THE BOOLEAN PRIME IDEAL THEOREM + COUNTABLE CHOICE DO NOT IMPLY DEPENDENT CHOICE

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Abstract. Two Fraenkel-Mostowski models are constructed in which the Boolean Prime Ideal Theorem is true. In both models, AC for countable sets is true, but AC for sets of cardinality $2^{\aleph_0}$ and the $2m = m$ principle are both false. The Principle of Dependent Choices is true in the first model, but false in the second.

Introduction.

In this paper we prove a conjecture of David Pincus first mentioned in Brunner [2]: Is there a Fraenkel-Mostowski model in which the Boolean prime ideal theorem and the countable axiom of choice are true but in which the axiom of dependent choice is false? Our notation for these principles and their precise statements are:

BPI: The Boolean prime ideal theorem. Every Boolean algebra has a prime ideal.

$C(\aleph_0, \infty)$: The countable axiom of choice. Every countable set of non-empty sets has a choice function.

DC: The axiom of dependent choice. If S is a relation on A such that $(\forall x \in A)(\exists y \in A)(xSy)$ then there is a sequence $a(1), a(2), \dots$ of elements of A such that $a(n)Sa(n+1)$ for all $n \in \omega$.

There are very few Fraenkel-Mostowski models of the theory ZF^0 (Zermelo-Fraenkel set theory without the foundation axiom) in which the Axiom of Choice (AC) is false and the Boolean Prime Ideal Theorem (BPI) is known to be true. We know of only two such models: The Mostowski linear ordered model [6] ([3], [4 p49]) and a model due to Tsukada [9]. In this paper we construct two Fraenkel-Mostowski models $\mathcal{N}1$ and $\mathcal{N}2$ in which the Boolean Prime Ideal Theorem is true. These two models are constructed using the same set of atoms A and the same group G of permutations of A. They differ only in their supports. It is also the case that in both models the Axiom of Choice for countable families of sets $(C(\aleph_0, \infty))$ is true, but both the Axiom of Choice for families of cardinality $2^{\aleph_0}$ $(C(2^{\aleph_0}, \infty))$ and

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the a $2m = m$ principle are false. By using a slight modification of the proof of the Tychonoff Compactness Theorem (see, for example, [5 p143]), it can be shown that BPI and $C(\aleph_0, \infty)$ imply that the product of a countable number of compact spaces is compact. Thus, the Countable Tychonoff Compactness Theorem is true in both models. The two models differ with regard to the truth of the Principle of Dependent Choices (DC). We will show that DC is false in $\mathcal{N}1$ and true in $\mathcal{N}2$. Thus $\mathcal{N}1$ provides a proof of the conjecture of Pincus mentioned above that there is a Fraenkel-Mostowski model for BPI $+C(\aleph_0, \infty) + \neg$ DC. We would like to thank David Pincus for reading a preliminary version of this paper and making several helpful suggestions. His transfer theorems ([7]) can be used to show that the results of this paper transfer to models of ZF as follows: Clearly the negations of BPI, $C(\aleph_0, \infty)$, and DC are boundable in the sense of [7] and therefore, by theorem 4 of [7] a conjunction of these negations and one or more of $C(\aleph_0, \infty)$ and DC is transferable. Further, by the note added in proof to [7], BPI can be added to this last list. Pincus has proved some results related to those above. In [8], he has constructed a ZF model for DC + BPI + $\neg C(2^{\aleph_0}, \infty)$ and in [7] a ZF model for $C(\aleph_0, \infty)$ + O + $\neg$DC where O is the ordering principle: "Every set can be linearly ordered".

Section 1. The Models.

Let $(A, \leq)$ be an ordered set of atoms which is order isomorphic to $\mathbb{Q}^\omega$, the set of all functions from ω into $\mathbb{Q}$, (where $\mathbb{Q}$ is the set of rational numbers) ordered by the lexicographic ordering. That is, if $a, b \in \mathbb{Q}^\omega$ then $a < b$ if and only if there is some $n \in \omega$ such that $(\forall j < n)(a_j = b_j)$ and $a_n < b_n$.

Definition. Assume $b \in A$ and $n \in \omega$

1. $A^n_b = \{a \in A : a_i = b_i \text{ for } 0 \leq i \leq n\}$ is the *n–level block containing b*. (A^n_b will not be in the models we are defining.)
2. The sequence $\langle b_{n+1}, b_{n+2}, \dots\rangle$ is the *position of b in its n–level block.*
3. $\mathfrak{B}^n = \{A^n_a : a \in A\}$ is the set of n–level blocks.
4. $\leq_n$ is the relation on $\mathfrak{B}^n$ defined by $A^n_a \leq_n A^n_b$ if and only if $a \leq b$.
5. Let f be an order automorphism of $(\mathfrak{B}^n, \leq_n)$ (see Lemmas A and B below) we define ϕ_f to be the unique order automorphism of $(A, \leq)$ which satisfies $\phi''_f A^n_a = f(A^n_a)$ for all $a \in A$ and such that for all $a \in A$, a and $\phi_f(a)$ have the same position in their n-level blocks. (By definition 2 above, this means that $(\forall a \in A)(\forall i > n)(a_i = \phi_f(a)_i)$.)
6. For $n \in \omega$, G_n is the group $\{\phi_f : f \text{ is an order automorphism of } (\mathfrak{B}^n, \leq_n)\}$.
7. G is the group $\bigcup_{n\in\omega} G_n$. Note that for $n < m$, $G_n \subseteq G_m$.
8. A subset $E \subseteq A$ is a support for the model $\mathcal{N}1$ if it satisfies (a), (b) and (c) below. E is a support for the model $\mathcal{N}2$ if it satisfies (b) and (c) below:
   (a) E is well ordered by the ordering $\leq$ on A.
   (b) For each $n \in \omega$, $\{A^n_a : a \in A \wedge A^n_a \cap E \neq \emptyset\}$ is finite. (That is, for each $n \in \omega$, the set of nth coordinates of elements of E is finite.)
   (c) E is countable.
9. $\mathcal{N}1$ and $\mathcal{N}2$ are the Fraenkel-Mostowski models determined by the set A, the group G, and the supports defined above.
10. If H is a subgroup of G and $E \subseteq A$, then $\text{fix}_H(E)$ denotes the subgroup $\{\phi \in H : (\forall a \in E)(\phi(a) = a)\}$.
11. If $n \in \omega$ and $E \subseteq A$ then $\mathfrak{B}^n(E) = \{A^n_a : a \in E\}$. It follows from definition 8 (b) that if E is a support then $\mathfrak{B}^n(E)$ is finite. We also note that $\mathfrak{B}^n(A) = \mathfrak{B}^n$

