VERSIONS OF NORMALITY AND SOME WEAK FORMS OF THE AXIOM OF CHOICE

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Abstract. We investigate the set theoretical strength of some properties of normality, including Urysohn's Lemma, Tietze-Urysohn Extension Theorem, normality of disjoint unions of normal spaces, and normality of F_σ subsets of normal spaces.

Introduction. The notion of a normal topological space has been of interest to topologists for many years. (See, for example, [2], [4], [7], [12], [13], [18], and [20].) In [4] it has been shown that there exists a close connection between properties of normality and the axiom of choice. In particular, in [4], van Douwen established that the proposition: "The disjoint union of a family $\mathcal{X} = \{X_n : n \in \omega\}$ of orderable topological spaces such that each X_n is ordered like the integers (hence, each X_n is normal with the induced ordered topology) is normal" is equivalent to a set theoretical statement which cannot be proved without the axiom of choice and which is identified bellow as **van Douwen's choice principle** (vDCP(ω)). Since the assertion DUN(ω) = "the disjoint union of a denumerable number of normal spaces is normal" clearly implies vDCP(ω), there remains the question of the equivalence of vDCP(ω) and DUN(ω).

In this paper we study the relationships between some properties of normal spaces and what roll various weak forms of the Axiom of Choice (AC) play in their proofs. We show that some properties of normal spaces are actually equivalent to weak forms of AC. For example, "The disjoint union of normal spaces is normal" (DUN) is equivalent to the Multiple Choice Axiom (MC), and the Countable Multiple Choice Axiom (CMC) is equivalent to "An F_σ subset of a space satisfying Urysohn's Lemma satisfies Urysohn's Lemma". See section 1 for definitions and section 2 for proofs. All our proofs are in ZF^0, Zermelo-Fraenkel (ZF) set theory without the axiom of foundation. Zermelo-Fraenkel set theory with the addition of AC is designated by ZFC.

In section 3, we give some independence results using both Cohen models of ZF and Fraenkel-Mostowski (FM) models of ZF^0, and in section 4 we summarize the results of the paper and list some unsolved problems.

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Section 1. Definitions.

We start with some definitions. Let (X, T) be a topological space.

Definition 1.

1. X is *normal* if for every pair of disjoint closed sets A and B, there exist disjoint open sets C and D such that $A \subseteq C$ and $B \subseteq D$. We say that X is *effectively normal* if there is a function F such that for every pair (A, B) of disjoint closed sets in X, $F(A, B) = (C, D)$ where C and D are disjoint open sets in X, $A \subseteq C$ and $B \subseteq D$. (F is called a *normality operator.*) (See [17].)
2. X is a U *space* iff it satisfies:

If A, B are closed and disjoint in X then there exists a continuous function $f : X \to [0, 1]$ separating A and B. i.e., $A \subseteq f^{-1}(0)$ and $B \subseteq f^{-1}(1)$.

3. X is a T *space* iff it satisfies:

If A is closed in X and $f : A \to [0, 1]$ a continuous function then there exists a continuous extension $\bar{f}$ of f to all of X taking values in $[0, 1]$.

4. The *disjoint union* of the disjoint family $\{(X_i, T_i) : i \in k\}$ of topological spaces is the set $X = \cup\{X_i : i \in k\}$ such that O is open in X iff $O \cap X_i$ is open in X_i for all $i \in k$.
5. An F_σ *set* is a countable union of closed sets.

Below we give our notation for the principles we use and their precise statements.

AC: The Axiom of Choice. For every family $\mathcal{A} = \{A_i : i \in k\}$ of non-empty pairwise disjoint sets there exists a set C which consists of one and only one element from each element of $\mathcal{A}$.

CAC: The Countable Axiom of Choice. AC restricted to countable families.

DC: The Axiom of Dependent Choices. If R is a non-empty relation on a non-empty set X such that $\forall x \exists y : xRy$, then there exists a function $f : \omega \to X$ such that $f(n)Rf(n+1)$ for all $n \in \omega$.

MC: The Multiple Choice Axiom. For every family $\mathcal{A} = \{A_i : i \in k\}$ of non-empty sets there exists a family $\mathcal{F} = \{F_i : i \in k\}$ of finite non-empty sets such that for every $i \in k$ $F_i \subseteq A_i$.

CMC: The Countable Multiple Choice Axiom. MC restricted to countable families.

CU: The Countable Union Theorem. The denumerable union of denumerable sets is denumerable. (We use the word "denumerable" to mean "countably infinite".)

NU: Urysohn's Lemma. Normal spaces are U spaces.

NT: The Tietze-Urysohn Extension Theorem. Normal spaces are T spaces.

NEFN: Every normal space is effectively normal.

vDCP(ω): van Douwen's Choice Principle. If $\mathcal{A} = \{A_i : i \in \omega\}$ is a family of non-empty disjoint sets and f a function such that for each $i \in \omega$, $f(i)$ is an ordering of A_i of type $\mathbb{Z}$ (= the integers), then $\mathcal{A}$ has a choice function.

PvDCP(ω): If $\mathcal{A} = \{A_i : i \in \omega\}$ is a family of non-empty disjoint sets and f a function such that for each $i \in \omega$, $f(i)$ is an ordering of A_i of type $\mathbb{Z}$ then some infinite subfamily of $\mathcal{A}$ has a choice function.

UT: Every U space is a T space.

LN: Every linearly ordered topological space is normal.

DUN (DUN(ω)): The (countable) disjoint union of normal spaces is normal.

DUU (DUU(ω)): The (countable) disjoint union of U spaces is a U space.

DUT (DUT(ω)): The (countable) disjoint union of T spaces is a T space.

NF$_\sigma$: An F$_\sigma$ subset of a normal space is normal.

UF$_\sigma$: An F$_\sigma$ subset of a U space is a U space.

TF$_\sigma$: An F$_\sigma$ subset of a T space is a T space.

Section 2. Theorems.

Lemma 1. *(i) A T space is also a U space.*
(ii) A T space is normal.
(iii) A U space is normal.
(iv) vDCP(ω) iff PvDCP(ω).

Proof. (i) Let X be a T space, G, $Q \subseteq X$ disjoint closed sets and, f the function which takes on the value 0 on G and 1 on Q. Clearly f is continuous on $G \cup Q$. As X is a T space, f has a continuous extension $\bar{f}$ to all of X separating G and Q. (ii) and (iii) can be proved similarly.

