1. Associated graded rings

**Question 1.1.** Let $G = k[G_1] = \bigoplus_{n \geq 0} G_n$ be a one-dimensional standard graded ring over the field $k = G_0$. What are necessary and sufficient conditions in order that $G$ be the associated graded ring, $\text{gr}_m(R)$, of a one-dimensional Cohen-Macaulay local ring $(R, m)$?

The standard graded ring $G = k[G_1]$ is isomorphic to the associated graded ring of the local ring $G_{G_+}$ and if $G$ is Cohen-Macaulay, then $G_{G_+}$ is Cohen-Macaulay. Hence every Cohen-Macaulay standard graded ring $k[G_1]$ is the associated graded ring of a Cohen-Macaulay local ring.

If $\dim_k G_1 \leq 2$, then $G = \text{gr}_m(R)$, where $(R, m)$ is Cohen-Macaulay, implies $G$ is Cohen-Macaulay. For one may assume that the embedding dimension of $R$ is at most two and that $R$ is complete in the $m$-adic topology. It follows that $R$ is either a DVR, or of the form $S/fS$, where $S$ is a two-dimensional RLR. If $R$ is a DVR, then $\text{gr}_m(R)$ is a polynomial ring $k[t]$, while if $R = S/fS$, where $(S, n)$ is a two-dimensional RLR, then $\text{gr}_m(R) = \text{gr}_n(S)/(f^*)$, where $\text{gr}_n(S) = k[x, y]$ is a polynomial ring in two variables over the field $k$ and $f^*$ is the initial form of $f$ and thus a nonzero homogeneous element of degree $e$, where $e$ is the multiplicity of $R$ and $G$.

There exist one-dimensional standard graded rings $G = k[G_1]$ with $\dim_k G_1 = 3$ that are not Cohen-Macaulay, but are of the form $\text{gr}_m(R)$, where $(R, m)$ is a one-dimensional Cohen-Macaulay local domain.

**Example 1.2.** Let $R = k[[t^4, t^5, t^{11}]]$, and define a $k$-algebra homomorphism of $S = k[[x, y, z]]$ onto $R$ by $x \mapsto t^4$, $y \mapsto t^5$, and $z \mapsto t^{11}$. Then $R = S/I$, where $I = (xz - y^4, yz - x^4, z^2 - x^3 y^2)$. In an abuse of notation, we let $x, y, z$ denote their own initial forms in the associated graded ring $\text{gr}_n(S)$. Thus $\text{gr}_n(S)$ is the polynomial ring $k[x, y, z]$ and the ideal $I^*$ of initial forms of elements of $I$ contains $(xz, yz, z^2)$. Since these initial forms are all multiples of $z$, they do not generate $I^*$.

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One computes that
\[ I^* = (xz, yz, z^2, y^4)k[x, y, z], \]
and therefore that
\[ G = \text{gr}_m(R) = \frac{k[x, y, z]}{(xz, yz, z^2, y^4)}. \]
Notice that \( \text{Ann}_G z = (x, y, z)G \), the graded maximal ideal of the one-dimensional standard graded ring \( G \). Thus \( G \) is a one-dimensional standard graded ring that is not Cohen-Macaulay, but yet has the form \( \text{gr}_m(R) \), where \( (R, m) \) is Cohen-Macaulay. We have \( e(R) = e(G) = 4 \) and \( t^4R = J \) is a principal reduction of \( m \).

Since \( R \) is Cohen-Macaulay, we have \( \ell(R/t^4R) = 4 = e(R) \). Notice that \( xG \) is a principal reduction of the maximal graded ideal \( G_+ \) of \( G \). One computes that
\[ \ell(G/xG) = \ell(k[y, z]) = 5, \]
and
\[ \ell(0 :_G x) = 1. \]
This confirms the equation
\[ e(G) = \ell(G/xG) - \ell(0 :_G x). \]

Tom Marley [1] proved a result that implies as a special case that if \( (R, m) \) is a one-dimensional Cohen-Macaulay local ring and \( G = \text{gr}_m(R) \) is not Cohen-Macaulay, then \( a_0(G) > a_1(G) \), where
\[ a_0(G) = \max\{n \mid H^0_{G_+}(G)_n \neq 0\}, \]
and \( H^0_{G_+}(G) = \bigcup_{i \geq 0} (0) :_G G^i_+ \).

