DISJOINT UNIONS OF TOPOLOGICAL SPACES AND CHOICE

Paul Howard, Kyriakos Keremedis, Herman Rubin, and Jean E. Rubin

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Abstract. We find properties of topological spaces which are not shared by disjoint unions in the absence of some form of the Axiom of Choice.

Introduction and Terminology This is a continuation of the study of the roll the Axiom of Choice plays in general topology. See also [vd], [gt], [wgt], and [hkrr]. Our primary concern will be the use of the axiom of choice in proving properties of disjoint unions of topological spaces (See Definition 1, part 11.) For example, in set theory with choice the disjoint union of metrizable topological spaces is a metrizable topological space. The usual proof of this fact begins with the choice of metrics for the component spaces. We will show that the use of some form of choice cannot be avoided in this proof and in fact without choice the disjoint union of metrizable spaces may not even be metacompact.

In section 1 we show that many assertions about disjoint unions of topological spaces are equivalent to the axiom of multiple choice. Models of set theory and corresponding independence results are described in section 2. In section 3, we study the roll the Axiom of Choice plays in the properties of disjoint unions of collectionwise Hausdorff and collectionwise normal spaces.

We begin with the definitions of the symbols and terms we will be using.

Definition 1.

1. A family $\mathcal{K}$ of subsets of a topological space (X, T) is *l.f.* (*locally finite)* iff each point of X has a neighborhood meeting a finite number of elements of $\mathcal{K}$.
2. X is *paracompact* iff X is T_2 and every open cover $\mathcal{U}$ of X has a *l.f.o.r.* (*locally finite open refinement*) $\mathcal{V}$. That is, $\mathcal{V}$ is a locally finite open cover of X and every member of $\mathcal{V}$ is included in a member of $\mathcal{U}$.
3. A family $\mathcal{K}$ of subsets of X is *p.f.* (*point finite)* iff each element of X belongs to only finitely many members of $\mathcal{K}$.
4. X is *metacompact* iff each open cover $\mathcal{U}$ of X has an *o.p.f.r.* (*open point finite refinement*).
5. An open cover $\mathcal{U} = \{U_i : i \in k\}$ of X is *shrinkable* iff there exists an open cover $\mathcal{V} = \{V_i : i \in k\}$ of non-empty sets such that $\overline{V}_i \subseteq U_i$ for all $i \in k$. $\mathcal{V}$ is also called a *shrinking* of $\mathcal{U}$.
6. X is a *PFCS space* iff every p.f. open cover of X is shrinkable.

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7. $D \subseteq X$ is *discrete* if every $x \in X$ has a neighborhood U such that $|U \cap D| = 1$. A set $C \subseteq \mathcal{P}(X)$ is *discrete* if $\forall x \in X$, there is a neighborhood U of x such that $U \cap A \neq \emptyset$ for at most one element $A \in C$. (Thus, $D \subseteq X$ is discrete iff $\{\{d\} : d \in D\}$ is discrete.)
8. X is cwH (*collectionwise Hausdorff*) iff if X is T_2 and for every closed and discrete subset D of X there exists a family $U = \{U_d : d \in D\}$ of disjoint open sets such that $d \in U_d$ for all $d \in D$.
9. X is cwH(B) (*collectionwise Hausdorff with respect to the base* B) in case $U \subseteq B$, where U is defined as in 8.
10. X is cwN (*collectionwise normal*) if X is T_2 and for every discrete C of pairwise disjoint closed subsets of X, there exists a family $U = \{U_A : A \in C\}$ of disjoint open sets such that $A \subseteq U_A$ for all $A \in C$. (cwN(B) is defined similarly to cwH(B) in 9.)
11. The *disjoint union* of the pairwise disjoint family $\{(X_i, T_i) : i \in k\}$ of topological spaces is the space $X = \bigcup\{X_i : i \in k\}$ such that O is open in X iff $O \cap X_i$ is open in X_i for all $i \in k$.

It is clear from Definition 1 that

$$(1) \qquad T_2 + \text{compact} \rightarrow \text{paracompact} \rightarrow \text{metacompact}.$$

None of the implications in (1) are reversible. It is provable in ZF^0, (ZF minus Foundation) that paracompact spaces are normal. (For example, the proof of Theorem 20.10 p. 147 given in [w] goes through in ZF^0 with some minor changes.) However, this conclusion does not hold for metacompact spaces. Dieudonné's Plank (Example 89, in [ss] p. 108) is an example of a metacompact, non-normal space. Any infinite set X endowed with the discrete topology is an example of a non-compact, paracompact space.

Below we give our notation for the principles we use and their precise statements:

Definition 2.

1. AC: The *Axiom of Choice*. For every family $\mathcal{A} = \{A_i : i \in k\}$ of non-empty pairwise disjoint sets there exists a set C which consists of one and only one element from each element of $\mathcal{A}$.
2. MC: The *Multiple Choice Axiom*: For every family $\mathcal{A} = \{A_i : i \in k\}$ of non-empty pairwise disjoint sets there exists a family $\mathcal{F} = \{F_i : i \in k\}$ of finite non-empty sets such that for every $i \in k$, $F_i \subseteq A_i$.
3. CMC: The *Countable Multiple Choice Axiom*. MC restricted to a countable family of sets.
4. ω-MC: For every family $\mathcal{A} = \{A_i : i \in k\}$ of non-empty pairwise disjoint sets there exists a family $\mathcal{F} = \{F_i : i \in k\}$ of countable non-empty sets such that for every $i \in k$, $F_i \subseteq A_i$.
5. CMC_ω: (Form 350 in [rh]) CMC restricted to countable sets.
6. MP: Every metric space is paracompact.
7. MM: Every metric space is metacompact.
8. DUM: The disjoint union of metrizable spaces is metrizable.
9. DUP: The disjoint union of paracompact spaces is paracompact.
10. DUMET: The disjoint union of metacompact spaces is metacompact.
11. DUPFCS: The disjoint union of PFCS spaces is PFCS.

12. DUPN: The disjoint union of paracompact spaces is normal.
13. DUMN: The disjoint union of metrizable spaces is normal.
14. vDCP(ω): (van Douwen's Choice Principle.) If $\mathcal{A} = \{A_i : i \in \omega\}$ is a family of non-empty disjoint sets and f a function such that for each $i \in \omega, f(i)$ is an ordering of A_i of type $\mathbb{Z}$ (= the integers), then $\mathcal{A}$ has a choice function.
15. DUN: The disjoint union of normal spaces is normal.
16. DUcwH: The disjoint union of cwH spaces is cwH.
17. DUcwN: The disjoint union of cwN spaces is cwN.
18. DUcwNN: The disjoint union of cwN spaces is normal.

All the proofs below are in ZF^0 and wlog abbreviates "without loss of generality".

1. Disjoint Unions and Multiple Choice

In this section we show that many "disjoint union" theorems are equivalent to MC. As a consequence of the following well known lemma, these theorems are equivalent to AC in ZF.

Lemma 1. [j]. *In ZF, AC is equivalent to MC.*

It is also known, [j], that in ZF^0, MC does not imply AC.