(iv) As ($\rightarrow$) is obvious it suffices to show only ($\leftarrow$). As $\mathbb{Z} \times \mathbb{Z}$ has clearly an ordering of type $\mathbb{Z}$, it follows by a straightforward induction that

$$B_n = \prod_{i=0}^{n} A_i$$

also has an ordering of type $\mathbb{Z}$ which is completely defined in terms of f . Put $B = \{B_n : n \in \omega\}$ and let by PvDCP(ω) $C = \{c_{n_j} : j \in \omega\}$ be a choice set for the family $\{B_{n_j} : j \in \omega\}$ where $\langle n_j \rangle_{j\in\omega}$ is a strictly increasing sequence. Clearly,

$$c = \mathrm{Range}(c_{n_0}) \cup (\cup\{\mathrm{Range}(c_{n_{j+1}}|(n_{j+1} + 1 \backslash n_j + 1)) : j \in \omega\}$$

is a choice set for $\mathcal{A}$ finishing the proof of the proposition.

We have defined the notion of effectively normal and a normal operator in Definition 1.

Definition 2. A topological space X is *effectively Urysohn* if there is a function F such that for every pair (A, B) of disjoint closed sets in X, $F(A, B)$ is a continuous function from X to $[0, 1]$ such that F is 0 on A and 1 on B. (F is called a *Urysohn operator.*) Similarly, X is *effectively Tietze* if there is a function F such that for each closed set A and each continuous function $g : A \rightarrow [0, 1]$, $F(A, g) : X \rightarrow [0, 1]$ is a continuous extension of g. (F is called a *Tietze operator.*)

Lemma 2. *If X is effectively normal then X satisfies the Tietze-Urysohn extension theorem.*

Proof. If X is effectively normal, then the definition of the Urysohn separating function is constructive and this is what is needed to prove the Tietze-Urysohn extension theorem. (See, for example [12] pp 115 and 142 or [18] p212.) Thus, the notions of effectively normal, effectively Urysohn, and effectively Tietze coincide in ZF^0. However, in section 3, we give a permutation model ($\mathcal{N}_3$) in which there is a normal space which in not a U space nor a T space.

Metric spaces are easily seen to be normal. Furthermore, if we modify the details of the proof that metric spaces are normal (see, for example, [20] p 100 or [12] p 120), then one can easily establish that metric spaces are effectively normal. On the other hand compact T_2 spaces are normal (without appealing to AC), but one cannot prove in ZF without AC that they are effectively normal. The statement that compact T_2 spaces are effectively normal implies $\mathrm{vDCP}(\omega)$. (The one point compactification of the space of Theorem 2.2 in [4] is a compact T_2 space which is not effectively normal.)

Theorem 1. *MC implies each of the statements:*
(i) DUT
(ii) DUU
(iii) DUN
(iv) NU
(v) NT
(vi) UT
(vii) NEFN

Proof. (i) Let X be the disjoint union of the family $\{(X_i, T_i) : i \in k\}$ of T spaces, A a closed subset of X and $f : A \to [0, 1]$ a continuous function. Then

$$A = \cup\{Y_i = X_i \cap A : i \in k\}.$$

As $f_i = f|Y_i$ is continuous and X_i is a T space, $B_i = \{\bar{f} : \bar{f} : X_i \to [0, 1]$ is a continuous extension of $f_i\} \neq \emptyset$. By MC, there exists a set $\mathcal{F} = \{F_i \subseteq B_i : i \in k\}$ of finite non-empty sets. Define $\bar{f} : X \to [0, 1]$ by requiring that for each $x \in X_i$,

$$\bar{f}|X_i(x) = \max(\{h(x) : h \in F_i\}).$$

Clearly $\bar{f} : X \to [0, 1]$ is a continuous extension of f.

(ii) and (iii) can be proved similarly.

(iv) $MC \to NU$ is also discussed in [1]. We prove something stronger, namely that X is effectively normal, then use Lemma 2 to establish (v). Suppose X is a normal topological space. Then, if A and B are disjoint closed sets, there exist disjoint open sets C and D such that $A \subseteq C$ and $B \subseteq D$. MC gives us a rule for choosing a finite number of pairs (C, D) with the above property. Since the intersection of a finite number of open sets is open, MC implies that there is a function F such that for each pair of disjoint closed sets (A, B), $F(A, B) = (C, D)$, where C and D are disjoint open sets such that $A \subseteq C$ and $B \subseteq D$. Thus, X is effectively normal.

(vi) follows from (v) and Lemma 1 (iii), and (vii) follows from the proof of (iv). □

Corollary. *(i) CMC implies DUT(ω).*
(ii) CMC implies DUU(ω).
(iii) CMC implies DUN(ω).

It follows from Theorem 1 that under MC, normal spaces, U spaces, and T spaces coincide. Furthermore, if all orderable spaces (spaces whose topology comes from a linear order) are normal and we restrict ourselves to this class of spaces, then these three notions also coincide (see [4]). In [15], Läuchli supplies an example of normal locally compact space on which every continuous real valued function is constant and in [7] the authors provide a consistency example of a compact Suslin line $(X, \leq)$ on which every continuous real valued function is constant. Both examples demonstrate the fact that normal spaces need not be U or T spaces if MC fails. (See, also, [2].)

In addition, while MC implies DUN in ZF^0, in ZFC the product of two normal spaces may not be normal. An example is given in [19]. (Take X to be the real line in which a base for the topology is the set of all half-open intervals $[a, b)$. Then in $X \times X$, let $A = \{(x, y) : x + y = 1 \ \& \ [(x-1)^2 + y^2]^{1/2} \text{ is rational}\}$ and $B = \{(x, y) : x + y = 1 \ \& \ [(x-1)^2 + y^2]^{1/2} \text{ is irrational}\}$. The sets A and B are disjoint closed sets which cannot be separated by disjoint open sets.)

Let z be any infinite set. In what follows we shall use $\mathcal{P}_{fin}(z)$ to denote the collection of finite subsets of z and $\mathcal{P}_{fin}(z)^+$ to denote $\mathcal{P}_{fin}(z) \setminus \{\emptyset\}$. Also let a_z and b_z be any two objects not in $\mathcal{P}_{fin}(z)^+$. (We could take, for example, $a_z = (0, z)$ and $b_z = (1, z)$.) For our next three theorems we will make use of the topology T_z defined below on the set $X_z = \mathcal{P}_{fin}(z)^+ \cup \{a_z, b_z\}$.

