1. An integral extension of integral domains where going-down fails

Definition 1. Let A be a subring of an integral domain B. The *conductor* of B into A is

$$\mathbf{c} \ = \ \{a \in A \mid aB \subset A\ \}.$$

Notice that $\mathbf{c}$ is an ideal of both the ring A and the ring B and is the largest ideal of A that is also an ideal of B. We have $\mathbf{c} = 0$ unless B is contained in the field of fractions of A, and B is contained in a cyclic A-module.

The conductor of B into A is useful for describing prime ideals $\mathbf{p}$ of A such that $B \subset A_{\mathbf{p}}$. The following lemma is from Zariski-Samuel Ch V, Section 5.

Lemma 2. *Let A be an integral domain, let B denote the integral closure of A and let $\mathbf{c}$ be the conductor of B into A. If S is a multiplicative system in A, then $S^{-1}B$ is the integral closure of $S^{-1}A$, and, for $S^{-1}A$ to be integrally closed, it is sufficient that $\mathbf{c} \cap S \neq \emptyset$. Furthermore, if B is a finite A-module, then the conductor of $S^{-1}B$ into $S^{-1}A$ is $S^{-1}\mathbf{c}$ and if, moreover, $S^{-1}A$ is integrally closed, then $\mathbf{c} \cap S \neq \emptyset$.*

Remark 3. Lemma 2 implies that if the integral closure B of A is a finite A-module, then the prime ideals $\mathbf{p}$ of A such that $B \subset A_{\mathbf{p}}$ are precisely the prime ideals $\mathbf{p}$ that do not contain the conductor $\mathbf{c}$ of B into A. Thus the closed set $\mathcal{V}(\mathbf{c}) = \{\mathbf{p} \in \operatorname{Spec} A \mid \ \mathbf{c} \subset \mathbf{p}\ \}$ is the nonnormal (or non-integrally closed) locus of $\operatorname{Spec} A$.

Example 4. Let x and y be indeterminates over a field k and consider the inclusion map of rings

$$A \ = \ k[x(x-1),\ x^2(x-1),\ y] \ \hookrightarrow \ k[x,y] \ = \ B.$$

Notice that x is integral over A. Thus $B = A[x]$ is integral over A. Since B is integrally closed, it follows that B is the integral closure of A. Moreover $B = A + Ax$. Thus B is a finite A-module. It is not difficult to show that $x(x-1)B$ is an ideal in A and that

$$\mathbf{c} \ = \ x(x-1)B \ = \ (x(x-1),\ x^2(x-1))A$$

is the conductor of B into A.

Let $\mathbf{q} = (x-y)B$ and let $\mathbf{p} = \mathbf{q} \cap A$. Since $\mathbf{c} \not\subseteq \mathbf{p}$, we have $B \subset A_{\mathbf{p}}$. Hence $\mathbf{q}\,B_{\mathbf{q}} = \mathbf{p}\,A_{\mathbf{p}}$ and $\mathbf{q}$ is the unique prime of B lying over $\mathbf{p}$. Let $M_1 = (x,y)B$ and $M_2 = (x-1,y)B$. Notice that

$$M := M_1 \cap A = (x(x-1),\ x^2(x-1),\ y)A = M_2 \cap A.$$

Since $\mathbf{q} \subset M_1$, it follows that $\mathbf{p} \subset M \subset M_2$. Hence $\mathbf{p}\,B \subset M_2$. Therefore M_2 contains a minimal prime of $\mathbf{p}\,B$. Since $\mathbf{q}$ is the unique prime of B lying over $\mathbf{p}$ and $\mathbf{q} \not\subseteq M_2$, it follows that the going-down property fails for the chain $\mathbf{p} \subset M$ of A and the prime ideal M_2 of B that lies over M in A. Indeed, in this example it can be seen that M_2 is a minimal prime of $\mathbf{p}\,B$.