Section 2. Preliminary Lemmas.

In this section we will use $\mathcal{N}$ for either $\mathcal{N}1$ or $\mathcal{N}2$. The following Lemmas, with the exception of Lemma I, apply to both $\mathcal{N}1$ and $\mathcal{N}2$.

Lemma A. For each $n \in \omega$ and $a \in A$, A^n_a is an interval in the ordering $\leq$ on A (in the sense that if $c, d \in A^n_a$ and $c \leq b \leq d$ then $b \in A^n_a$.

Lemma B. The ordering $\leq_n$ defined on $\mathfrak{B}^n$ by $A^n_a \leq_n A^n_b \Leftrightarrow a \leq b$ is well defined and the ordered set $(\mathfrak{B}^n, \leq_n)$ is order isomorphic to the rationals with the usual ordering.

Lemma C. (Characterization of G in terms of coordinates) Let ϕ be an order automorphism of $(A, \leq)$ then ϕ is in G if and only if for some $n \in \omega$, the following two conditions are satisfied:

a. For all a and b in A, if $a_i = b_i$ for $0 \leq i \leq n$ then $\phi(a)_i = \phi(b)_i$ for $0 \leq i \leq n$.

b. For all $a \in A$, for all $i > n$, $a_i = \phi(a)_i$.

Lemma D. Fix $n \in \omega$. For each (finite) sequence $\sigma \in \mathbb{Q}^{n+1}$, let p_σ be an order preserving permutation of $\mathbb{Q}$, then the permutation ϕ of A defined by

$$(\phi(a))_j = \begin{cases} a_j & \text{if } j \neq n+1 \\ p_{\langle a_0, \ldots, a_n \rangle}(a_j) & \text{if } j = n+1 \end{cases}$$

that is,

$$\phi(\langle a_0, \ldots, a_n, a_{n+1}, a_{n+2}, \ldots \rangle) = \langle a_0, \ldots, a_n, p_\sigma(a_{n+1}), a_{n+2}, \ldots \rangle$$

(where $\sigma = \langle a_0, a_1, \ldots, a_n \rangle$) is in G_{n+1} and satisfies $(\forall a \in A)(\phi(A^n_a) = A^n_a)$.

Definition. If $n \in \omega$ and $x \in \mathcal{N}$ then $E \subseteq \mathfrak{B}^n$ is an *n-level support for x* if E is finite and $\forall \phi \in G_n$, if $\phi(E) = E$ then $\phi(x) = x$.

Lemma E. If $x \in \mathcal{N}$, then x has an n-level support for every $n \in \omega$.

Lemma F. If $x \in \mathcal{N}$ and $n \in \omega$ then the intersection of two n-level supports for x is an n-level support for x.

Lemma G. $\forall n \in \omega,\ \forall C, D \subseteq A,\ \mathfrak{B}^n(C \cup D) = \mathfrak{B}^n(C) \cup \mathfrak{B}^n(D)$.

Lemma H. Assume $n, m \in \omega$, $m < n$, $C, D \subseteq A$ and $\mathfrak{B}^n(C) \subseteq \mathfrak{B}^n(D)$ then $\mathfrak{B}^m(C) \subseteq \mathfrak{B}^m(D)$.

Lemma I. ($\mathcal{N}1$ only) Assume that E is a support and that a and b are chosen in A so that $a < b$ and the interval $(a, b]$ (in the ordering $(A, \leq)$) is disjoint from E. Then $\exists \phi \in \text{fix}_G(E)$ such that $\phi(b) \neq b$.

Proof. Let e be the least element of E which is $> b$. Choose n large enough so that a, b and e are in different n-level blocks. Then there is a $\phi \in \text{fix}_{G_n}(E)$ which moves the n-level block containing b.

Lemma J. Each support is bounded. (Follows from Definition 8 (b).)

Lemma K. If $B,\ C,\ D \subseteq A$ and k and i are in ω with $k \geq i$, then

$$\left[\mathfrak{B}^k(B) \subseteq \mathfrak{B}^k(C) \wedge \mathfrak{B}^i(C) \subseteq \mathfrak{B}^i(D)\right] \Rightarrow \mathfrak{B}^i(B) \subseteq \mathfrak{B}^i(D).$$

(Lemma K is a consequence of Lemma H.)

Lemma L. If $\phi \in G_n$ and E is a support, then $|\mathfrak{B}^k(E)| = |\mathfrak{B}^k(\phi(E))|$ for all $k \geq n$.

