonnegative integers? Justify your answer.
16. Let R be an integral domain.
- (a) Define “ R is a Euclidean domain”.
 - (b) Prove that the ring $R = \mathbb{Z}[i]$ of Gaussian integers is a Euclidean domain.
17. Diagram the lattice of subgroups of the dihedral group D_8 and circle precisely those subgroups that are not normal in D_8 .
18. For n a positive integer, let Z_n denote a cyclic group of order n .
- (a) What is the order of the group $\text{Aut}(Z_n)$?
 - (b) Among the groups $\text{Aut}(Z_8)$, $\text{Aut}(Z_{10})$ and $\text{Aut}(Z_{12})$ which are isomorphic and which are not? Justify your answer.
19. Let H and K be subgroups of a group G . If $H \cup K$ is a subgroup of G , prove that either $H \subseteq K$ or $K \subseteq H$.