Theorem 1. *Each of the following are equivalent to MC :*
(i) DUP.
(ii) DUMET.
(iii) DUPFCS.
(iv) If X is the disjoint union of a family $\{X_i : i \in k\}$ of compact T_2 spaces, then every open cover $\mathcal{U}$ of X has an open refinement $\mathcal{V}$ such that for every $i \in k$ only a finite number of elements of $\mathcal{V}$ meet X_i non-trivially.
(v) Every open covering of a topological space X which is the disjoint union of spaces X_i, $i \in k$, each of which is the one point compactification of a discrete space, can be expressed as a well ordered union of sets $U_\alpha \subseteq \mathcal{P}(X)$ where each U_α is locally finite (point finite).
(vi) Every open covering of a topological space can be expressed as the well ordered union of locally finite (point finite) sets.
If we restrict (i) through (v) to countable families of topological spaces, then the resulting statements are equivalent to CMC.

Proof. MC$\to$(i). Fix a family $\mathcal{X} = \{X_i : i \in k\}$ of pairwise disjoint paracompact spaces and let X be their disjoint union. Fix $\mathcal{U}$ an open cover of X. Wlog we assume that each member of $\mathcal{U}$ meets just one member of $\mathcal{X}$. Put $\mathcal{U}_i = \{u \in \mathcal{U} : u \cap X_i \neq \emptyset\}$ and $A_i = \{V : V$ is a l.f.o.r. of $\mathcal{U}_i\}$. As X_i is paracompact, $A_i \neq \emptyset$. Use MC to obtain a family $\mathcal{F} = \{F_i : i \in k\}$ of finite sets satisfying $F_i \subseteq A_i$, $i \in k$. Then, $\mathcal{V} = \cup\{\cup F_i : i \in k\}$ is a l.f.o.r. of $\mathcal{U}$.
(i)$\to$MC. Fix $A = \{A_i : i \in k\}$ a family of pairwise disjoint non-empty sets. Wlog we assume that each A_i is infinite. Let $\{y_i : i \in k\}$ be distinct sets so that for each $i \in k$, $y_i \notin \bigcup A$. Put the discrete topology on A_i and let A_i^* denote the one point compactification of A_i by adjoining the point y_i to A_i. Let X be the disjoint union of the family $\{A_i^* : i \in k\}$. As A_i^* is paracompact, (A_i^* is a compact T_2 space) our hypothesis implies that X is of the same kind. Let $\mathcal{V}$ be a l.f.o.r of the open cover $\mathcal{U} = \{U : U$ is a neighborhood of y_i in A_i^*, $U \neq A_i^*$ for some $i \in k\}$. Since $\mathcal{V}$ is a l.f.

open cover, it follows that $F_i = \bigcup\{A_i \backslash v : v \in \mathcal{V} \wedge y_i \in v\}$ is finite and non-empty. Thus, $F = \{F_i : i \in k\}$ satisfies MC for A.

MC$\leftrightarrow$(ii). This can be proved exactly as in MC$\leftrightarrow$(i). MC$\rightarrow$(iii) and MC$\rightarrow$(iv) can be proved as in MC$\rightarrow$(i).

(iii)$\rightarrow$MC. Fix $A = \{A_i : i \in k\}$ a family of infinite pairwise disjoint sets and let A_i^* and X be as in the proof of (i) $\rightarrow$ MC. Clearly, A_i^* is a PFCS space. Thus, by (iii), X is also a PFCS space. Let $\mathcal{V}$ be a shrinking of the p.f. open cover $\mathcal{U} = \{u : u = A_i \text{ or } u = A_i^*, i \in k\}$ of X. Clearly $F_i = v$, $v \in \mathcal{V}$ and $\overline{v} \subset A_i$ is a finite non-empty subset of A_i. Hence, $F = \{F_i : i \in k\}$ satisfies MC for A.

(iv)$\rightarrow$MC. Let A and X be as in (iii) $\rightarrow$MC. Let $\mathcal{V}$ be a refinement of the open cover $\mathcal{U} = \{A_i^* \backslash \{a\} : a \in A_i, i \in k\}$ of X which is guaranteed by (iv). Clearly, $F_i = \cup\{A_i^* \backslash v : v \in \mathcal{V}, y_i \in v\}$ is a non-empty finite set included in A_i and $F = \{F_i : i \in k\}$ satisfies MC for A.

MC$\rightarrow$(vi). Since MC is equivalent to the statement "Every set is the union of a well ordered family of finite sets" ([le]), it is clear that MC implies (vi). It is also clear that (vi) implies (v).

(v)$\rightarrow$MC. Let $\mathcal{A} = \{A_i : i \in k\}$ a family of disjoint non-empty infinite sets. Let X be as in (i)$\rightarrow$MC and let G be the set of proper neighborhoods of y_i, for some $i \in k$. (If $g \in G$, then there is an $i \in k$ such that g is a neighborhood of y_i and $g \subsetneq A_i^*$.) G is an open covering of X, so by (v), $G = \bigcup\{U_\alpha : \alpha \in \gamma\}$, where each U_α is locally (point) finite. For $i \in k$, let α_i be the smallest α such that $y_i \in \bigcup U_\alpha$. (Notice that $y_i \in g$, for $g \in G$ iff every neighborhood of y_i has a non-empty intersection with g, so it does not matter whether the U_α's are locally finite or point finite.) It follows that there are only a finite number of sets, $g_1, g_2, \cdots, g_n$, in U_{α_i} such that $y_i \in g_j$, for $j = 1, 2 \cdots, n$. Let $h_i = \bigcap_{j=1}^n g_j$, then h_i is a neighborhood of y_i and $h_i \subsetneq A_i^*$. Consequently, $A_i \setminus h_i$ is a finite non-empty subset of A_i so that it follows that $\{A_i \setminus h_i : i \in k\}$ is a multiple choice set for $\mathcal{A}$. □

Remark 1. We remark here that the space X in (i)$\rightarrow$MC is normal (also completely normal (every subspace of it is normal)), but if MC is false it is neither paracompact nor metacompact. Also, since A_i^* in the proof of (i)$\rightarrow$MC is a compact T_2 space, it follows that MC is equivalent to the statement:

(1) The disjoint union of compact T_2 spaces is paracompact.

If in (1) we replace *disjoint union* with *product* and *paracompact* with *compact*, then the resulting statement, Tychonoff's Compactness Theorem for compact Hausdorff spaces,

(2) The product of compact T_2 spaces is compact.

is equivalent to the Boolean Prime Ideal Theorem, ([ln]) and (2) clearly implies:

(3) The product of compact T_2 spaces is paracompact.

We do not know if (3) implies (2).

Similarly, the statements:

(4) The disjoint union of compact T_2 spaces is metacompact.

and

(5) The disjoint union of compact T_2 spaces is a PFCS space.

are equivalent to MC and (3) implies

(6) The product of compact T_2 spaces is metacompact.

We do not know the relationship between (3), (6) and

(7) The product of compact T_2 spaces is PFCS.

(However, the product of paracompact spaces may not be normal, and therefore, may not be paracompact. See [so].)

Remark 2. Working as in Theorem 2.2 from [vd] one can easily establish that DUPN implies vDCP(ω). Hence DUPN cannot be proved in ZF without AC. Since paracompact spaces are normal (without appealing to AC) it follows that DUP implies DUPN and consequently, MC implies DUPN. In the next theorem using ideas from [hkrr] we show that the converse is also true. In fact we could deduce it as corollary to Theorem 1 of [hkrr], but we won't do it. We prefer to give a new proof and exploit some new ideas.