LEMMA M. If E and E' are supports, $n \in \omega$, and $\mathcal{B}^n(E) \not\subseteq \mathcal{B}^n(E')$ then there is a $\phi \in G_n$ which fixes E' pointwise and for which $\phi(E) \neq E$.

Section 3. The proof that DC fails in $\mathcal{N}1$.

Let R be the relation on A defined by $a \, R \, b$ if and only if $b \leq a$. If there is an infinite sequence $\langle a_j \rangle_{j \in \omega}$ such that $a_j \, R \, a_{j+1}$ for all $j \in \omega$, then for all $j \in \omega$, $a_j > a_{j+1}$. We will assume there is such a sequence in $\mathcal{N}1$ and arrive at a contradiction. Let E be a support for such a sequence. Then for all $\phi \in \text{fix}_G(E)$ and for all $j \in \omega$, $\phi(a_i) = a_i$. Since E is well ordered by $\leq$, $\exists j_0 \in \omega$ such that the interval $(a_{j_0+1}, a_{j_0}]$ contains no points of E. By I there is a permutation $\phi \in \text{fix}_G(E)$ such that $\phi(a_{j_0}) \neq a_{j_0}$.

Section 4. The proof of DC and $\mathbf{C}(\aleph_0, \infty)$ in $\mathcal{N}2$.

Assume that R is a relation on a set X which is in $\mathcal{N}2$ and which satisfies the hypothesis of dependent choice: $(\forall x \in X)(\exists y \in Y)(x \, R \, y)$. Suppose R has support E. Applying the principle of dependent choice in the ground model we obtain a sequence $\langle x_k \rangle_{k \in \omega}$ of elements of X such that $x_k \, R \, x_{k+1}$ for all $k \in \omega$. Since each x_k is in $\mathcal{N}2$ we can choose a set $E_k \subseteq A$ such that $E_k \cup E$ is a support for x_k and such that $E_k \cap E = \emptyset$ (although the function $x_k \to E_k$ may not be in $\mathcal{N}2$).

For $n \in \omega$, let $\mathfrak{K}^n$ be the set of open intervals in the ordering $(\mathcal{B}^n, \leq_n)$ determined by the finite subset $\mathcal{B}^n(E)$ of $\mathcal{B}^n$. We may assume that for each $n \in \omega$

$$(*) \qquad (\forall \mathcal{K} \in \mathfrak{K}^n)(\mathcal{K} \cap \mathcal{B}^n(E_n) \neq \emptyset).$$

(by adding finitely many elements to each E_n if necessary.)

We will define a sequence $\langle \phi_n \rangle_{n \in \omega}$ of elements of G such that following four conditions hold:

(1) $\phi_n \in G_n$.
(2) ϕ_n fixes E pointwise.
(3) For $n \geq 1$, ϕ_n fixes $\psi_{n-1}(E_{n-1})$ pointwise, where ψ_k is the composition $\phi_k \circ \phi_{k-1} \circ \cdots \circ \phi_0$.
(4) For $n \geq 1$, $\mathcal{B}^{n-1}(\psi_n(E_n)) \subseteq \mathcal{B}^{n-1}(\psi_{n-1}(E_{n-1}))$.

LEMMA. Conditions (1) through (4) above imply that the sequence $\langle \psi_n(x_n) \rangle_{n \in \omega}$ is in $\mathcal{N}2$ and that $(\forall n \in \omega)\,(\psi_n(x_n) \, R \, \psi_{n+1}(x_{n+1}))$.

PROOF. To show that the sequence $\langle \psi_n(x_n) \rangle_{n \in \omega}$ is in $\mathcal{N}2$, we show that $E_\infty = \bigcup_{n \in \omega} \psi_n(E_n)$ is a support. (This will suffice since for each $n \in \omega$, $\psi_n(E_n)$ is a support of $\psi_n(x_n)$.)

We first note that E_∞ is countable since it is a countable union of countable sets. Secondly we show that for all $n \in \omega$, $\mathcal{B}^n(E_\infty)$ is finite: Assume $n \in \omega$. It will suffice to show that $\forall k \in \omega$, $\mathcal{B}^n(\psi_{n+k}(E_{n+k})) \subseteq \mathcal{B}^n(\psi_n(E_n))$. This is an argument by mathematical induction using condition (4) and Lemma K.

For the argument that $\psi_n(x_n) \, R \, \psi_{n+1}(x_{n+1})$, first note that $\psi_n(x_n) \, R \, \psi_n(x_{n+1})$. This is because $x_n \, R \, x_{n+1}$ and ψ_n fixes a support of R (namely E) pointwise. By condition (3), $\phi_{n+1}(\psi_n(x_n)) = \psi_n(x_n)$ since $\psi_n(E_n)$ is a support of $\psi_n(x_n)$. The permutation ϕ_{n+1} also fixes R so $\phi_{n+1} \circ \psi_n(x_n) \, R \, \phi_{n+1} \circ \psi_n(x_{n+1})$. But the left hand side of this relational statement is $\psi_n(x_n)$ and the right hand side is $\psi_{n+1}(x_{n+1})$. This completes the proof of the lemma.

The definition of $\langle \phi_n \rangle_{n\in\omega}$ is by induction. ϕ_0 is the identity permutation on A. Assume that $\phi_0, \ldots, \phi_n$ have been defined satisfying conditions (1) through (4). Since $\psi_n \in G_n$ and fixes E pointwise and since E_n satisfies (*), $\psi_n(E_n)$ also satisfies (*), that is, $(\forall \mathcal{K} \in \mathfrak{K}^n)(\mathcal{K} \cap \mathcal{B}^n(\psi_n(E_n)) \neq \emptyset)$.

