These are exercises from Chapter 3 of Jacobson. By a basis or base for a module is meant a linearly independent subset which spans the module, cf. page 164 of Hoffman and Kunze. From Section 3.6 of Jacobson.

1. Find a base for the submodule of \( \mathbb{Z}^{(3)} \) generated by
\[
 f_1 = (1, 0, -1), \quad f_2 = (2, -3, 1), \quad f_3 = (0, 3, 1), \quad f_4 = (3, 1, 5).
\]

2. Find a base for the submodule of \( \mathbb{Q}[\lambda]^{(3)} \) generated by
\[
 f_1 = (2\lambda - 1, \lambda, \lambda^2 + 3), \quad f_2 = (\lambda, \lambda, \lambda^2), \quad f_3 = (\lambda + 1, 2\lambda, 2\lambda^2 - 3).
\]

3. Find a base for the \( \mathbb{Z} \)-submodule of \( \mathbb{Z}^{(3)} \) consisting of all \((x_1, x_2, x_3)\) satisfying the conditions
\[
 x_1 + 2x_2 + 3x_3 = 0, \quad x_1 + 4x_2 + 9x_3 = 0.
\]

From Section 3.7, 1. Obtain a normal form over \( \mathbb{Z} \) for the integral matrix
\[
 B = \begin{bmatrix}
 6 & 2 & 3 & 0 \\
 2 & 3 & -4 & 1 \\
 -3 & 3 & 1 & 2 \\
 -1 & 2 & -3 & 5
\end{bmatrix}
\]

From Section 3.8, 1. Determine the structure of \( \mathbb{Z}^{(3)}/K \) where \( K \) is generated by \( f_1 = (2, 1, -3) \), and \( f_2 = (1, -1, 2) \).

From Section 3.9, 1. Let \( D = \mathbb{R}[\lambda] \) and suppose \( M \) is a direct sum of cyclic \( D \)-modules whose order ideals are the ideals generated by the polynomials
\[
 (\lambda - 1)^3, \quad (\lambda^2 + 1)^2, \quad (\lambda - 1)(\lambda^2 + 1)^4, \quad (\lambda + 2)(\lambda^2 + 1)^2.
\] Determine the elementary divisors and invariant factors of \( M \).

Let \( D \) be a principal ideal domain (PID) and let \( M \) be a \( D \)-module. A submodule \( N \) of \( M \) is said to be pure in \( M \) if for any \( y \in N \) and \( a \in D \), if there exists \( x \in M \) with \( ax = y \), then there exists \( x' \in N \) with \( ax' = y \). The module \( M \) is said to be a torsion module if for each \( m \in M \) there exists a nonzero \( d \in D \) such that \( dm = 0 \).

7. Show that if \( N \) is a direct summand of \( M \), then \( N \) is pure in \( M \). Show that if \( N \) is a pure submodule of \( M \) and \( \text{ann}(x + N) = (d) \) then \( x \) can be chosen in its coset \( x + N \) so that \( \text{ann} x = (d) \).

8. Show that if \( N \) is a pure submodule of a finitely generated torsion module \( M \) over a PID, then \( N \) is a direct summand of \( M \).
Some remarks about $T$-annihilators.

In connection with Exercise 4 on page 225 of Hoffman and Kunze, I suggest you go back and review the remark on page 202 concerning a vector $\alpha \in V$ and $W$ a $T$-invariant subspace of $V$. The $T$-conductor of $\alpha$ into $W$ is by definition

$$S_T(\alpha; W) = \{ g(x) \in F[x] : g(T)(\alpha) \in W \}.$$ 

$S_T(\alpha; W)$ is an ideal of the polynomial ring $F[x]$. One also calls the unique monic generator of $S_T(\alpha; W)$ the $T$-conductor of $\alpha$ into $W$. A useful fact is that for every $\alpha \in V$ and $T$-invariant subspace $W$ of $V$, the $T$-conductor of $\alpha$ into $W$ divides the minimal polynomial of $T$.

In Exercise 4 on page 225, we are given that $p = (x - c_1)^{r_1} \cdots (x - c_k)^{r_k}$ is the minimal polynomial of $T$. If $\alpha \in V$ is such that $(T - c_i I)^m(\alpha) = 0$ for some positive integer $m$, then the $T$-annihilator of $\alpha$ is a divisor of $(x - c_i)^m$ and of $p$ and therefore has the form $(x - c_i)^s$, where $s \leq r_i$ and $s \leq m$. Thus $(T - c_i I)^{r_i}(\alpha) = 0$.

Given a vector space $V$ over a field $F$ and $T : V \to V$ a linear operator, we give to $V$ the structure of a module over the polynomial ring $F[x]$ by defining $g(x)(\alpha) = g(T)(\alpha)$ for each $g(x) \in F[x]$ and $\alpha \in V$. The submodules of $V$ are precisely the $T$-invariant subspaces of $V$. Suppose $V$ is finite-dimensional and $p = p_1^{r_1} \cdots p_k^{r_k}$ is the minimal polynomial for $T$ where the $p_i$ are distinct monic irreducible polynomials in $F[x]$. Let $W_i$ be the null space of $p_i(T)^{r_i}$. The primary decomposition theorem tells us that $V = W_1 \oplus \cdots \oplus W_k$. Moreover, as is asserted in Exercise 10 on page 226 of Hoffman and Kunze, if $W$ is a $T$ invariant subspace of $V$, then

$$W = (W \cap W_1) \oplus \cdots \oplus (W \cap W_k).$$

Note that this tells us a great deal about the submodules of $V$. It says, for example, that if $\dim(V) = n$ and if $T$ has $n$ distinct characteristic values, then $V$ has precisely $2^n$ submodules. Thus an easy way to prove Exercise 7 (b) on page 231 of Hoffman and Kunze (which asks to show that if $\{\alpha_1, \ldots, \alpha_n\}$ is a basis for $V$ of characteristic vectors having distinct characteristic values, then $\alpha = \alpha_1 + \cdots + \alpha_n$ is a cyclic vector for $V$) is to observe that $\alpha$ is in no proper submodule of $V$. 