Theorem 2. *DUPN can be added to the list of Theorem 1.*

Proof. In view of Remark 2, it suffices to show that DUPN implies MC. Let $\mathcal{A} = \{A_i : i \in k\}$ be a family of infinite pairwise disjoint sets. Let

$$A = \{a_i : i \in k\} \text{ and } B = \{b_i : i \in k\}$$

be any two disjoint sets, each of which is disjoint from $\bigcup \mathcal{A}$. Let T_i be the topology on $X_i = \{a_i, b_i\} \cup [A_i]^{<\omega}$, (where $[A_i]^{<\omega}$ is the set of all finite subsets of A_i) which is generated by $\mathcal{B}_i = \{C_y(a_i) : y \in [A_i]^{<\omega}\} \cup \{D_w(b_i) : w \in [A_i]^{<\omega}\} \cup \{\{x\} : x \in [A_i]^{<\omega}\}$, where, $C_y(a_i) = \{a_i\} \cup \{x \in [A_i]^{<\omega} : x \supseteq y\}$ and $D_w(b_i) = \{b_i\} \cup \{x \in [A_i]^{<\omega} : x \cap w = \emptyset\}$.

Claim 1. *X_i is paracompact.*

Proof of Claim 1. First we show that X_i is a T_2 space. Fix $x, y \in X_i$. We consider the following cases.
(i) $x, y \in \{a_i, b_i\}$. Assume that $x = a_i$ and $y = b_i$. Then for any $w \in [A_i]^{<\omega}$, $C_w(a_i)$ and $D_w(b_i)$ are disjoint neighborhoods of x and y respectively.
(ii) Assume $x = a_i$ and $y \in [A_i]^{<\omega}$. Let $w \in [A_i]^{<\omega}$ be disjoint from y. Then $\{y\}$ and $C_w(a_i)$ are the required disjoint neighborhoods. If $x = b_i$ then $\{y\}$ and $D_y(b_i)$ are the required disjoint neighborhoods.
(iii) $x, y \in [A_i]^{<\omega}$ then $\{x\}, \{y\}$ are disjoint neighborhoods of x and y respectively.

Thus, X_i is a T_2 space as required.

To see that (X_i, T_i) is paracompact, we fix $\mathcal{U}$ an open cover of X_i. If $E(a_i), O(b_i) \in \mathcal{U}$ are neighborhoods of a_i and b_i respectively, then it follows easily that

$$\mathcal{V} = \{E(a_i), O(b_i)\} \cup (\{\{x\} : x \in [A_i]^{<\omega}\}$$

is a locally finite open refinement of $\mathcal{U}$.

Claim 2. *If $C_y(a_i)$ and $D_w(b_i)$ are disjoint neighborhoods of a_i and b_i, then $w \cap y \neq \emptyset$*

Proof of Claim 2. If $y \cap w = \emptyset$, then y is in $C_y(a_i) \cap D_w(b_i)$ which is a contradiction.

Let X be the disjoint union of the family $\{X_i : i \in k\}$. Then X, in view of the hypothesis, is a normal space and A and B are closed and disjoint sets in X. Hence, there exists disjoint open sets $O_A \supseteq A$ and $Q_B \supseteq B$. For every $i \in k$ put

$$E_{A_i}(a_i) = O_A \cap X_i,\ O_{B_i}(b_i) = Q_B \cap X_i,$$
$$\mathcal{Z}_i = \{y \in [A_i]^{<\omega} : C_y(a_i) \subseteq E_{A_i}(a_i)\},$$
$$\mathcal{W}_i = \{w \in [A_i]^{<\omega} : D_w(b_i) \subseteq O_{B_i}(b_i)\},$$
$$n_i = min\{|y| : y \in \mathcal{Z}_i\},\ \mathcal{Z}_{in_i} = \{y \in \mathcal{Z}_i : |y| = n_i\}.$$

Claim 3. *Either $\mathcal{Z}_{in_i}$ is finite or,*
(*) *there exists a finite $Q \subseteq \bigcup \mathcal{W}_i$ satisfying $\forall w \in \mathcal{W}_i, Q \cap w \neq \emptyset$.*

Proof of Claim 3. Assume, to the contrary, that $\mathcal{Z}_{in_i}$ is infinite and (*) fails. We shall construct inductively a pairwise disjoint set of finite non-empty sets $\{w_0, w_1, ..., w_{n_i+1}\}, w_j \subseteq \bigcup \mathcal{W}_i$, such that the set $P = \{y \in \mathcal{Z}_{in_i} : w_j \subseteq y$ for all $j \leq n_i + 1\}$ is infinite. This contradiction (any $y \in P$ will satisfy $n_i = |y| \geq |\bigcup\{w_v : v \leq n_i + 1\}| > n_i$) will establish the claim.

Fix $w^0 \in \mathcal{W}_i$. Since $D_{w^0}(b_i)$ is disjoint from every $C_y(a_i)$, $y \in \mathcal{Z}_{in_i}$, it follows from Claim 2, that each $y \in \mathcal{Z}_{in_i}$ meets non-trivially w^0. Hence, $\mathcal{Z}_{in_i} = \bigcup\{\{y \in \mathcal{Z}_{in_i} : w^0 \cap y = w\} : w \subseteq w^0 \wedge w \neq \emptyset\}$ and, consequently, there exists a non-empty $w \subseteq w^0$ such that $H_1 = \{y \in \mathcal{Z}_{in_i} : w \subseteq y\}$ is infinite. Put $w_0 = w$. Assume that pairwise disjoint and non-empty sets $w_0, w_1, ..., w_n$ have been chosen so that the set $\{y \in \mathcal{Z}_{in_i} : w_0, w_1, ..., w_n \subseteq y\}$ is infinite. By the negation of (*), there exists $w^n \in \mathcal{W}_i$ such that $(w_0 \cup w_1 \cup ... \cup w_n) \cap w^n = \emptyset$. By Claim 2 again, there exists a non-empty $w \subseteq w^n$ such that $H_n = \{y \in \mathcal{Z}_{in_i} : w_0, w_1, ..., w_n, w \subseteq y\}$ is infinite. Put $w_{n+1} = w$. This terminates the induction and the proof of the claim.

Claim 4. *Assume $\mathcal{Z}_{in_i}$ is infinite and let l_0 be the least integer for which there is a $Q \in [\bigcup \mathcal{W}_i]^{l_0}$ satisfying (*). Then $|\mathcal{Q}_{il_0}| < \omega$, where $\mathcal{Q}_{il_0} = \{Q \in [\cup \mathcal{W}_i]^{l_0} : Q$ satisfies (*)$\}$.*

Proof of Claim 4. Assume, to the contrary, that $\mathcal{Q}_{il_0}$ is infinite. We construct inductively a finite sequence of finite, pairwise disjoint sets $w_0, w_1, \ldots, w_r$, for some $r \in \omega$, with the property that $W = \bigcup_{i=0}^{r} w_i$ has cardinality l_0 and infinitely many elements of $\mathcal{Q}_{il_0}$ include W. This contradiction will establish the claim.