Fix a $\mathcal{K} \in \mathfrak{K}^n$. Since the set $\mathcal{B}^{n+1}(\bigcup \mathcal{K})$ of $n+1$-level blocks which are subsets of $\bigcup \mathcal{K}$ is an interval in $\left(\mathcal{B}^{n+1}, \leq_{n+1}\right)$ and $(\bigcup \mathcal{K}) \cap E = \emptyset$, there is a permutation $\eta_{\mathcal{K}} \in G_{n+1}$ which fixes $E \cup \psi_n(E_n)$ and such that for all $n+1$-level blocks Y such that $Y \in \mathcal{B}^{n+1}(\bigcup \mathcal{K}) \cap \mathcal{B}^{n+1}(\psi_n(E_{n+1}))$, there is an n-level block $Z \in \mathcal{B}^n(\bigcup \mathcal{K}) \cap \mathcal{B}^n(\psi_n(E_n))$ such that $\eta_{\mathcal{K}}(Y) \subseteq Z$. Hence

$$(**) \qquad \mathcal{B}^n(\bigcup \mathcal{K}) \cap \mathcal{B}^n(\eta_{\mathcal{K}}(\psi_n(E_{n+1}))) \subseteq \mathcal{B}^n(\bigcup \mathcal{K}) \cap \mathcal{B}^n(\psi_n(E_n))$$

and we may assume that $\eta_{\mathcal{K}}$ is the identity on $A - \bigcup \mathcal{K}$.

If we let $\phi_{n+1} = \prod_{\mathcal{K} \in \mathfrak{K}^n} \eta_{\mathcal{K}}$, then ϕ_{n+1} is in G_{n+1} and fixes $E \cup \psi_n(E_n)$ pointwise. Hence ϕ_{n+1} satisfies conditions (1), (2) and (3). Also by (**) and the fact that ϕ_{n+1} fixes $\bigcup \mathcal{B}^n(E)$ pointwise, we get $\mathcal{B}^n(\psi_{n+1}(E_{n+1})) \subseteq \mathcal{B}^n(\psi_n(E_n)) \cup \mathcal{B}^n(E)$ which is condition (4) for ϕ_{n+1}. This completes the definition of the sequence $\langle \phi_n \rangle_{n\in\omega}$ satisfying (1) through (4).

Since DC implies $C(\aleph_0, \infty)$ we also have that $C(\aleph_0, \infty)$ is true in $\mathcal{N}2$.

Section 5. The proof of $\mathbf{C}(\aleph_0, \infty)$ in $\mathcal{N}1$.

THEOREM. $C(\aleph_0, \infty)$ is true in $\mathcal{N}1$.

PROOF. Let $S = \{s_i : i \in \omega\}$ be a set of non-empty sets which is countable in $\mathcal{N}1$. Assume that E is a support of s_i for all $i \in \omega$. We may assume without loss of generality that

$$(1) \qquad (\forall a \in A)[((\forall n \in \omega)(\exists e_n \in E)(e_n \in A_a^n)) \to a \in E].$$

The reason is that if E is a support then so is $E \cup \{a \in A : (\forall n \in \omega)(\exists e_n \in E)(e_n \in A_a^n)\}$.

Assume that $\{e_n : n \in \omega\}$ is an enumeration of E and define for $n \in \omega$

$$H_n = \{a \in A : a < e_n \wedge (\forall t \in E)(t < e_n \to t < a)\}$$
$$\text{and } H_\infty = \{a \in A : (\forall t \in E)(t < a)\}.$$

H_n for $n \in \omega$ and H_∞ are the open intervals in $(A, \leq)$ determined by E. Note, however, that H_n for $n > 0$ need not be the interval (e_{n-1}, e_n) since the function $n \mapsto e_n$ may not be order preserving from ω to $(A, \leq)$, i.e., the order type of E may not be ω. In fact some of the H_n may be empty. (However as a consequence of Lemma J, H_∞ is not empty.) Note also that A is the disjoint union

$$(2) \qquad A = \left(\bigcup_{n\in\omega} H_n\right) \cup H_\infty \cup E.$$

For each $i \in \omega$ choose $t_i \in s_i$ (The function $i \mapsto t_i$ need not be in $\mathcal{N}1$.) and choose $E_i \subseteq A$ so that $E_i \cap E = \emptyset$ and $E_i \cup E$ is a support of t_i. We may assume (by adding finitely many points to each E_i if necessary) that

$$(3) \qquad E_i \cap H_\infty \neq \emptyset \wedge (\forall i \in \omega)(\forall n, 0 \leq n < i)(H_n \neq \emptyset \to E_i \cap H_n \neq \emptyset)$$

Let $m_0 = 0$ and choose a positive integer $m_1 > 1$ so that $\forall m \geq m_1$,

$$\text{(a)} \quad (\forall b \in E_1 \cap H_\infty)(\forall t \in E)(A_t^m <_m A_b^m)$$

$$\text{(b)} \quad (\forall b \in E_1 \cap H_0)(\forall t \in E)(t < e_0 \to A_t^m <_m A_b^m)$$

$$\text{(c)} \quad (\exists b \in E_1 \cap H_0)(A_b^m <_m A_{e_0}^m)$$

Such an m_1 exists: If $H_0 \neq \emptyset$ let b_0 be the smallest element of $E_1 \cap H_0$ and let b_∞ be the smallest element of $E_1 \cap H_\infty$ (in the ordering $(A, \leq)$). By (1) there is an $m' \in \omega$ such that $A_{b_0}^{m'} \cap E = \emptyset$ and an $m'' \in \omega$ such that $A_{b_\infty}^{m''} \cap E = \emptyset$. Let $m_1 = \max\{m', m''\}$.