Here \( x \in G_1 \) generates a principal reduction of \( G_+ \) and \( \varphi : G \to G[1/x] \) is the canonical homomorphism.

For \( R \) and \( G \) as in Example 1.2, one sees that \( H^0_{G_+}(G) = zG = (z) \) and \( \deg z = 1 \), so \( a_0(G) = 1 \). Also \( G[1/x] = k[x, y][1/x] \), where \( y^4 = 0 \). Hence \( y^3/x \in G[1/x] \) has a nonzero image in \( G[1/x] \) and is a homogeneous element of maximal degree with this property. Since \( \deg y^3/x = 2 \), we have \( a_1(G) = 2 > 1 = a_0(G) \), as is to be expected because of Marley’s result.

**Question 1.3.** Does there exist a one-dimensional standard graded ring \( G = k[G_1] \) as in Question 1.1 that also has \( a_0(G) < a_1(G) \) and yet \( G \) is not an associated graded ring \( \text{gr}_m(R) \), where \( (R, m) \) is Cohen-Macaulay? Or said another way, could it be that the necessary condition \( a_0(G) < a_1(G) \) is also a sufficient condition, and thus an answer to Question 1.1?
Example 1.4. Let $R = k[[t^6, t^7, t^{15}]]$, and define a $k$-algebra homomorphism of $S = k[[x, y, z]]$ onto $R$ by $x \mapsto t^6$, $y \mapsto t^7$, and $z \mapsto t^{15}$. Then $R = S/I$, where $I = (xz - y^3, z^2 - x^5)S$. Thus $R$ is a complete intersection and therefore is Gorenstein. As in Example 1.2, we let $x, y, z$ denote their own initial forms in the associated graded ring $\text{gr}_n(S)$. Thus $\text{gr}_n(S)$ is the polynomial ring $k[ x, y, z]$ and one computes that the ideal $I^*$ of initial forms of elements of $I$ is

$$I^* = (xz, z^2, zy^3, y^6)k[x, y, z],$$

and therefore that

$$G = \text{gr}_m(R) = \frac{k[x, y, z]}{(xz, z^2, zy^3, y^6)}.$$  

Notice that $\text{Ann}_G z = (x, z, y^3)G$ is primary for the graded maximal ideal $G_+$ of the one-dimensional standard graded ring $G$. Thus $G$ is not Cohen-Macaulay, but yet has the form $\text{gr}_m(R)$, where $(R, m)$ is Gorenstein. We have $e(R) = e(G) = 6$ and $t^6R = J$ is a principal reduction of $m$. Since $R$ is Cohen-Macaulay, we have $\ell(R/t^6R) = 6 = e(R)$. Notice that $xG$ is a principal reduction of $G_+$. One computes that

$$\ell\left(\frac{G}{xG}\right) = \ell\left(\frac{k[y, z]}{(z^2, zy^3, y^6)}\right) = 9, \quad \text{and} \quad \ell(0 :_G x) = 3.$$ 

This confirms the equation

$$e(G) = \ell(G/xG) - \ell(0 :_G x).$$

One also computes that

$$(0 :_G x = (z, zy, zy^2)G = H^0_{G_+}(G) \quad \text{and thus} \quad a_0(G) = 3,$$

while

$$G\left[\frac{1}{x}\right] = k[x, y]\left[\frac{1}{x}\right], \quad \text{where} \quad y^6 = 0, \quad \text{so} \quad \frac{y^5}{x} \in G\left[\frac{1}{x}\right]$$

has a nonzero image in

$$H^1_{G_+}(G) = G\left[\frac{1}{x}\right] \varphi(G).$$

Since $\deg\frac{y^5}{x} = 4$, $a_1(G) \geq 4$,

and since $y^5/x$ has maximal degree among homogeneous elements of $G[1/x] \setminus \varphi(G)$, we have $a_1(G) = 4$.