Fix $w^0 \in \mathcal{W}_i$. As w^0 and each member of $\mathcal{Q}_{il_0}$ meets w^0 non-trivially, it follows that there exists a non-empty $w_0 \subseteq w^0$ such that $K_0 = \{Q \in \mathcal{Q}_{il_0} : w_0 \subseteq Q\}$ is infinite. If $|w_0| = l_0$, then $r = 0$ and the induction terminates.

Assume that pairwise disjoint finite non-empty sets $w_0, w_1, ..., w_n$ have been chosen so that $|w_0 \cup w_1 \cup ... \cup w_n| < l_0$ and $K_n = \{Q \in K_{n-1} : w_0, w_1, ..., w_n \subseteq Q\}$ is infinite. Fix $w^{n+1} \in \mathcal{W}_i$ such that $(w_0 \cup w_1 \cup ... \cup w_n) \cap w^{n+1} = \emptyset$. (Such a $w^{n+1} \in \mathcal{W}_i$ exists, for otherwise, $w_0 \cup w_1 \cup ... \cup w_n$ will satisfy (*) with $|w_0 \cup w_1 \cup ... \cup w_n| < l_0$.) By the hypothesis again, there exists $w_{n+1} \subseteq w^{n+1}$ such that

$$K_{n+1} = \{Q \in K_n : w_0, w_1, ..., w_n, w_{n+1} \subseteq Q\}$$

is infinite, terminating the induction and the proof of the claim.

By Claims 3 and 4, it follows that $\mathcal{F} = \{F_i : i \in k\}$ where,

$$F_i = \begin{cases} \bigcup \mathcal{Z}_{in_i} & \text{if } \mathcal{Z}_{in_i} \text{ is finite,} \\ \bigcup \mathcal{Q}_{il_0} & \text{if } \mathcal{Z}_{in_i} \text{ is infinite.} \end{cases}$$

satisfies MC for $\mathcal{A}$, finishing the proof of the theorem. □

Since the space X_i of Theorem 2 is also normal, cwH, and cwN, we get as a corollary to Theorem 2.

Corollary. [hkrr] *The following can be added to the list of Theorem 1:*
(i) DUN.
(ii) DUcwH.
(iii) DUcwN.
(iv) DUcwNN.

It is easy to see that closed subspaces of paracompact (metacompact) spaces are also paracompact (metacompact) without appealing to AC. The next result shows that CMC is needed for F_σ sets (countable unions of closed subsets).

Theorem 3. *The following are equivalent:*
(i) CMC.
(ii) F_σ subsets of paracompact spaces are paracompact.
(iii) F_σ subsets of metacompact spaces are metacompact.

Proof. (i)$\rightarrow$(ii). Follow the proof of Theorem 20.12 ([w], p. 148).
(ii)$\rightarrow$(i). Fix $A = \{A_i : i \in \omega\}$ a family of disjoint non-empty sets. Note that if X^* is the one point compactification of X as given in (i)$\rightarrow$MC of Theorem 1 then X is F_σ in X^* and thus, paracompact. Continue the proof as in Theorem 1.
(i)$\rightarrow$(iii). Let (X,T) be metacompact, let $G = \bigcup\{G_n : n \in \omega\}$ be an F_σ set in X, and let $\mathcal{U}$ be an open cover of G. For every $u \in \mathcal{U}$ pick $v_u \in T$ such that $u = v_u \cap G$. (Note that we do not need AC in order to pick the element v_u. We can pick all such elements and then take their union. Clearly the union is such an element.) For every $n \in \omega$ put $U_n = \{v_u : u \in \mathcal{U} \wedge v_u \cap G_n \neq \emptyset\} \cup \{X \backslash G_n\}$. Clearly U_n is an open cover of X. As X is metacompact $A_n = \{W : W$ is an o.p.f.r. of $U_n\} \neq \emptyset$. By CMC, there is a family $F = \{F_n \subseteq A_n : n \in \omega\}$ of finite non-empty sets. For each $n \in \omega$, let $B_n = \{G \cap s \setminus \bigcup_{m<n} G_m : s \in \bigcup F_n \wedge s \cap G_n \neq \emptyset\}$. Then B_n is p.f. and each element of B_n is open in G. We claim that $\mathcal{B} = \bigcup_{n\in\omega} B_n$ is an open p.f. refinement of $\mathcal{U}$ that covers G. To show that $\mathcal{B}$ covers G, suppose $x \in G$. Take n to be the smallest natural number such that $x \in G_n$. It follows from the construction of F_n that there is an $s \in \bigcup F_n$ such that $x \in s$. Consequently, $x \in \bigcup \mathcal{B}$.

The proof that (iii) implies (i) is similar to the proof that (ii) implies (i).

DUM implies DUMN because metric spaces are normal in ZF^0. We do not know if DUMN implies DUM. However, in Theorem 4 we show that DUM + ω-MC iff MC and in Theorem 5 we show that DUMN + ω-MC iff MC.

Theorem 4. *DUM + ω-MC iff MC.*

Proof. ($\leftarrow$) MC implies ω-MC is obvious. To see that MC implies DUM, fix $\mathcal{X} = \{X_i : i \in k\}$ a disjoint family of metrizable spaces. For every $i \in k$ let A_i denote the set of all metrics on X_i producing its topology. Put $A = \{A_i : i \in k\}$ and use MC to find for each $i \in k$ a finite subset $G_i = \{m_1, m_2, \cdots, m_{n_i}\}$ of A_i. For all x, y in X_i, let $d_i(x,y)$ be defined as follows: $d_i(x,y) = \dfrac{\sum_{j=1}^{n_i} m_j(x,y)}{1 + \sum_{j=1}^{n_i} m_j(x,y)}$. Then d_i produces the topology of X_i. Clearly, the function d, where $d : \bigcup \mathcal{X} \times \bigcup \mathcal{X} \rightarrow \mathbb{R}$, $d(x,y) = 1$ if $x \in X_i$ and $y \notin X_i$ for some $i \in k$ and $d(x,y) = d_i(x,y)$ otherwise, is a metric on $\bigcup \mathcal{X}$ producing the topology of the disjoint union on $X = \bigcup \mathcal{X}$. Thus, X is metrizable as required.
($\rightarrow$) Fix $\mathcal{A} = \{A_i : i \in k\}$ a family of pairwise disjoint non-empty sets. By ω-MC we may assume that the members of A are countably infinite. Let X be as in

Theorem 1, (i)→MC. Since each A_i^* is metrizable, it follows by DUM that X is also metrizable. Let d be a metric on X producing its topology. For every $i \in k$ let $n_i = min\{n : A_i \neq D(y_i, 1/n) \cap A_i\}$, where $D(y_i, 1/n)$ is the open ball of radius $1/n$ centered at y_i. Clearly, $F_i = A_i \backslash D(y_i, 1/n_i)$ is a finite non-empty subset of A_i and $F = \{F_i : i \in k\}$ satisfies MC for $\mathcal{A}$. □

Corollary. *DUM implies* CMC_ω.

Proof. The proof is similar to the second part of the proof of Theorem 4.