Similarly for each $i \in \omega$, $i > 1$, choose $m_i \in \omega$ so that $\forall m \geq m_i$,

$$\text{(a)} \quad (\forall b \in E_i \cap H_\infty)(\forall t \in E)(A_t^m <_m A_b^m)$$

$$\text{(b)} \quad \text{for } n = 0, \ldots, i-1,\ (\forall b \in E_i \cap H_n)(\forall t \in E)(t < e_n \to A_t^m <_m A_b^m)$$

$$\text{(c)} \quad \text{for } n = 0, \ldots, i-1,\ (\exists b \in E_i \cap H_n)(A_b^m <_m A_{e_i}^m)$$

We assume that $m_{i+1} > m_i + 1$ and therefore $\langle m_i \rangle_{i \in \omega}$ is an increasing sequence.

For each $i \in \omega$, let $\mathfrak{K}^{m_{i+1}}$ be the set of open intervals in the order $\left(\boldsymbol{\mathcal{B}}^{m_{i+1}}, \leq_{m_{i+1}}\right)$ determined by $\boldsymbol{\mathcal{B}}^{m_{i+1}}(E)$. By Definition 8 (b), $\boldsymbol{\mathcal{B}}^{m_{i+1}}(E)$ is finite and therefore $\mathfrak{K}^{m_{i+1}}$ is finite.

Claim. It is possible to add finitely many points to E_i to obtain a set E_i' for which

(i) $(\forall \boldsymbol{\mathcal{K}} \in \mathfrak{K}^{m_{i+1}})(\boldsymbol{\mathcal{K}} \cap \boldsymbol{\mathcal{B}}^{m_{i+1}}(E_i') \neq \emptyset)$ and

(ii) (a) and (b) in the definition of m_i hold with E_i replaced by E_i'.

Proof. Assume $\boldsymbol{\mathcal{K}} \in \mathfrak{K}^{m_{i+1}}$. Then there is an $n \in \omega \cup \{\infty\}$ such that $\bigcup \boldsymbol{\mathcal{K}} \subseteq H_n$. If $n \in \omega$ then $\boldsymbol{\mathcal{K}}$ is the open interval $(A_t^{m_{i+1}}, A_{e_n}^{m_{i+1}})$ in the ordering $(\boldsymbol{\mathcal{B}}^{m_{i+1}}, \leq_{m_{i+1}})$ where $t \in E$. If $n \leq i-1$, then this interval contains an element $A_b^{m_{i+1}}$ where $b \in E_i$ by conditions (b) and (c) in the definition of m_i. Similarly if $n = \infty$ then $\boldsymbol{\mathcal{K}} \cap \boldsymbol{\mathcal{B}}^{m_{i+1}}(E_i) \neq \emptyset$. If $n > i-1$, then adding an element b to E_i such that $A_b^{m_{i+1}}$ is in the interval $(A_t^{m_{i+1}}, A_{e_n}^{m_{i+1}})$ does not affect condition (b) in the definition of m_i. This proves the claim.

We will therefore assume

$$(4) \qquad (\forall \boldsymbol{\mathcal{K}} \in \mathfrak{K}^{m_{i+1}})(\boldsymbol{\mathcal{K}} \cap \boldsymbol{\mathcal{B}}^{m_{i+1}}(E_i) \neq \emptyset).$$

We now define two sequences $\langle \phi_i \rangle_{i \in \omega}$ and $\langle F_i \rangle_{i \in \omega}$ by induction. For each $i \in \omega$, ϕ_i will be in G_{m_i+1} and will fix E pointwise and F_i will be $\phi_i(E_i)$. After we give the definition, we will show that $F = \bigcup_{i \in \omega} F_i$ is a support. It follows that the function $f : S \to \bigcup S$ defined by $f(s_i) = \phi_i(s_i)$ (which is a choice function for S and which has support F) is in $\mathcal{N}1$.

For $i = 0$

$$\phi_0 = id_A \text{ the identity on } A \text{ and } F_0 = \phi_0(E_0) = E_0.$$

For $i = 1$, let ϕ_1 be an element of G_{m_1+1} which fixes E pointwise and such that $\forall \boldsymbol{\mathcal{K}} \in \mathfrak{K}^{m_1}$, if X is the largest element of $\boldsymbol{\mathcal{K}} \cap \boldsymbol{\mathcal{B}}^{m_1}(F_0)$ then

$$(\forall a \in E_1) \text{ if } A_a^{m_1+1} \subseteq \bigcup \boldsymbol{\mathcal{K}} \text{ then}$$

$$(5) \qquad \phi_1(A_a^{m_1+1}) \subseteq X \text{ and}$$

$$(6) \qquad (\forall b \in (\bigcup \boldsymbol{\mathcal{K}}) \cap F_0)(b < \phi_1(a))$$

and let $F_1 = \phi_1(E_1)$. (Note that $\mathcal{K} \cap \mathcal{B}^{m_1}(E_0)$ is finite since E_0 is a support and $\mathcal{K} \cap \mathcal{B}^{m_1}(E_0)$ is non-empty by (4).)