Definition 3. T_z is the topology on X_z which has as a basis the following sets
1. $\{t\}$ such that $t \in \mathcal{P}_{fin}(z)^+$.
2. $\{a_z\} \cup \{t \in \mathcal{P}_{fin}(z)^+ : w \subseteq t\}$ for any finite set $w \subseteq z$. We call this set $N(w, a_z)$.
3. $\{b_z\} \cup \{t \in \mathcal{P}_{fin}(z)^+ : w \cap t = \emptyset\}$ for any finite set $w \subseteq z$. We call this set $M(w, b_z)$.

Claim. (X_z, T_z) is a T space (and therefore, a U space and a normal space).

Proof. Assume that A is closed in (X_z, T_z) and that $f : A \to [0, 1]$ is continuous. First we note that for any $t \in X_z$ different from a_z and b_z, $\{t\}$ is open and therefore any function from X_z to $[0, 1]$ is continuous at such a t. Therefore to show that a function $f^* : X_z \to [0, 1]$ is continuous we only have to show that it is continuous at a_z and at b_z. We consider several cases.

Case 1. Neither a_z nor b_z is in A. In this case A is also open since each one element subset of A is open. Therefore if we define $f^* : X_z \to [0, 1]$ by $f^*(t) = f(t)$ if $t \in A$ and $f(t) = 0$ if $t \notin A$, then f^* is a continuous extension of f.

Case 2. Both a_z and b_z are in A. Choose open disjoint neighborhoods N and M of a_z and b_z respectively. Define $f^* : X_z \to [0, 1]$ by $f^*(t) = f(t)$ for $t \in A$, $f^*(t) = f(a_z)$ for $t \in N \setminus A$, $f^*(t) = f(b_z)$ for $t \in M \setminus A$, and $f^*(t) = 0$ for $t \in X_z \setminus (M \cup N \cup A)$.

Case 3. $a_z \in A$ and $b_z \notin A$. As above, choose disjoint open neighborhoods N and M of a_z and b_z respectively. We may assume that $M \cap A = \emptyset$ by replacing M with $M \cap A^c$ if necessary. Define f^* by $f^*(t) = f(t)$ for $t \in A$, $f^*(t) = f(a_z)$ for $t \in N \setminus A$, $f^*(t) = 0$ for $t \in X_z \setminus (A \cup N)$.

Case 4. $b_z \in A$ and $a_z \notin A$. This is similar to case 3.

This proves the claim.

Lemma 3. *There is a definable function F such that if z is an infinite set and U_z and V_z are disjoint open sets in (X_z, T_z) containing a_z and b_z respectively, then $F(U_z, V_z)$ is a finite non-empty subset of z.*

Proof. Our proof of Lemma 3 requires two combinatorial lemmas (Lemma 4 and Lemma 5 below). We begin with

Definition 4. A Δ-*system* is a collection D of finite sets such that $|D| \geq 2$ and for some finite set r, $x \cap y = r$ for all x and y in D such that $x \neq y$. The set r is called the *root* of the Δ-system D.

(More information on Δ-systems can be found in [14] p 49, [10], and [21] pp 141-146. The referee informed us that there is a proof of Lemma 4 in [21]. However, this proof uses AC. Because of the nature of our results, we need a proof which does not use any form of AC. Since we did not see any easy way to eliminate AC from the proof in [21], we have included our own short proof.)

Lemma 4. *For every positive integer n and every infinite set A of n-element sets, there is a finite subset r of $\bigcup A$ such that for every positive integer k, there is a subset of A of size k which is a Δ-system with root r.*

Proof. The proof is by induction on n. For $n = 1$ we only have to note that any infinite collection of 1-element sets is a Δ-system with root $\emptyset$.

Assume that $n > 1$ and that the lemma is true for all $j < n$. Let A be an infinite set of n-element sets. For each $x \in A$, define $E_0(x) = \{x\}$, $E_1(x) = \{y \in A : y \cap x \neq \emptyset\}$, and in general $E_n(x) = \{y \in A : y \cap (\bigcup E_{n-1}(x)) \neq \emptyset\}$. It is not hard to verify that the relation $\sim$ defined on A by

$$x \sim y \Leftrightarrow (\exists n \in \omega)(y \in E_n(x))$$

is an equivalence relation. For each $x \in A$, let $[x]$ denote the $\sim$ equivalence class of x. Clearly if $x, y \in A$ and $[x] \neq [y]$ then $x \cap y = \emptyset$. Therefore if there are infinitely many $\sim$ equivalence classes, $r = \emptyset$ will satisfy the conclusion of Lemma 4. We therefore assume that there are only finitely many $\sim$ equivalence classes. Then, since A is infinite, there is an $x \in A$ such that $[x]$ is infinite. Fix such an x. We now consider two cases.

Case 1. For every finite $z \subseteq \bigcup[x]$, there is a $y \in [x]$ such that $y \cap z = \emptyset$. In this case we can take $r = \emptyset$ and prove by induction on k that there are k pairwise disjoint elements of $[x]$. (For the induction step, assuming that $\{y_1, \ldots, y_k\}$ are pairwise disjoint, choose y_{k+1} so that $y_{k+1} \cap (y_1 \cup \cdots \cup y_k) = \emptyset$.)

Case 2. There is a finite $z \subseteq \bigcup[x]$ such that $\forall y \in [x]$, $y \cap z \neq \emptyset$. Since $[x]$ is infinite, there is a finite non-empty subset $z' \subseteq z$ such that $B = \{y \in [x] : y \cap z = z'\}$ is infinite. Let $B' = \{y \setminus z' : y \in B\}$. B' is an infinite collection of sets of cardinality $n - |z'|$. By the induction hypothesis there is a set r_0 such that for all $k > 0$, there is a subset D of B' of size k which is a Δ-system with root r_0. Let $r = r_0 \cup z'$. Then for any k-element subset D of B' which is a delta system with root r_0, the set $\{y \cup z' : y \in D\}$ is a k-element subset of A which is a Δ-system with root r. This completes the proof in case 2 and therefore completes the proof of Lemma 4.