Theorem 5. *DUMN +* ω*-MC iff MC.*

Proof. (←) It is shown in Theorem 4 that MC implies DUM. The result follows because DUM implies DUMN and MC implies ω-MC.
(→) Let $\mathcal{A} = \{A_i : i \in k\}$ be a family of pairwise disjoint non-empty sets. By ω-MC we may assume that the members of $\mathcal{A}$ are countably infinite. If A_i is countable so is $[A_i]^{<\omega}$. Let (X_i, T_i) be the corresponding topological space as defined in Theorem 2. The topology, T_i as defined in Theorem 2, clearly has a countable base. It is shown in Theorem 2 that X_i is paracompact, and, therefore, regular. Thus, it follows from [gt] Corollary 4.8, that X_i is metrizable. Using DUMN it follows that the disjoint union of the X_i's is normal. Using the same ideas as in Theorem 2, we can construct a multiple choice function for $\mathcal{A}$. □

Remark 3. In the Cohen-Pincus model $\mathcal{M}1(\langle\omega_1\rangle)$, see [rh] and [pi], ω-MC holds. Since MC is false in this model, it follows from Theorem 4 that DUM is also false. Thus, ω-MC does not imply DUM.

2. Examples and Models

Example 1. (A non-metrizable, non-paracompact countable disjoint union of paracompact spaces). The space L of Theorem 2.2 from [vd] is clearly a non-normal (hence, non-paracompact) countable disjoint union of paracompact spaces.

Example 2. (A metrizable non-paracompact countable disjoint union of paracompact spaces). Let $\mathcal{N}$ be the permutation model with set of atoms: $A = \bigcup\{Q_n : n \in \omega\}$, where $Q_n = \{a_{n,q} : q \in \mathbb{Q}\}$.

Suppose $<$ is the lexicographic ordering on A, i.e.

$$a_{n,q} < a_{m,p} \text{ if } n < m, \text{ or } n = m \text{ and } q < p.$$

The group of permutations G, is the group of all permutations on A which are a rational translation on Q_n, i.e. if $\phi \in G$, then $\phi | Q_n(a_{n,q}) = a_{n,q+r_n}$ for some $r_n \in \mathbb{Q}$, and supports are finite. Let d be the metric on A given by:

$$d(a_{n,q}, a_{m,p}) = 1 \text{ if } n \neq m$$

and

$$d(a_{n,q}, a_{m,p}) = |q - p|/(1 + |q - p|) \text{ if } n = m.$$

Good/Tree/Watson ([wgt]) prove that (A, d) is a linearly ordered, zero dimensional, metric space, but is not paracompact (MP is false). Each Q_n, being a second countable metric space, is paracompact, and the disjoint union topology on A

coincides with the metric topology which is normal. Hence, A can be considered as a normal non-paracompact disjoint union of paracompact spaces.

The disjoint union of the Q_n's is also not metacompact because $\mathcal{V} = \{(a,b) : (\exists n \in \omega)(a, b \in Q_n \wedge a < b)\}$ is an open cover without a point finite refinement. in the model. (Suppose $\mathcal{U}$ is a point finite refinement of $\mathcal{V}$ with support E and $Q_n \cap E = \emptyset$. Then, for some $U \in \mathcal{U}$, $U \subseteq (a,b) \subseteq Q_n$. Choose $x \in Q_n$ and an interval $(c,d) \subseteq (a,b)$ such that $x \in (c,d) \subset U$. Then it is easy to see that x is contained in a countably infinite number of translates of U.) Thus, MM is false in $\mathcal{N}$.

DUMN is false in $\mathcal{N}$. For each $n \in \omega$, let $X_n = Q_n \cup \{c_n, d_n\}$, where c_n and d_n are distinct sets in the model which are not in A, and let $<_n$ be the ordering on Q_n extended so that c_n is the smallest element and d_n is the largest element. $(X_n, <_n)$ is metrizable because it is the subspace of a metric space. (It is isomorphic to the closed interval of rational numbers $[0,1]$.) Let X be the disjoint union of the X_n's. Let $C = \{c_n : n \in \omega\}$ and $D = \{d_n : n \in \omega\}$. The sets C and D are disjoint closed sets in X. If X were normal there would be disjoint open sets, U and V, containing C and D respectively, such that their closures, $\overline{U}$ and $\overline{V}$, are disjoint. For each $n \in \omega$, let $Y_n = Q_n \setminus \overline{U}$, then $Y_n \subsetneq Q_n$. Since supports are finite, it is easy to show that there is no such function $f : n \mapsto Y_n$ in the model. Thus, DUMN is false, and since DUM implies DUMN, DUM is also false.

The Axiom of Choice for pairs, $C(\infty, 2)$, is also false in $\mathcal{N}$. First note that if a permutation leaves any element of any Q_n fixed, then it has to leave the whole set fixed because the permutations are translations. Therefore, every subset of each Q_n is in the model. Let B be the set of all unordered pairs $\{C, D\}$ where $C \subseteq Q_n$ and $D \subseteq Q_n$, $C \neq D$, for some $n \in \omega$. The set B has empty support so it is in the model. Suppose f is a choice function on B with support E. Since E is finite, there is an $n \in \omega$ such that $E \cap Q_n = \emptyset$. Let $O_n = \{a_{n,2i+1} : i \in \mathbb{Z}\}$ and let $E_n = \{a_{n,2i} : i \in \mathbb{Z}\}$. Then $\{O_n, E_n\} \in B$. Let π be a permutation in $\text{fix}(E)$ such that $\pi(a_{mr}) = a_{mr}$ if $m \neq n$, and $\pi(a_{nr}) = a_{n,r+1}$. Then π leaves f and $\{O_n, E_n\}$ fixed, but interchanges O_n and E_n, which is a contradiction. Thus, $C(\infty, 2)$, is false in the model. Since, the Boolean Prime Ideal Theorem and the Ordering Principle (Every set can be linearly ordered.) each imply $C(\infty, 2)$, they are also false.

However, in $\mathcal{N}$, every infinite set has a countably infinite subset. Let X be an infinite non-well-orderable set with support E and let $x_1 \in X$ be such that E is not a support of x_1. (Such an element exists because otherwise, X can be well ordered.) Let E_1 be a support of x_1. For $n, m \in \omega$ and $p, q \in \mathbb{Q}$, we define $\pi_q^n \in G$ as follows. For $a_{m,p} \in A$,

$$\pi_q^n(a_{m,p}) = \begin{cases} a_{m,p}, & \text{if } m \neq n \\ a_{m,p+q}, & \text{if } m = n \end{cases}$$

We claim that there is an $n \in \omega$ and $q \in \mathbb{Q}$ such that $\pi_q^n \in \text{fix}E$ and $\pi_q^n(x_1) \neq x_1$. (Otherwise, E would be a support of x_1.) We shall show that that the set $S = \{\pi_q^n(x_1) : q \in \mathbb{Q}\}$ is a countably infinite subset of X in $\mathcal{N}$. S has support E_1 so it is in $\mathcal{N}$. Also, $S \subseteq X$ because $\pi_q^n \in \text{fix}E$. It remains to show that S is infinite. Let $H = \{q \in \mathbb{Q} : \pi_q^n(x_1) = x_1\}$. It follows from the definition of n that H is a proper subgroup of $\mathbb{Q}$. We shall show that S is infinite by showing that the factor group $\mathbb{Q}/H$ is infinite. Suppose the factor group has order n and suppose $p \notin H$. Then $n(p/n) + H = H$, because n is the order of $\mathbb{Q}/H$, but $n(p/n) + H = p + H$, which is a contradiction because $p \notin H$.