By (5) and Lemma H,

$$\mathcal{B}^j(F_1) \subseteq \mathcal{B}^j(F_0 \cup E) \text{ for } j = 0, 1, \ldots, m_1 \tag{7}$$

and by (6)

$$\begin{aligned} &(\forall a \in H_\infty \cap F_1)(\forall b \in H_\infty \cap F_0)(b < a) \text{ and} \\ &(\forall a \in H_0 \cap F_1)(\forall b \in H_0 \cap F_0)(b \notin A^{m_1}_{e_0} \to b < a) \end{aligned} \tag{8}$$

In general, we choose $\phi_i \in G_{m_i+1}$ so that ϕ_i fixes E pointwise and for all $\mathcal{K} \in \mathfrak{K}^{m_i}$, if X is the largest element of $\mathcal{K} \cap \mathcal{B}^{m_i}(F_0 \cup \cdots \cup F_{i-1})$ then

$$(\forall a \in E_i) \text{ if} A^{m_i+1}_a \subseteq \bigcup \mathcal{K} \text{ then}$$

$$\phi_i(A^{m_i+1}_a) \subseteq X \text{ and} \tag{9}$$

$$(\forall b \in (\bigcup \mathcal{K}) \cap (F_0 \cup \cdots \cup F_{i-1}))(b < \phi_i(a)) \tag{10}$$

Note that $\mathcal{K} \cap \mathcal{B}^{m_i}(F_0 \cup \cdots \cup F_{i-1})$ is finite since $F_0 \cup \cdots \cup F_{i-1}$ is a support. Also $\mathcal{K} \cap \mathcal{B}^{m_i}(F_0 \cup \cdots \cup F_{i-1})$ is non-empty by (4). Let $F_i = \phi_i(E_i)$. By (9),

$$\text{for } j = 0, 1, \ldots, m_i,\ \mathcal{B}^j(F_i) \subseteq \mathcal{B}^j(F_0 \cup \cdots \cup F_{i-1} \cup E) \tag{11}$$

and by (10)

$$\begin{aligned} &(\forall a \in H_\infty \cap F_i)(\forall b \in H_\infty \cap (F_0 \cup \cdots \cup F_{i-1}))(b < a) \text{ and} \\ &\text{for } j = 0, 1, \ldots, i-1,\ (\forall b \in H_j \cap (F_0 \cup \cdots \cup F_{i-1}))(b \notin A^{m_i}_{e_j} \to b < a) \end{aligned} \tag{12}$$

We now argue that $F = \bigcup_{i \in \omega} F_i$ is a support. Referring to definition 8, we must show:

a. F is well ordered by $\leq$.

Suppose that F is not well ordered. Then there is an infinite decreasing sequence $\langle \gamma_j \rangle_{j \in \omega}$ of elements of F. (Decreasing in the order $(A, \leq)$) Each F_i is well ordered by $\leq$ so we may assume (by taking a subsequence of $\langle \gamma_j \rangle_{j \in \omega}$ is necessary) that $\gamma_j \in F_{k_j}$ where $\langle k_j \rangle_{j \in \omega}$ is a strictly increasing sequence of natural numbers (and hence $k_j \geq j$). Since each F_i is disjoint from E, we get $F \cap E = \emptyset$ so $F \subseteq H_\infty \cup \left(\bigcup_{n \in \omega} H_n\right)$ by (2). We now consider two cases:

Case 1. $\{\gamma_j : j \in \omega\} \cap \left(\bigcup_{n \in \omega} H_n\right) \neq \emptyset$.

Let e_{n_0} be the least element of E (in the ordering $(A, \leq)$) such that $H_{n_0} \cap \{\gamma_j : m \in \omega\}$ is not empty and assume that $\gamma_{i_0} \in H_{n_0}$. Then for all $i > i_0$, $\gamma_i < \gamma_{i_0} < e_{n_0}$ and further for all $e_n \in E$, if $e_n < e_{n_0}$ then $e_n < \gamma_i$. (If not then the least $e_n \in E$ such that $\gamma_i < e_n$ is $< e_{n_0}$. Hence $\gamma_i \in H_n$ where $e_n < e_{n_0}$ contradicting our choice of n_0.) Therefore, $\gamma_i \in H_{n_0}$ for all $i \geq i_0$. We may therefore assume that $i_0 > n_0$.

Choose k so that $A^k_{\gamma_{i_0}} <_k A^k_{e_{n_0}}$, then for all $m \geq k$, $A^m_{\gamma_{i_0}} <_m A^m_{e_{n_0}}$. In addition, for all $i \geq i_0$, $\gamma_i \leq \gamma_{i_0}$ and therefore $A^m_{\gamma_i} <_m A^m_{e_{n_0}}$. It follows that $(\forall m \geq k)\,(\forall i \geq i_0)\left(\gamma_i \notin A^m_{e_{n_0}}\right)$. Now choose $i_1 > i_0$ so that $m_{i_1} \geq k$. Then

(13) $(\forall i \geq i_1)\,(\forall m \geq m_{i_1})\,(\gamma_i \notin A^{m_{i_1}}$ …

Consider $\gamma_{i_1} \in F_{k_{i_1}}$ and $\gamma_{i_1+1} \in F_{k_{i_1+1}}$. Since the sequence $\langle \gamma_j \rangle_{j\in\omega}$ is decreasing we have $\gamma_{i_1+1} < \gamma_{i_1}$. However, $k_{i_1+1} \geq i_1 + 1 > n_0$ and therefore by (12) with $i = k_{i_1+1}$ and $j = n_0$ we get

$$\left(\forall a \in H_{n_0} \cap F_{k_{i_1+1}}\right)\left(\forall b \in H_{n_0} \cap (F_0 \cup \cdots \cup F_{k_{i_1+1}-1})\right)\left(b \notin A^{m_1+1}_{e_{n_0}} \to b < a\right).$$

If we let $a = \gamma_{i_1+1}$ and $b = \gamma_{i_1}$ we arrive at the contradiction $\gamma_i < \gamma_{i_1+1}$.