Lemma 5. *Let A be an infinite collection of finite sets such that there is a finite set $x \subseteq \bigcup A$ such that for some natural number k, x intersects all but k members of A in a non-empty set. (That is, $|\{y \in A : y \cap x = \emptyset\}| \leq k$. We denote this statement by $\Phi(A, k, x)$.) Let n_0 be the least natural numbers for which there is a $k \geq 0$ and a finite $x \subseteq \bigcup A$ of size n_0 such that $\Phi(A, k, x)$. Fix such a k, say k_0 then the set $\{x : |x| = n_0$ and $\Phi(A, k_0, x)\}$ is finite.*

Proof. (Although not strictly necessary, we give the proof in the case that $n_0 = 1$ since it uses the same ideas as the general proof but in a less complicated setting.) If $n_0 = 1$ assume (for purposes of proof by contradiction) that $B = \{x : |x| = 1 \text{ and } \Phi(A, k_0, x)\}$ is infinite. Choose k_0 distinct elements $y_1, \ldots, y_{k_0}$ of A. Since y_i is finite for $1 \leq i \leq k_0$ and $|x| = 1$ for all $x \in B$, the set $\{x \in B : x \cap y_i \neq \emptyset\}$ is finite for $1 \leq i \leq k_0$. Hence $C = \bigcup_{i=1}^{k_0} \{x \in B : x \cap y_i \neq \emptyset\}$ is finite. If $z \in B \setminus C$ then $z \cap y_i = \emptyset$ for $1 \leq i \leq k_0$. Therefore, because $\Phi(A, k_0, z)$ holds,

$$(*) \qquad z \cap y \neq \emptyset \text{ for all } y \in A \setminus \{y_1, \ldots, y_{k_0}\}$$

Since $B \setminus C$ is an infinite set of one element sets, $(*)$ implies that y is infinite for all $y \in A \setminus \{y_1, \ldots, y_{k_0}\}$. This is a contradiction since A is an infinite set of finite sets.

Now assume that the n_0 of the lemma is any positive integer. We again assume that $B = \{x : |x| = n_0 \text{ and } \Phi(A, k_0, x)\}$ is infinite and try to arrive at a contradiction. By Lemma 5 there is a finite subset r of $\bigcup B$ such that for every positive integer k, there is a subset of B of size k which is a Δ-system of size k with root r. Since n_0 is the least positive integer n for which there is an x of size n and a k such that $\Phi(A, k, x)$ and since $|r| < n_0$, we conclude that the set $C = \{x \in A : x \cap r = \emptyset\}$ is infinite. Choose $y_1, \ldots, y_{k_0}$ in C. Let D be any Δ-system with root r which is a subset of B. Then

$$(**) \qquad (\forall y \in C)(|\{x \in D : x \cap y \neq \emptyset\}| \leq |y|)$$

(because for x and x' in D, $x \cap x' = r$ and $r \cap y = \emptyset$). Since this is true, in particular, for $y = y_i$, $1 \leq i \leq k_0$, we may conclude

$$\left| \bigcup_{i=1}^{k_0} \{x \in D : x \cap y_i \neq \emptyset\} \right| \leq k_0 \cdot (\max\{|y_i| : 1 \leq i \leq k_0\}).$$

Choose a positive integer N and let D be a subset of B which is a Δ-system with root r of size $k_0 \cdot (\max\{|y_i| : 1 \leq i \leq k_0\}) + N$. Then the set $D' = D \setminus \left(\bigcup_{i=1}^{k_0} \{x \in D : x \cap y_i \neq \emptyset\} \right)$ has cardinality at least N. Further, for each $x \in D'$, $x \cap y_i = \emptyset$ for $1 \leq i \leq k_0$. Since such an x is in B, $|\{y \in A : y \cap x = \emptyset\}| \leq k_0$. Therefore, for every $y \in C \setminus \{y_1, \ldots, y_{k_0}\}$, $x \cap y \neq \emptyset$. Hence by $(**)$ if $y \in C \setminus \{y_1, \ldots, y_{k_0}\}$, then $N \leq |D'| \leq |y|$. But N was arbitrary so y is infinite, a contradiction. This completes the proof of Lemma 5.

To complete the proof of Lemma 3, let U_z and V_z be disjoint open sets in (X_z, T_z) containing a_z and b_z respectively. (See Definition 3.) Let $A = \{w \subset z : w$ is finite and $N(w, a_z) \subseteq U_z$ and for no proper subset w' of w is $N(w', a_z) \subseteq U_z\}$ and $B = \{w \subset z : w$ is finite and $M(w, b_z) \subseteq V_z$ and for no proper subset w' of w is $M(w', b_z) \subseteq V_z\}$. A and B are non-empty. To show that A is non-empty, we note first that because of the definition of the topology, Definition 3, there is a finite subset $w \subset z$ such that $N(w, a_z) \subseteq U_z$. However, $N(\emptyset, a_z)$ is not a subset of U_z because $N(\emptyset, a_z) \cap V_z = V_z \setminus \{b_z\} \neq \emptyset$. Similarly, $B \neq \emptyset$. Since $N(\emptyset, a_z)$ is not a subset of U_z and similarly, $M(\emptyset, b_z)$ is not a subset of V_z, it follows that $\emptyset \notin A \cup B$. We also claim that if $w_1 \in A$ and $w_2 \in B$, then $w_1 \cap w_2 \neq \emptyset$. For if $w_1 \cap w_2 = \emptyset$ then $w_1 \in N(w_1, a_z)$ and $w_1 \in M(w_2, b_z)$ which is a contradiction since $N(w_1, a_z) \cap M(w_2, b_z) = \emptyset$.

If A is infinite then for $x \in B$, $\Phi(A, 0, x)$. (See Lemma 5 for a definition of Φ.) Hence there is a least natural number n_0 for which $\exists x$ and k such that $|x| = n_0$ and $\Phi(A, k, x)$. Let k_0 be the least natural number k such that the set $D = \{x : |x| = n_0$ and $\Phi(A, n_0, k_0)\}$ is non-empty. By Lemma 5, D is finite. We can now define $F(U_z, V_z)$ by $F(U_z, V_z) = \bigcup D$ if A is infinite and $F(U_z, V_z) = \bigcup A$ otherwise.

Theorem 2. *Each of DUN, DUU and DUT imply MC.*

Proof. Assume DUN (or DUU or DUT) and let Y be a set of infinite disjoint sets. We may assume wlog that the sets $\{X_z : z \in Y\}$ are pairwise disjoint. Let $X^* = \bigcup_{z \in Y} X_z$. Using the topology T_z on each X_z, let T^* be the 'disjoint union' topology on X^*. The sets $F_1 = \{a_z : z \in Y\}$ and $F_2 = \{b_z : z \in Y\}$ are disjoint and closed in the space (X^*, T^*). We showed that the spaces (X_z, T_z) for $z \in Y$ are T spaces therefore by DUN (or DUU or DUT) there are disjoint open sets G_1 and G_2 such that $F_1 \subseteq G_1$ and $F_2 \subseteq G_2$. For each $z \in Y$, let $U_z = X_z \cap G_1$ and $V_z = X_z \cap G_2$. Using the function F from Lemma 3 we can define a multiple choice function H on Y by $H(z) = F(U_z, V_z)$ for each $z \in Y$.