Thus, we have shown that $\mathcal{N} \models$ "Every infinite set has a countably infinite subset." + ¬ MM + ¬ DUMN + $\neg C(\infty, 2)$.

Example 3. In the Mostowski linearly ordered model, the disjoint union of metrizable spaces is metrizable.

Proof. Let $(A, <)$ be the set of atoms in the the Mostowski linearly ordered model $\mathcal{M}$. (A is countable and $<$ is a dense linear ordering on A without first or last elements.) For notational convenience we add first and last elements $-\infty$ and ∞ to the ordering $(A, <)$. Let G denote the group of order automorphisms of $(A, <)$ and for each $E \subseteq A$ let $G_E = \{\phi \in G : \phi$ fixes E pointwise$\}$. Supports in this model are finite subsets of A. For each $x \in \mathcal{M}$ we denote the minimal support of x by $\sup(x)$. The crucial lemma is

Lemma 2. *If (X, T) is a metrizable topological space in $\mathcal{M}$, then there is a metric m on X such that*
(a) m is in $\mathcal{M}$,
(b) T is the topology induced by m, and
(c) $\sup(m) = \sup(X, T)$.

We shall show first that it follows from Lemma 2 that DUM is true in $\mathcal{M}$. Assume that $\{(X_i, T_i) : i \in K\}$ is a collection of disjoint metrizable topological spaces in $\mathcal{M}$ with support D. For each $i \in K$ choose a metric m_i on X_i so that m_i is in $\mathcal{M}$, m_i induces the topology T_i and $\sup(m_i) = \sup(X_i, T_i)$. (We may assume wlog that each m_i is bounded by one.) It follows from the last of these properties that for all $\phi \in G_D$, $\phi(m_i) = m_i$ if and only if $\phi(X_i, T_i) = (X_i, T_i)$. (This would not be true, for example, in the basic Fraenkel model where "ϕ fixes $\sup(x)$ pointwise" implies "ϕ fixes x", but not conversely.) Therefore, the function $(X_i, T_i) \mapsto m_i$ has support D so it is in $\mathcal{M}$. We can use this function to define a metric m on $\bigcup_{i \in K} X_i$ which induces the disjoint union topology. ($m(x, y) = m_i(x, y)$ if $x, y \in X_i$ for some $i \in K$; and $m(x, y) = 1$ otherwise.)

To prove Lemma 2, let (X, T) be a metrizable topological space in $\mathcal{M}$ and let d be a metric on X which is in $\mathcal{M}$ and which induces the topology T. If $\phi \in G$ fixes d then ϕ fixes X and T and therefore $\sup(X, T) \subseteq \sup(d)$. Assuming that $\sup(X, T) \neq \sup(d)$, it suffices to construct a metric m satisfying (a) and (b) from the statement of Lemma 2 and

(c′) $\sup(m) \subsetneq \sup(d)$.

By our assumption that $\sup(X, T) \neq \sup(d)$ there is an element $t \in \sup(d) \setminus \sup(X, T)$. Let e_0 and e_1 be the unique elements of $\sup(d) \cup \{-\infty, \infty\}$ such that $e_0 < t < e_1$ and both of the open intervals (e_0, t) and (t, e_1) are disjoint from $\sup(d)$. Let $E = \sup(d) \setminus \{t\}$. Since $E \subsetneq \sup(d)$ the proof can be completed by constructing a metric m with support $\subseteq E$ and satisfying (a) and (b).

At this point we need a well known property of linear orders.

Lemma 3. *If $(C, <_1)$ and $(D, <_2)$ are countable dense linear orders without first or last elements and C' and D' are finite subsets of C and D respectively with $|C'| = |D'|$ then there is an order isomorphism f from C onto D such that $f(C') = D'$.*

Using Lemma 3, let Δ' be an order isomorphism from (e_0, e_1) onto (e_0, t) and define $\Delta : A \to A \setminus [t, e_1)$ by

$$\Delta(a) = \begin{cases} \Delta'(a) & \text{if } a \in (e_0, e_1) \\ a & \text{otherwise} \end{cases}.$$

We first note that for any $x \in \mathcal{M}$, there is a $\psi \in G_E$ such that ψ agrees with Δ on $\sup(x)$. (This is a consequence of Lemma 3.) Further, if ψ and ϕ are in G_E and ψ and ϕ agree with Δ on $\sup(x)$ then $\psi(x) = \phi(x)$. This allows us to define a function Δ^* from the topological space X to itself by $\Delta^*(x) = \psi(x)$ where $\psi \in G_E$ and ψ agrees with Δ on $\sup(x)$. We leave to the reader the proof that Δ^* is one to one and onto the set $X^* = \{x \in X : sup(x) \cap [t, e_1) = \emptyset\}$. There are two natural ways of putting a topology on the set X^*:

$$T_1 = \{U^* : U \in T\} \text{ where } U^* = \{\Delta^*(x) : x \in U\}$$

and

$$T_2 = \{U \cap X^* : U \in T\}.$$

The topology T_1 is the topology under which the function $\Delta^* : X \to X^*$ is a homeomorphism (between the spaces (X, T) and (X, T_1)). The topology T_2 is the relative topology on X^* inherited from (X, T) and is therefore the topology induced by the metric d restricted to X^*. We will call this restricted metric d^*.

We are now able to outline our plan for finding a metric m for the topology T with support contained in E: We will show that $T_1 = T_2$. It follows that if we "pull the metric d^* back to X using $(\Delta^*)^{-1}$" we get a metric which induces the topology T on X. (This "pull back" of d^* is the metric m defined below.) The metric d^* has support $E \cup \{t\}$, however in pulling d^* back to m, t is identified with e_1 and therefore m will have support E. The most difficult part of the argument is the proof that $T_1 = T_2$. We postpone this part and begin with the definition of m.

$$m(x, y) = d(\Delta^*(x), \Delta^*(y))$$

The fact that m is a metric follows from the fact that d is a metric. To prove the triangle inequality, for example, choose x, y and $z \in X$ and a $\psi \in G_E$ such that ψ agrees with Δ on $\sup(x) \cup \sup(y) \cup \sup(z)$. Then $m(x, y) + m(y, z) = d(\psi(x), \psi(y)) + d(\psi(y), \psi(z)) \geq d(\psi(x), \psi(z)) = m(x, z)$.

We now argue that $\forall \eta \in G_E$, η fixes m. This can be accomplished by showing $(\forall x, y \in X)(m(\eta(x), \eta(y)) = m(x, y))$. Assume $\eta \in G_E$ and $x, y \in X$. Choose $\phi \in G_E$ such that ϕ agrees with Δ on $\sup(x) \cup \sup(y) \cup \sup(\eta(x)) \cup \sup(\eta(y))$. By the definition of m, $m(x, y) = d(\phi(x), \phi(y))$ and $m(\eta(x), \eta(y)) = d(\phi(\eta(x)), \phi(\eta(y)))$. The permutation $\phi\eta\phi^{-1}$ takes $\phi(x)$ to $\phi(\eta(x))$ and $\phi(y)$ to $\phi(\eta(y))$. Further, since the range of Δ is $A \setminus [t, e_1)$, the four sets

$$\begin{gathered} \sup(\phi(x)) = \phi(\sup(x)), \\ \sup(\phi(y)) = \phi(\sup(y)), \\ \sup(\phi(\eta(x))) = \phi(\sup(\eta(x))), \text{ and} \\ \sup(\phi(\eta(y))) = \phi(\sup(\eta(y))) \end{gathered}$$

are all subsets of $A \setminus [t, e_1)$. Therefore, we can find a permutation $\sigma \in G_E$ such that σ agrees with $\phi\eta\phi^{-1}$ on $\sup(\phi(x)) \cup \sup(\phi(y))$ and such that σ fixes $[t, e_1)$ pointwise. It follows that $\sigma(\phi(x)) = \phi(\eta(x))$ and $\sigma(\phi(y)) = \phi(\eta(y))$. Also, since σ fixes $E \cup \{t\}$,

$\sigma(d) = d$. Therefore $d(\phi(x),\phi(y)) = d(\sigma(\phi(x)),\sigma(\phi(y))) = d(\phi(\eta(x)),\phi(\eta(y)))$. Hence $m(x,y) = m(\eta(x),\eta(y))$.