Case 2. $\{\gamma_j : j \in \omega\} \cap \left(\bigcup_{n\in\omega} H_n\right) = \emptyset$.

In this case $\{\gamma_j : j \in \omega\} \subseteq H_\infty$. The argument is similar to that of case 1. This completes the proof that F is well ordered.

b. For all n $\in \omega$, F intersects only finitely many n level blocks in a non-empty set.

We first note that b can be written $(\forall n \in \omega)(\mathfrak{B}^n(F)$ is finite). It suffices to show that for all $i \in \omega$, $\mathfrak{B}^{m_i}(F)$ is finite since m_i is a strictly increasing sequence of natural numbers. Assume $i \in \omega$. Using Lemma G, this will be proved if we show that $\mathfrak{B}^{m_i}(F) \subseteq \mathfrak{B}^{m_i}(F_0 \cup \cdots \cup F_{i-1} \cup E)$. Since $\mathfrak{B}^{m_i}(F) = \bigcup_{n\in\omega} F_n$ it will be enough to show that $\mathfrak{B}^{m_i}(F_n) \subseteq \mathfrak{B}^{m_i}(F_0 \cup \cdots \cup F_{i-1} \cup E)$ for all $n \in \omega$. This is clear if $0 \leq n \leq i-1$. So we have to show that $\mathfrak{B}^{m_i}(F_{i+k}) \subseteq \mathfrak{B}^{m_i}(F_0 \cup \cdots \cup F_{i-1} \cup E)$ for each $k \in \omega$. We do this by induction on k.

For $k = 0$, (11) gives the desired relation. Now assume that $\mathfrak{B}^{m_i}(F_{i+k})$ is a subset of $\mathfrak{B}^{m_i}(F_0 \cup \cdots \cup F_{i-1} \cup E)$ for all $k \leq n$. Applying (11) with $i = n+1$ we get

$$\mathfrak{B}^j(F_{n+1}) \subseteq \mathfrak{B}^j(F_0 \cup \cdots \cup F_n \cup E)$$

for $j = 0, 1, 2, \ldots, m_{n+1}$. In particular,

$$\mathfrak{B}^{m_i}(F_{n+1}) \subseteq \mathfrak{B}^{m_i}(F_0 \cup \cdots \cup F_n \cup E) = \mathfrak{B}^{m_i}(F_0 \cup \cdots \cup F_{i-1} \cup E)$$

Where the last equality holds by the induction hypotheses and Lemma G. This completes the proof of b.

c. F is countable.

F is countable since F is the union of countably many countable sets.

This completes the proof of $C(\aleph_0, \infty)$ in $\mathcal{N}1$.

Section 6. The proof of BPI in $\mathcal{N}1$ and $\mathcal{N}2$.

The argument is identical in $\mathcal{N}1$ and in $\mathcal{N}2$. We use $\mathcal{N}$ to denote either model in this section.

Theorem. The Boolean Prime Ideal Theorem is true in $\mathcal{N}$.

Proof. We shall use Blass' theorem [1, Theorem 2] which characterizes Fraenkel-Mostowski models in which BPI is true. According to this theorem BPI holds in a Fraenkel-Mostowski model if and only if the model is determined by a filter of subgroups Γ with the Ramsey property described below. (We should like to thank the referee for sugesting this method of proof.)

Definition. 1. A subgroup H of a group K is a Ramsey subgroup if for every finite $F \subseteq K/H$ (the set of left cosets of H in K) there is a finite $Y \subseteq K/H$ such that whenever Y is partitioned into two pieces there is a $g \in K$ such that gF is included in one of the pieces.

2. A filter Γ of subgroups of G has the Ramsey property if it has a basis consisting of Ramsey subgroups of G.

Let E be a support. By Blass' theorem it suffices to show that $\mathrm{fix}_G(E)$ is a Ramsey subgroup of G. Let F be a finite subset of $G/\mathrm{fix}_G(E)$. We will find a Y satisfying definition 1 above. Say $F = \{\phi_1 \mathrm{fix}_G(E), \ldots, \phi_k \mathrm{fix}_G(E)\}$. Let n be the smallest natural number for which $\phi_1, \ldots, \phi_k$ are all in G_n. By definitions 5 and 6 from section 1, there are automorphisms $f_1, \ldots, f_k$ of $(\mathfrak{B}^n, \leq_n)$ such that $\phi_i = \phi_{f_i}$ for $1 \leq i \leq k$. Let $\mathfrak{E} = \{ A_a^n \in \mathfrak{B}^n : A_a^n \cap E \neq \emptyset \}$. $\mathfrak{E}$ is finite by definition 8 a. It follows from Rado's Corollary [2, p 61] that if K is the automorphism group of the rationals $\mathbb{Q}$ with their usual ordering and D is any finite subset of $\mathbb{Q}$ then $\mathrm{fix}_K(D)$ is a Ramsey subgroup of K. Therefore, by lemma B, if K is $\mathrm{Aut}(\mathfrak{B}^n, \leq_n)$ then $\mathrm{fix}_K(\mathfrak{E})$ is a Ramsey subgroup of K. (We let K denote $\mathrm{Aut}(\mathfrak{B}^n, \leq_n)$ in what follows.) We denote by $\mathfrak{F}$ the finite subset $\{f_1 \mathrm{fix}_K(\mathfrak{E}), \ldots, f_k \mathrm{fix}_K(\mathfrak{E})\}$ of $K/\mathrm{fix}_K(\mathfrak{E})$. By the definition of Ramsey subgroup there is a finite subset $\mathfrak{Y}$ of $K/\mathrm{fix}_K(\mathfrak{E})$ such that for any partition of $\mathfrak{Y}$ into two pieces there is an $h \in K$ with $h\mathfrak{F}$ included in one of the pieces. Assume $\mathfrak{Y} = \{g_1 \mathrm{fix}_K(\mathfrak{E}), \ldots, g_r \mathrm{fix}_K(\mathfrak{E})\}$ and let $Y = \{\phi_{g_1} \mathrm{fix}_G(E), \ldots, \phi_{g_r} \mathrm{fix}_G(E)\}$. To prove that this Y satisfies definition 1, assume that $\{C, D\}$ is a partition of Y. Then $\{\mathfrak{C}, \mathfrak{D}\}$ is a partition of $\mathfrak{Y}$ where $\mathfrak{C} = \{ g_i \mathrm{fix}_K(\mathfrak{E}) : \phi_{g_i} \mathrm{fix}_G(E) \in C \}$ and $\mathfrak{D} = \{ g_i \mathrm{fix}_K(\mathfrak{E}) : \phi_{g_i} \mathrm{fix}_G(E) \in D \}$. Hence there is an $h \in K$ such that $h\mathfrak{F} \subseteq \mathfrak{C}$ or $h\mathfrak{F} \subseteq \mathfrak{D}$. A straightforward argument shows that $\phi_h F \subseteq C$ in the first case and $\phi_h F \subseteq D$ in the second.