Corollary 1. *(To the proof) Each of DUN(ω), DUU(ω), and DUT(ω) imply CMC.*

Corollary 2. *Each of the following are equivalent:*

(i) DUN

(ii) If $\{X_i : i \in k\}$ is a family of normal spaces then there exists a set of functions $\{f_i : i \in k\}$ such that for each $i \in k$, f_i is a normality operator on X_i.

(iii) If $\{X_i : i \in k\}$ is a family of normal spaces then there exists a set of functions $\{f_i : i \in k\}$ such that for each $i \in k$, f_i is a Urysohn operator on X_i.

(iv) If $\{X_i : i \in k\}$ is a family of normal spaces then there exists a set of functions $\{f_i : i \in k\}$ such that for each $i \in k$, f_i is a Tietze operator on X_i.

Proof. Marianne Morillon brought these equivalences to our attention. It is clear that each implies DUN. To prove the converse, use the fact that DUN implies MC (Theorem 2) which, in view of Theorem 1 (v), implies that a normal topological space is effectively normal. Then the theorem follows from Lemma 2.

Theorem 3. *UT (and therefore NT) implies CMC*

Proof. Let $Y = \{z_i : i \in \omega\}$ be a countable set of infinite sets. Let (X^*, T^*) be the disjoint union of the spaces (X_{z_i}, T_{z_i}) for $z_i \in Y$ with the disjoint union topology and let

$X^{**} = X^* \cup \{\infty\}$ where ∞ is an object not in X^*. Let T^{**} be the topology generated by T^* and for each $i \in \omega$, the set $\{\infty\} \cup (\bigcup_{j>i} X_{z_j})$.

Claim 1. (X^{**}, T^{**}) is a U space.

Proof. Let A and B be two closed disjoint sets in X^{**}. If both only have a non-empty intersection with a finite number of the X_{z_i}'s, then since each X_{z_i} is a U space, the claim is true. Suppose A has a non-empty intersection with an infinite number of the X_{z_i}'s. Then since A is closed, ∞ must be in A and B can have a non-empty intersection with at most a finite number of the X_{z_i}'s. Then we can find a neighborhood of A which is disjoint from B which contains a set of the form $V = \{\infty\} \cup \bigcup_{i>j} X_{z_i}$. This is a clopen set. We can define a function f on X^{**} so that f is 0 on V. The complement of V, W, is a union of a finite number of U spaces. Thus, we can extend f to W so that f is continuous, f is 0 on $W \cap A$, and f is 1 on $W \cap B$.

Claim 2. The set $C = \{\infty\} \cup \{a_i, b_i : i \in \omega\}$ is closed in the topology.

Claim 3. The function g defined on C by $g(\infty) = 0$, $g(a_i) = \frac{1}{2^{2i}}$, $g(b_i) = \frac{1}{2^{2i+1}}$ is continuous on C.

Claim 4. If we apply UT to (X^{**}, T^{**}) to get a continuous extension $g^* : X^{**} \to [0,1]$ of g, we can use g^* to define a sequence $(U_i, V_i)_{i\in\omega}$ such that U_i and V_i are disjoint and open in T_{z_i} with $a_{z_i} \in U_i$, and $b_{z_i} \in V_i$. ($U_i = (g^*)^{-1}([0, \frac{1}{2^{2i}} + \frac{1}{2^{2i+2}})) \cap X_{z_i}$ and $V_i = (g^*)^{-1}((\frac{1}{2^{2i}} + \frac{1}{2^{2i+2}}, 1]) \cap X_{z_i}$.)

Then, using the F from Lemma 3, we can define a multiple choice function G on Y by $G(z_i) = F(U_i, V_i)$.

Theorem 4. *NEFN implies MC.*

Proof. The proof is similar to that of Theorem 3. Let Y be a set of infinite sets. Let (X^*, T^*) be the disjoint union of the spaces (X_z, T_z)'s for $z \in Y$ with the disjoint union topology Let $X^{**} = X^* \cup \{\infty\}$ where ∞ is an object not in X^*. Let T^{**} be the topology generated by T^* and for each cofinite subset W of Y the set $\{\infty\} \cup (\bigcup_{z\in W} X_z)$. The proofs of the following claims are fairly straightforward.

Claim 1. (X^{**}, T^{**}) is normal.

Claim 2. For each $z \in Y$, the pair $\{a_z, b_z\}$ is closed in X^{**}.

Applying effective normality gives two functions P_1 and P_2 each with domain Y such that for every $z \in Y$, $P_1(z)$ and $P_2(z)$ are disjoint open sets in X_z containing a_z and b_z respectively. Using the F from Lemma 3 we can define a multiple choice function G on Y by $G(z) = F(P_1(z), P_2(Z))$.

It follows from Theorems 1, 2 and 4 and their corollaries that

Theorem 5.

(a) Each of the following are equivalent
  - *(i) MC*
  - *(ii) DUN*
  - *(iii) DUU*

(iv) DUT
(v) NEFN

(b) Each of the following are equivalent
(i) CMC
(ii) DUN(ω)
(iii) DUU(ω)
(iv) DUT(ω)

A topological space is called *effectively Hausdorf* if there is a function f such that for every pair (x, y) of distinct points in X, $f(x, y) = (U, V)$ where U and V are disjoint open sets in X with $x \in U$ and $y \in V$. Using similar proofs as for NEFN, it is easy to see that MC is equivalent to the statement "Hausdorf spaces are effectively Hausdorf". In addition, if (X, T) is a topological space, we define:

1. X is cwH, *collectionwise Hausdorf,* iff for every closed and discrete subset D of X there exists a family $U = \{U_d : d \in D\}$ of pairwise disjoint open sets such that $d \in U_d$ for all $d \in D$.
2. X is cwN, *collectionwise normal,* iff for every collection of pairwise disjoint closed sets $\mathcal{D}$ of X there exists a family $\mathcal{U} = \{U_D : D \in \mathcal{D}\}$ of pairwise disjoint open sets such that for each $D \in \mathcal{D}$ there is a $U_D \in \mathcal{U}$ such that $D \subseteq U_D$.

   It is quite easy to prove that the following are equivalent:
   (i) MC
   (ii) The disjoint union of cwH spaces is cwH.
   (iii) The disjoint union of cwN spaces is cwN.
   (iv) The disjoint union of cwN spaces is normal.

(Since the spaces X_z of Theorem 2 are both cwH and cwN, the proof of Theorem 2 can be used to prove that each of (ii), (iii), and (iv) implies (i). The converses follow using a proof similar to the proof given in Theorem 1 (i).)