We conclude that (a) and (c$'$) are true of m.

We shall complete the proof of Lemma 2 and also show that $T_1 = T_2$ by proving part (b) of Lemma 2, T is the topology induced by m. We first show

(i) Every $U \in T$ is in the topology induced by m.

Assume $U \in T$ and $x \in U$. Choose $\phi \in G_E$ so that ϕ agrees with Δ on $\sup(x) \cup \sup(U)$. Since ϕ fixes T, $\phi(x) \in \phi(U) \in T$. Since the metric d induces T, there is an $\epsilon > 0$ such that the open ball $B_1 = \{y \in X : d(\phi(x), y) < \epsilon\} \subseteq \phi(U)$. We claim that $B_2 = \{y \in X : m(x,y) < \epsilon\} \subseteq U$. For suppose $y \in B_2$, then (by definition of m) $d(\phi'(x),\phi'(y)) < \epsilon$, where ϕ' agrees with Δ on $\sup(x) \cup \sup(y) \cup \sup(U)$. Since $\phi'(x) = \phi(x)$, $\phi'(y) \in B_1$. Therefore, $\phi'(y) \in \phi(U) = \phi'(U)$ and hence $y \in U$.

Now we show

(ii) If U is open in the topology induced by m then $U \in T$.

It suffices to show that for every $x \in X$ and every $\epsilon > 0$,

$$(*) \qquad \exists\delta > 0 \text{ such that } \{z \in X : d(x,z) < \delta\} \subseteq \{z \in X : m(x,z) < \epsilon\}.$$

Assume that there is an $x \in X$ and an $\epsilon > 0$ for which $(*)$ is false. Fix an s in the interval (t, e_1) so that $\sup(x) \cap [s, e_1) = \emptyset$. The failure of $(*)$ gives, for each positive $n \in \omega$, an element z_n of X such that $d(x, z_n) < \frac{\epsilon}{n}$ and $m(x, z_n) > \epsilon$. (The function $n \mapsto z_n$ may not be in $\mathcal{M}$.)

Claim. We may assume that $\sup(z_n) \cap [s, e_1) = \emptyset$.

If this is not the case, choose an s' in the interval (t, s) which also satisfies $\sup(x) \cap [s', e_1) = \emptyset$. Let γ' be an order isomorphism of (s', e_1) for which $\gamma'(\sup(z_n) \cap (s', e_1)) \subseteq (s', s)$. The permutation γ which agrees with γ' on (s', e_1) and is the identity outside of (s', e_1) then has the property that $\sup(\gamma(z_n)) \cap [s, e_1) = \gamma(\sup(z_n)) \cap [s, e_1) = \emptyset$. In addition, γ fixes $E \cup \{t\} \cup \sup(x)$ pointwise. This means that γ fixes d, m, and x. Hence $d(x, \gamma(z_n)) < \frac{\epsilon}{n}$ and $m(x, \gamma(z_n)) > \epsilon$. We may therefore replace z_n by $\gamma(z_n)$. This proves the claim.

Choose $\phi \in G_E$ which agrees with Δ on (e_0, s). (This can be accomplished by using Lemma 2 to obtain an order isomorphism β from $[s, e_1)$ onto $[\Delta(s), e_1)$. Then define $\phi(a) = \beta(a)$ for $a \in [s, e_1)$ and $\phi(a) = \Delta(a)$ otherwise.) For $n \in \omega \setminus \{0\}$, define

$$Z_n = \{\psi(z_n) : \psi \text{ fixes } \sup(d) \cup \sup(x) \cup \{s\} \text{ pointwise}\}$$

Then

(A) Z_n has support $\sup(d) \cup \sup(x) \cup \{s\}$.

(B) $(\forall w \in Z_n)(d(x, w) < \frac{\epsilon}{n})$.

(Since $w = \psi(z_n)$ where ψ fixes x and d and $d(x, z_n) < \frac{\epsilon}{n}$.) Similarly,

(C) $(\forall w \in Z_n)(m(x, w) > \epsilon)$ and

(D) $(\forall w \in Z_n)(\sup(w) \cap [s, e_1) = \emptyset)$.

By (D), for all $w \in Z_n$, ϕ (which is in G_E) agrees with Δ on $\sup(w) \cup \sup(x)$. Hence, using (C) and the definition of m we conclude that

(E) $d(\phi(x), \phi(w)) > \epsilon$.

By (A), $\bigcup_{n=1}^{\infty} Z_n \in \mathcal{M}$. By (B), x is in the T closure of $\bigcup_{n=1}^{\infty} Z_n$ since d induces T. Similarly, by (E), $\phi(x)$ is not in the T closure of $\phi(\bigcup_{n=1}^{\infty} Z_n)$. This is a contradiction since $\phi \in G_E$ and therefore fixes T. This completes the proof of (b) and the theorem. $\square$

3. Disjoint Unions of Collectionwise Hausdorff, Collectionwise Normal Spaces and the Axiom of Choice

Theorem 6. *The following are equivalent:*
(i) AC.
(ii) If a topological space (X, T) is cwH, then it is cwH(B) for every basis B..
(iii) The disjoint union of cwH spaces is cwH(B) for any basis B.
(iv) If a topological space (X, T) is cwN, then it is cwH(B) for any basis B.

Proof. (i)→(ii). Let (X, T) be cwH, $G = \{g_i : i \in k\}$ a closed discrete subset of X and B a basis for X. As X is cwH, there exists a disjoint family $\{O_i : i \in k\} \subseteq T$ such that $g_i \in O_i$ for all $i \in k$. Since B is a basis it follows that $Q_i = \{b \in B : g_i \in b \subseteq O_i\} \neq \emptyset$. Put $Q = \{Q_i : i \in k\}$ and let v be a choice function on Q. Clearly $V = \{v(i) : i \in k\} \subseteq B$ is the required family of open sets.

The proofs (ii)→(iii) and (iii)→(iv) are straight forward. It therefore suffices to prove (iv)→(i).