It is interesting to also consider the principal reduction $x - y$ of $G_+$. We have

$$\ell\left(\frac{G}{(x - y)G}\right) = \ell\left(\frac{k[y, z]}{(z^2, zy, y^6)}\right) = 7, \quad \text{and} \quad \ell(0 :_G x - y) = 1.$$ 

Since $G$ is not Cohen-Macaulay, for each $w \in G_1$ that generates a reduction of $G_+$, we must have $\ell(G/wG) > e(G) = 6$, but the length varies depending on the length of $(0) :_G w$. 
Remark 1.5. Let $G = k[G_1] = \bigoplus_{n \geq 0} G_n$ be a one-dimensional standard graded ring over the field $k = G_0$. If $G$ is not Cohen-Macaulay and $\dim_k G_1 = 2$, then $a_0(G) > a_1(G)$. For $G = k[x, y] = S/I$, where $S = k[X, Y]$ is a polynomial ring and $I$ is a homogeneous ideal with $\text{ht} I = 1$. Since the polynomial ring $S$ is a UFD, $I = fL$, where $f$ is a homogeneous polynomial of degree $d \geq 1$ and either $L = S$, or $L$ is a homogeneous ideal that is primary for the maximal ideal $(X, Y)S$. Since $G$ is not Cohen-Macaulay, $L \neq S$. We may assume that the field $k$ is infinite and that $f$ is monic as a polynomial in $Y$. Then $xG$ is a principal reduction of $G +$ and $H^1_G + (G) = G[1/x]/\varphi(G)$, where $\varphi(G) = G/fG = S/fS$ and $a_1(G) = d - 2$ since $y^{d-1} \in G[1/x] \setminus \varphi(G)$ is a homogeneous element of degree $d - 2$ and this is the maximal degree of a homogeneous element of $G[1/x] \setminus \varphi(G)$. On the other hand, $H^0_G + (G) = J/I$, where $J$ is the saturation in $S = k[X, Y]$ of $I = fL$, i.e.,

$$J = \bigcup_{n \geq 0} (fL :_S (X, Y))^n = fS,$$

and thus $H^0_G + (G) = fS/fL$.

It follows that $a_0(G) = \deg f + \text{socdeg}(L)$, where $\text{socdeg}(L)$ is the maximal degree of a homogeneous element in $(L :_S (X, Y)) \setminus L$. Since $L$ is $(X, Y)S$-primary, $\text{socdeg}(L)$ is a well-defined nonnegative integer.

We conclude that $a_0(G) + 2 \leq a_0(G)$ if $\dim_k G_1 = 2$ and $G$ is not Cohen-Macaulay.

**Question 1.6.** Let $S = k[x, y, z]$ be the graded polynomial ring in the variables $x, y, z$ over the field $k$. Can one describe or classify in some way the homogeneous ideals $I$ of $S$ such that

1. the radical of $I$ is the prime ideal $(y, z)S$,
2. $I$ has the maximal ideal $(x, y, z)S$ as an associated prime,
3. $G = S/I$ has the property that $a_0(G) < a_1(G)$?

Example 1.2 shows that the ideal $(xz, yz, z^2, y^4)S$ has the properties enumerated in Question 1.6, but one sees that the ideals $(xz, yz, z^2, y^2)S$ and $(xz, yz, z^2, y^3)S$ fail to satisfy the condition $a_0(G) < a_1(G)$. Example 1.4 shows that the ideal $(xz, z^2, yz^3, y^6)S$ has the properties enumerated in Question 1.6, but one sees that the ideals $(xz, z^2, yz^3, y^4)S$ and $(xz, z^2, yz^3, y^5)S$ fail to satisfy the condition $a_0(G) < a_1(G)$.
2. THE ASSOCIATIVITY FORMULA FOR MULTIPLICITIES

REFERENCES


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