Definition 5. A subset Q of a poset (= a partially ordered set) $(P, \leq)$ is *dense* iff every element $p \in P$ has a lower bound in Q. A set $F \subseteq P$ is a *filter* iff every two element subset of F has a lower bound in F and, if $a \in F$ and $b \geq a$ then $b \in F$.

In the language of posets, DC is given by:

Lemma 6. [9] *DC iff for every poset $(P, \leq)$ and every family D, $|D| \leq \omega$ of dense subsets of P there exists a filter meeting non trivially each member of D.*

It is easy to see that a closed subset A of a T space X is again a T space. Indeed, if B is a closed subset of A in the relative topology then B is also closed in X. Thus, if $f : B \to [0, 1]$ is a continuous real valued function then f has a continuous extension F to all of X. Hence, the restriction of F to A is the required continuous extension of f to A. Similarly, closed subsets of U spaces are U spaces and closed subsets of normal spaces are normal. In the next theorem we show that DC extends this property to F_σ subsets of T spaces, as well as, of U spaces and normal spaces.

Theorem 6. *DC implies each of the statements:*
(i) NF$_\sigma$

(ii) NU
(iii) NT
(iv) UF_σ
(v) TF_σ
(vi) UT

Proof. Let (X,T) be a normal space, $G=\cup\{G_n : n\in\omega\}$ F_σ in X and A, $B\subseteq G$ disjoint closed sets in G. Then, $A=A^*\cap G$ and $B=B^*\cap G$ for some A^*, B^* closed in X. Clearly

$$A=\cup\{A^*\cap G_n : n\in\omega\} \text{ and } B=\cup\{B^*\cap G_n : n\in\omega\}.$$

Wlog we may assume that for every $n\in\omega$, $A^*\cap G_n$ and $B^*\cap G_n$ are non-empty. We make

$$P=\{(O,Q) : O,Q\in T\backslash\{\emptyset\}, \bar{O}\cap\bar{Q}=\emptyset, \bar{O}\cap B=\bar{Q}\cap A=\emptyset\}$$

into a poset by requiring:

$$(O,Q)\leq(F,H) \text{ iff } O\supseteq F \text{ and } Q\supseteq H.$$

We observe that the elements (O,Q) and (F,H) are *compatible* (they have a common extension) iff

$$(O\cup F\bar{)}\cap(Q\cup H\bar{)}=\emptyset.$$

We claim that

$$D_n=\{(O,Q)\in P : (A^*\cap G_n)\subseteq O \text{ and } (B^*\cap G_n)\subseteq Q\}$$

is dense in $(P,\leq)$ for every $n\in\omega$. Indeed, fix $(O,Q)\in P$. Then, $\bar{O}\cup(A^*\cap G_n)$ and $\bar{Q}\cup(B^*\cap G_n)$ are disjoint closed sets in X ($(A^*\cap G_n)$ and $(B^*\cap G_n)$ are disjoint and closed in X.). Thus, there exist open sets F,H including $\bar{O}\cup(A^*\cap G_n)$ and $\bar{Q}\cup(B^*\cap G_n)$ respectively such that $\bar{F}\cap\bar{H}=\emptyset$. We may also, choose F and H in such a way that: $\bar{F}\cap B=\emptyset$ and $\bar{H}\cap A=\emptyset$. Let, by Lemma 5, $\mathcal{F}$ be a filter of P meeting each member of $D=\{D_n : n\in\omega\}$. Clearly,

$$O=\cup\{F : (F,H)\in\mathcal{F} \text{ for some open } H\}\cap G,$$

$$Q=\cup\{H : (F,H)\in\mathcal{F} \text{ for some open } F\}\cap G$$

are disjoint neighbourhoods of A and B respectively finishing the proof of (i).

(ii) A proof is given in [12] p 115.

(iii) In Theorem 3.2 ([18] p 212) use DC whenever it is needed.

To get (iv) and (v) simply combine Lemma 1 and the previous parts (ii) and (iii) respectively. (vi) follows from (iii) and Lemma 1 (iii). □

Theorem 7. *DUU(ω) implies UF$_\sigma$.*

Proof. Let X be a U space. Let $G = \{G_i : i \in \omega\}$ be a family of closed sets in X, $A, B \neq \emptyset$ closed and disjoint in $Y = \cup G$ and $X_i = X \times \{i\}$ be an isomorphic copy of X under the map

$$f_i : X \to X_i,\ f_i(x) = (x, i) \text{ for all } x \in X.$$

Let A^* be a closed set in X such that $A^* \cap Y = A$. Clearly $\bar{A}_i^* = A^* \times \{i\}, \bar{B}_i = B_i(= B \cap G_i) \times \{i\}$ are closed and disjoint in X_i. As each X_i is a U space, DUU(ω) implies that $Z = \cup\{X_i : i \in k\}$ is also a U space. Then, $\bar{A}^* = \cup\{\bar{A}_i^* : i \in \omega\}$ and $\bar{B} = \cup\{\bar{B}_i : i \in \omega\}$ are closed and disjoint in Z. Hence, there exists a function $h : Z \to [0, 1]$ taking on the value 0 on $\bar{A}^*$ and the value 1 on $\bar{B}$. Clearly

$$h_i : X \to [0, 1],\ h_i = h \circ f_i$$

is a continuous function taking on the value 0 on A^* $(\supseteq A)$ and the value 1 on B_i, and

$$g_a = \sum_{n \in \omega} 1/2^{n+1} h_n$$

is a continuous function that is 0 on A and positive on B. In a similar way construct a continuous function g_b that is 0 on B and positive on A. Let $C = \{x : g_a(x) + g_b(x) = 0\}$. Let $g_c : X \to [0, 1]$ be a continuous function which is 0 on $A \cup B$ and positive on C. Finally, we define a function $F : X \to [-1, 1]$ such that

$$F = \frac{g_a - g_b}{g_a + g_b + g_c}.$$

Now, it is easy to see the F is a continuous function which is -1 on A and 1 on B, thereby proving that Y is a U space. □

It follows from Theorem 5 (b) that CMC implies UF$_\sigma$.

In [4], van Douwen proved that the statement DUN(ω) implies vDCP(ω). Consequently, by Theorem 5 (b), CMC implies vDCP(ω). In the next theorem we also show that UF$_\sigma$ implies DUU(ω) and NF$_\sigma$ implies DUN(ω).

Theorem 8. *(i) UF$_\sigma$ implies DUU(ω).*
(ii) NF$_\sigma$ implies DUN(ω).
(iii) CMC implies vDCP(ω).