Fix $\mathcal{A} = \{A_i : i \in \omega\}$ a family of infinite pairwise disjoint sets. For each $A \in \mathcal{A}$, let x_A be a set not in $\mathcal{A} \cup \bigcup \mathcal{A}$ and chosen so that the function $A \mapsto x_A$ is one-to-one. (To be specific, we could take x_A to be the ordered pair $(A, \mathcal{A})$.) For each $A \in \mathcal{A}$, we define a topological space $(Y_A, \mathcal{T}_A)$ as follows. $Y_A = \{x_A\} \cup (\omega \times A)$ and $\mathcal{T}_A$ is the topology with basis sets

$$\mathcal{B}_A = \{\{(n, a)\} : (n, a) \in \omega \times A\} \cup$$
$$\{\{x_A\} \cup Z : (\omega \setminus m) \times A \subseteq Z \subseteq \omega \times A \text{ for some } m \in \omega\}.$$

Let $Y = \bigcup_{A \in \mathcal{A}} Y_A$ and let $\mathcal{T}$ be the disjoint union topology on Y. We claim that Y is cwN. Indeed, let Z be a discrete collection of closed sets. It can be readily verified that,

(1) For every $A \in \mathcal{A}$ either $x_A \notin z$ for all $z \in Z$ or if $x_A \in z^*$ for some $z^* \in Z$, then there exists $n_{z^*} \in \omega$ such that for every $z \in Z$, $z \neq z^*$, $z \subseteq n_{z^*} \times A$.

For every $z \in Z$, let

(2) $U_z = \bigcup\{(\omega \times A) \cap z : A \in \mathcal{A}, x_A \notin z\} \cup (\bigcup\{z \cup (Y_A \setminus \bigcup Z) : A \in \mathcal{A}, x_A \in z\})$.

In view of (1) and (2), it follows that $\mathcal{U} = \{U_z : z \in Z\}$ is a disjoint family of open sets separating the members of Z. It is easy to see that each Y_A is T_2, so that Y is also T_2. Thus, $(Y, \mathcal{T})$ is cwN as claimed.

Now we construct another basis $\mathcal{B}$ for the topology $\mathcal{T}$. The basis $\mathcal{B}$ consists of the sets $\{(n, a)\}$ such that $n \in \omega$ and $a \in \bigcup \mathcal{A}$ and the sets $B_{A,a,n} = \{x_A\} \cup \{(n, a)\} \cup \{(m, b) : m > n \wedge b \in A\}$ for each triple (A, a, n) such that $A \in \mathcal{A}$, $a \in A$, and $n \in \omega$.

Applying cwNH(B) to this basis and the discrete set $\{x_A : A \in \mathcal{A}\}$ gives for each $A \in \mathcal{A}$ a unique set $B_{A,a,n}$. The function f defined by $f(A) = a$ is a choice function for $\mathcal{A}$. $\square$

Theorem 7. *The following are equivalent*
(i) MC.
(ii) Every space (X, T) which is cwH, is also cwH(B) for any basis B closed under

finite intersections.
(iii) The disjoint union of cwH spaces is cwH(B) for any basis B closed under finite intersections.
(iv) Every space (X, T) which is cwN, is also cwH(B) for any basis B closed under finite intersections.

Proof. The proof of (i)→(ii) is similar to the proof of (i)→(ii) in Theorem 6. (Here v would be a multiple choice function, and $V = \{\bigcap v(i) : i \in k\} \subseteq B$ is the required family of open sets.) The proofs that (ii)→(iii) and (iii)→(iv) are again straight forward.

(iv)→(i). Let $\mathcal{A} = \{A_i : i \in k\}$ be a family of infinite pairwise disjoint sets and let X be as in Theorem 1 (i)→ MC. We claim that X is cwN. Indeed, let $D = \{d_j : j \in J\}$ be a discrete collection of closed sets. Clearly, for every $i \in k$ either $y_i \notin d_j$ for all $j \in J$ or if $y_i \in d_{j*}$ for some $j^* \in J$, (j^* is unique). Then, $|A_i \cap d_j| < \omega$ for all $j \in J$, $j \neq j^*$ and

$$|\{j \in J : A_i \cap d_j \neq \emptyset\}| < \omega.$$

For every $j \in J$, put

$$O_j = \cup\{A_i \cap d_j : i \in k, y_i \notin d_j\} \cup (\cup\{A_i^* \backslash d_v : y_i \in d_j, v \neq j\}).$$

Clearly $\mathcal{O} = \{O_j : j \in J\}$ is a disjoint family separating the members of D. It is easy to see that $\mathcal{D} = \{\{y_i\} : i \in k\}$ is a discrete family of closed subsets of X.

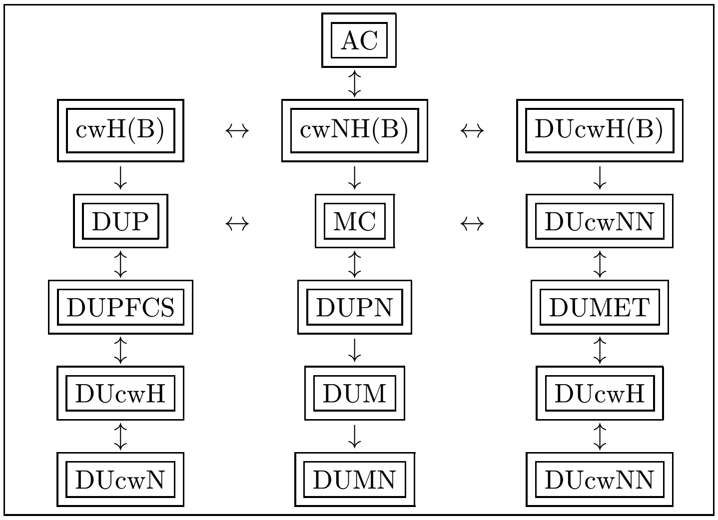
If B is defined as follows:

$$B = \{A_i^* \backslash a : a \in [A_i]^{<\omega} \backslash \{\emptyset\}, i \in k\} \cup \{\{x\} : x \in \cup\mathcal{A}\},$$

then B is a basis for X closed under finite intersections and $D = \{y_i : i \in k\}$ is a closed discrete subspace of X. It follows from cwH(B) that there is a pairwise disjoint set $V = \{V_i : i \in k\} \subseteq B$ such that for each $i \in k$, $y_i \in V_i$. Putting $F = \{F_i = A_i \backslash V_i : i \in k\}$, we see that each F_i, $i \in k$, is a finite non-empty subset of A_i, which proves MC. $\square$

Summary

We summarize our results in the following diagram. (The forms cwH(B), DUcwH(B), and cwNH(B) in the diagram are abbreviations for (ii), (iii), and (iv), respectively, in Theorem 6.)



We have shown that DUM does not imply ω-MC or MC in ZF^0. (See Example 3 above. DUM is true in $\mathcal{M}$, but ω-MC and MC are false.) However, we still have the following questions.

Questions: (i) Does DUM imply ω-MC or MC in ZF?
(ii) Does DUMN imply DUM?

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Paul Howard
Department of Mathematics, Eastern Michigan University, Ypsilanti, MI 48197 USA

email: phoward@emunix.emich.edu

Kyriakos Keremedis
Department of Mathematics, University of the Aegean, Karlovasi, Samos 83200 GREECE
email: kker@kerkis.math.aegean.gr.

Herman Rubin
Department of Statistics, Purdue University, West Lafayette, IN 47907 USA
email: hrubin@stat.purdue.edu.

Jean E. Rubin
Department of Mathematics, Purdue University, West Lafayette, IN 47907 USA
email: jer@math.purdue.edu.