Section 7. The proof that $\mathbf{C}(\mathbf{2}^{\aleph_0}, \infty)$ is false in $\mathcal{N}1$ and $\mathcal{N}2$.

Let $\mathcal{N}$ be either $\mathcal{N}1$ or $\mathcal{N}2$. Let $\prec$ be the ordering on ω^ω defined by $f \prec g$ if and only if $(\exists k \in \omega)(\forall j > k)(f(j) < g(j))$. Let $F \subseteq \omega^\omega$ be any subset satisfying $(\forall f \in \omega^\omega)(\exists g \in F)(f \prec g)$ (for example, $F = \omega^\omega$ would work). For each $g \in F$, let E_g be a support of g chosen so that $(\forall n \in \omega)\,(|\mathfrak{B}^n(E_g)| = g(n))$. (It is fairly clear that this can be done in $\mathcal{N}2$. In $\mathcal{N}1$, we have to work a little harder to insure that E_g is well ordered in the ordering $(A, \leq)$.) Define $x_g = \{\phi(E_g) : \phi \in G\}$ for each $g \in F$ then the set $X = \{x_g : g \in F\}$ is well ordered in $\mathcal{N}$ and has cardinality less than or equal to $|\omega^\omega| = 2^{\aleph_0}$.

We will show, by contradiction, that X has no choice function in $\mathcal{N}$. Assume H is such a choice function and that H has support E. Then for any $g \in F$, E is a support of $H(x_g)$. Define the function $f : \omega \to \omega$ by $f(n) = |\mathfrak{B}^n(E)|$ and choose a $g \in F$ so that $f \prec g$. Since H is a choice function of X, $H(x_g) = \phi(E_g)$ for some $\phi \in G$. Assume that $\phi \in G_{n_0}$.

Since $f \prec g$, there is a natural number $k_0 > n_0$ such that $g(k_0) > f(k_0)$. By Lemma L

$$|\mathfrak{B}^{k_0}(\phi(E_g))| = |\mathfrak{B}^{k_0}(E_g)| = g(k_0) > f(k_0) = |\mathfrak{B}^{k_0}(E)|.$$

Hence, by Lemma M, there is a $\psi \in G_{k_0}$ that fixes E pointwise and moves $\phi(E_g)$. This is a contradiction.

Section 8. The proof that the $\mathbf{2m} = \mathbf{m}$ principle is false in $\mathcal{N}1$ and $\mathcal{N}2$.

Let $\mathcal{N}$ be either $\mathcal{N}1$ or $\mathcal{N}2$. Suppose there is a $1-1$ function f in $\mathcal{N}$ mapping $2 \times A$ into A. Let E be a support of f and let $a \in A - E$. Then, since f is $1-1$, $f(i, a) \neq a$ for $i = 0$ or $i = 1$. Suppose that $f(i, a) = b \neq a$. Let $\sigma \in \mathrm{fix}_G(E)$ (so $\sigma(f) = f$) such that $\sigma(b) = b$ and $\sigma(a) = c$, where $c \neq a$ and $c \neq b$. Then $f(i, a) = b$ and $f(i, c) = b$, contradicting the fact that f is $1-1$.

References

1. A. Blass, *Prime ideals yield almost maximal ideals*, Fund. Math. **127** (1986), 57-66.
2. N. Brunner, *Amorphe Potenzen kompakter Räume*, Arch. Math. Logik Grundlagenforsch **24** (1984), 119-135.
3. J. D. Halpern, *The independence of the axiom of choice from the Boolean prime ideal theorem*, Fund. Math. **55** (1964), 57-66.
4. T. Jech, *The Axiom of Choice*, North-Holland, Amsterdam, 1973.
5. J. L. Kelley, *General Topology*, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1955.
6. A. Mostowski, *Axiom of choice for finite sets*, Fund. Math. **33** (1945), 137-168.
7. D. Pincus, *Adding dependent choice*, Annals of Math. Logic **11** (1977), 105-145.
8. D. Pincus, *Adding dependent choice to the prime ideal theorem*, Logic Colloquium 76, R.O. Gandy and J.M.E. Hyland eds., Studies in Logic and the Foundations of Mathematics **87** (1977), 547-565.
9. N. Tsukada, *Cardinals and the Boolean prime ideal theorem*, Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **13** (1977), 276-283.