Proof. (i) Fix $\{X_i : i \in \omega\}$, a family of U spaces. Let X be their disjoint union and $\bar{X} = X \cup \{a\}$, $a \notin X$, where neighbourhoods of a leave out only a finite number of spaces X_i. We claim that $\bar{X}$ is a U space. Indeed, fix G, Q two closed and disjoint sets in $\bar{X}$. Put $I = \{i \in \omega : G \cap X_i \neq \emptyset \text{ and } Q \cap X_i \neq \emptyset\}$. Clearly I is finite. As for each $i \in I$ $G \cap X_i$ and $Q \cap X_i$ are disjoint closed sets in X_i, it follows that there exists a continuous function f_i on X_i separating $G \cap X_i$ and $Q \cap X_i$. Since I is finite it is easy to find, using the f_i's, a continuous function f separating G and Q as required. By UF$_\sigma$ now X is a U space finishing the proof of the theorem.

Part (ii) can be proved similarly to the proof of (i). □

Corollary 1. *CMC iff* UF_σ

Proof. The proof follows from Theorem 8 (i), Theorem 5 (b), and the Corollary to Theorem 1.

Corollary 2. *(i) MC implies* NF_σ.
(ii) MC implies UF_σ.
(iii) MC implies TF_σ.

Proof. Use Theorem 1 (ii), (iv) and (vi), and Corollary 1 above.

Section 3. Models. In this section we shall give models of ZF (Cohen models) and models of ZF^0 (Fraenkel-Mostowski (FM) models) to demonstrate some independence results. For the Cohen models, we shall supply references and/or a brief description of the model. To describe the FM models, we shall give the set of atoms (urelements), A, the group of permutations, $\mathcal{G}$, and either the ideal of supports, S, or the filter of subgroups Γ. (See, for example, [11].)

Our first model, $\mathcal{M}_1$, is the original Cohen model. (Add a denumerable set of generic reals along with the set containing them to the base model. See for example, [11] p 66.) In this model vDCP(ω) and LN are true, but DC and AC are false. Therefore, MC is also false because AC and MC are equivalent in ZF.

Our second Cohen model, $\mathcal{M}_2$, is Feferman's variation of the original Cohen model. (Add a denumerable number of generic reals to the base model, but do not collect them. See [5].) Feferman has shown that DC and CMC are true in $\mathcal{M}_2$, but MC is false.

We shall next give an FM model, $\mathcal{N}_1$, in which vDCP(ω) is false. (Thus, by Theorem 3, it contains an example of a U space that is not a T space, UT is false.) The set of atoms, $A = \bigcup_{n\in\omega} B_n$, where the B_n's are pairwise disjoint sets each of which is ordered like the integers, $\mathbb{Z}$. $\mathcal{G}$ is the group of all permutations which fix each B_n and also preserve the ordering on each B_n, $n \in \omega$. S, the ideal of supports, is the set of all finite subsets of A. The set $\{B_n : n \in \omega\}$ has no choice set so vDCP(ω) is false. (This result is transferable to ZF.)

We shall construct a U space in this model that is not a T space. For each $n \in \omega$, let a_n and b_n be disjoint from A and let

$$\bar{B}_n = \{a_n\} \cup B_n \cup \{b_n\}.$$

We linearly order $\bar{B}_n$ by declaring a_n to be less that every element of B_n and b_n to be greater than every element of B_n. (B_n is ordered like the integers.) Let $\bar{A} = \bigcup_{n\in\omega} \bar{B}_n$.

Let $(X, \mathcal{T})$, be the topological space $X = \bar{A} \cup \{c\}$, $c \notin \bar{A}$ and $\mathcal{T}$ is the order topology defined by $x < y$ if

1. $x \in \bar{A}$ and $y = c$,
2. $x \in \bar{B}_i$, $y \in \bar{B}_j$, where $i, j \in \omega$ and $i < j$, or
3. $\exists i \in \omega$, $x, y \in \bar{B}_i$, and x is less than y with respect to the ordering on $\bar{B}_i$.

We claim that $(X, \mathcal{T})$ is a U space. The argument is similar to the argument given in Theorem 3 that (X^{**}, T^{**}) is a U space. In fact, $(X, \mathcal{T})$ is compact. Suppose $(X, \mathcal{T})$ is a T space. Let $A^* = \{a_n : n \in \omega\}$, $B^* = \{b_n : n \in \omega\}$, and $C = A^* \cup B^* \cup \{c\}$. C is a closed subset of X. Define a function $f : C \to [-1, 1]$ such that $f(c) = 0$, and for $n \in \omega$, $f(a_n) = -1/(n+1)$, $f(b_n) = 1/(n+1)$. f is clearly continuous so f has a continuous extension $\bar{f} : X \to [-1, 1]$. Let $f_n = \bar{f}|\bar{B}_n$, then for each $n \in \omega$, f_n is continuous. Let $c_n = \max\{x \in B_n : f_n(x) < 0\}$. Then, $\{c_n : n \in \omega\}$ is a choice set for $\{B_n : n \in \omega\}$, which is a contradiction.

Let $\mathcal{N}_2$ be the FM model defined as follows: $A = \{a_i : i \in \omega\}$; $B = \{\{a_{2i}, a_{2i+1}\} : i \in \omega\}$; $\mathcal{G}$ is the group of permutations on A that leaves B point-wise fixed; and S is the set of all finite subsets of A. It follows from [16] that MC is true in $\mathcal{N}_2$. The set B in $\mathcal{N}_2$ is a countable set of pairs with no choice function. It is clear that DC implies that a countable set of pairs has a choice function, so DC is false in $\mathcal{N}_2$. Consequently, we have Lemma 7.

Lemma 7. *MC is true in $\mathcal{N}_2$, but DC is false.*

Let $\mathcal{N}_3$ be the linearly ordered FM model due to Mostowski. (See [11] p 49. A is countable and ordered like the rational numbers, $\mathcal{G}$ is the group of all automorphisms of A, and supports are finite.) We shall show that in this model the countable union theorem, CU, is true. (It is clear the CU implies vDCP(ω).) It was shown in [2] that NU is false in $\mathcal{N}_3$ and is was shown in [13] that LN is false.

Lemma 8. *CU is true in $\mathcal{N}_3$ and LN, NU, and CMC are false.*

Proof. To show CMC is false, for each $i \in \omega \setminus \{0\}$, let c_i be the set of all i-element subsets of A. Then the set $\{c_i : i \in \omega \setminus \{0\}\}$ is a denumerable set that has no multiple choice function.

To show CU is true, assume that in $\mathcal{N}_3$, D is a denumerable set of denumerable sets. Let $E \subseteq A$ be a finite support for every $b \in D$. Such an E exists because D is well orderable in $\mathcal{N}_3$. Assume $b \in D$ and $x \in b$. The proof will be completed by showing that E is a support of x. (It would then follow that each element of $\bigcup D$ has E as a support so $\bigcup D$ can be well ordered.) We argue by contradiction. Assume that E is not a support of x. We will show that under this assumption b has a subset of the same cardinality as an infinite subset of A which contradicts the countability of b. (A has no countably infinite subset.)

Let $E' \supseteq E$ be a support of x. Then there is a $t \in E' \setminus E$ and a $\phi \in \mathcal{G}$ which fixes $E' \setminus \{t\}$ pointwise and such that $\phi(x) \neq x$. (It follows that $\phi(t) \neq t$.) It can be verified that

$$\{\langle \psi(t), \psi(x) \rangle : \psi \in \mathcal{G} \text{ and } \psi \text{ fixes } E' \setminus \{t\} \text{ pointwise } \}$$

is a one to one function from an infinite subset of A into b which is in $\mathcal{N}_3$. □

$\mathcal{N}_4$ is a model given in [8]. Let $(A, \leq)$ be an ordered set of atoms which is order isomorphic to $\mathbb{Q}^\omega$, the set of all functions from ω into $\mathbb{Q}$ ordered by the lexicographic ordering. That is, if $a, b \in \mathbb{Q}^\omega$ then $a < b$ if and only if there is some $n \in \omega$ such that $(\forall j < n)(a_j = b_j)$ and $a_n < b_n$.

For each $b \in A$ and $n \in \omega$,

1. $A^n_b = \{a \in A \mid a_i = b_i \text{ for } 0 \leq i \leq n\}$ is the *n–level block containing b.*
   (A^n_b will not be in the model we are defining.)
2. The sequence $\langle b_{n+1}, b_{n+2}, \ldots \rangle$ is the *position of b in its n–level block.*

3. $\mathfrak{B}^n = \{A_a^n \mid a \in A\}$ is the set of n–level blocks.
4. $\leq_n$ is the relation on $\mathfrak{B}^n$ defined by $A_a^n \leq_n A_b^n$ if and only if $a \leq b$.
5. Let f be an order automorphism of $(\mathfrak{B}^n, \leq_n)$. (See lemmas A and B in [8].) We define ϕ_f to be the unique order automorphism of $(A, \leq)$ which satisfies $\phi''_f A_a^n = f(A_a^n)$ for all $a \in A$ and such that for all $a \in A$, a and $\phi_f(a)$ have the same position in their n-level blocks. (By 2 above, this means that $(\forall a \in A)(\forall i > n)(a_i = \phi_f(a)_i)$.)
6. For $n \in \omega$, G_n is the group $\{\, \phi_f : f$ is an order automorphism of $(\mathfrak{B}^n, \leq_n) \,\}$.

$\mathcal{G}$ is the group $\bigcup_{n \in \omega} G_n$. (Note that for $n < m$, $G_n \subseteq G_m$.) S is the set of all $E \subseteq A$ which satisfy the following conditions:

(a) $(\forall n \in \omega)(E \cap A_a^n \neq \emptyset$ for only finitely many $a \in A)$
(b) E is countable.

It is shown in [8] that NT, DC, and CAC are true in $\mathcal{N}_4$ and Krom ([13]) shows that there is a linearly ordered topological space that is not normal. Thus, LN is false.

Lemma 9. *NT, DC, and CAC are true in* $\mathcal{N}_4$*, but LN is false.*

It follows from Theorem 1 that NU and NT are true in $\mathcal{N}_2$.

Section 4. Summary. In the diagram below we indicate the relationship between the statements discussed in this paper.

$$\begin{tikzcd}
 & \text{MC} \arrow[drr] \arrow[ddl] \arrow[dddr] \arrow[dddrrr] & & & \text{DC} \arrow[dl] \arrow[dddll] \arrow[ddd] \\
 & & & \text{TF}_\sigma & \\
\text{LN} & & & & \\
 & & \text{NT} \arrow[dl] \arrow[dd] & & \text{NF}_\sigma \arrow[dd] \\
 & \text{NU} & & & \\
 & & \text{UT} \arrow[rr] & & \text{CMC} \arrow[d] \\
 & & & & \text{vDCP}(\omega)
\end{tikzcd}$$

$$\text{MC} \leftrightarrow \text{DUN} \leftrightarrow \text{DUU} \leftrightarrow \text{DUT} \leftrightarrow \text{NEFN}$$
$$\text{CMC} \leftrightarrow \text{DUN}(\omega) \leftrightarrow \text{DUU}(\omega) \leftrightarrow \text{DUT}(\omega) \leftrightarrow \text{UF}_\sigma$$

$$\mathcal{M}_1 \models \text{ vDCP}(\omega) \ \& \ \text{LN} \ \& \ \neg\text{MC} \ \& \ \neg\text{DC}$$

$$\mathcal{M}_2 \models \text{ DC} \ \& \ \text{CMC} \ \& \ \neg\text{MC}$$

$$\mathcal{N}_1 \models \neg\text{vDCP}(\omega)$$

$$\mathcal{N}_2 \models \text{ MC} \ \& \ \neg\text{DC}$$

$$\mathcal{N}_3 \models \text{ vDCP}(\omega) \ \& \ \neg\text{NU} \ \& \ \neg\text{LN} \ \& \ \neg \text{ CMC}$$

$$\mathcal{N}_4 \models \text{ NT \& DC \& CMC \& } \neg\text{LN}$$

It follows that none of the statements in our diagram above are provable in ZF^0. Neither DC nor MC imply any statements in the diagram above except where indicated. None of the following implications are reversible: MC $\rightarrow$ LN, MC $\rightarrow$ NT, MC $\rightarrow$ NF_σ, DC $\rightarrow$ NT, DC $\rightarrow$ NF_σ, and CMC $\rightarrow$ vDCP(ω). In fact, vDCP(ω) does not imply any of the other statements. Otherwise, we do not know any other relationships between these statements.

QUESTIONS. (1) Does either of the statements TF_σ or NU imply vDCP(ω)?
(2) Does CMC imply TF_σ or NF_σ?
(3) Does NU imply NT or UT?
(4) Is a normal subspace of a T space a T space?
(5) Is a normal subspace of a U space a U space?
(6) Is a U subspace of a T space a T space?

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