KINNA-WAGNER SELECTION PRINCIPLES, AXIOMS OF CHOICE AND MULTIPLE CHOICE

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Abstract. We study the relationships between weakened forms of the Kinna-Wagner Selection Principle (KW), the Axiom of Choice (AC), and the Axiom of Multiple Choice (MC).

Introduction. In this paper we study the relationships between weakened forms of the Kinna-Wagner Selection Principle (KW), the Axiom of Choice (AC), and the Axiom of Multiple Choice (MC). The forms of these statements that we shall use are given below.

AC: For every set x of of non-empty sets there is a function f (choice function) such that for each $u \in x$, $f(u) \in x$.

MC: For every set x of non-empty sets there is a function f (multiple choice function) such that for each $u \in x$, $f(u)$ is a non-empty finite subset of u.

KW: For every set x of sets of cardinality at least 2 there is a function f (Kinna-Wagner selection function) such that for each $u \in x$, $f(u)$ is a non-empty proper subset of u.

We weaken these statements by applying them to well ordered sets. The notation we shall use is as follows:

AC^{WO}: Every well ordered set of non-empty sets has a choice function.

AC_{WO}: Every set of non-empty well orderable sets has a choice function.

AC^{WO}_{WO}: Every well ordered set of non-empty well orderable sets has a choice function.

$AC_{<\aleph_0}$: Every set of non-empty finite sets has a choice function.

$AC^{WO}_{<\aleph_0}$: Every well ordered set of non-empty finite sets has a choice function.

AC_n: Every set of n-element sets has a choice function.

AC^{WO}_n: Every well ordered set of n-element sets has a choice function.

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We use similar notation for MC and KW. In addition, we have the following two forms.

MC_{even}: For every set x of sets , each of which has at least two elements, there is a multiple choice function f such that for each $u \in x$, the cardinality of $f(u)$ is even. (Keremedis [17].)

MC_{odd}: For every set x of non-empty sets, there is a multiple choice function f such that for each $u \in x$, the cardinality of $f(u)$ is odd. (Keremedis [17].)

KW is due to Kinna/Wagner [19]. (We first saw the form KW^{WO} in Brunner [4] and that gave us the idea for this paper.) KW is equivalent to the statement: For every set x there is an ordinal α such that $x \precsim \mathcal{P}(\alpha)$. (Since the power set of a well ordered set can be well ordered in every Fraenkel-Mostowski permutation model, KW implies AC in each such model.) The independence of MC from AC is due to Levy [21] and it follows from H. Rubin [28] that MC implies AC in ZF. (See also Felgner/Jech [9].) Thus, MC implies AC in every Cohen model. All proofs we give, unless otherwise stated, will be in $\text{ZF}^0 = \text{ZF} -$ axiom of foundation. In the independence results below we use both Cohen models and Fraenkel-Mostowski (FM) models.

Relations Between Forms. In the diagram below we indicate the relationships between these forms.

$$\begin{tikzcd}
 & KW \arrow[d] & & & \\
AC^{W0} \arrow[r] \arrow[dr] & KW^{W0} \arrow[dr] & & & \\
AC^{W0}_{<\aleph_0} & AC^{W0}_{W0} \arrow[l] \arrow[r] & KW^{W0}_{W0} \arrow[r] & KW^{W0}_{<\aleph_0} \arrow[dr] & \\
AC_{W0} \arrow[r] \arrow[ur] & KW_{W0} \arrow[r] \arrow[ur] & AC_{<\aleph_0} \arrow[r] \arrow[dr] & AC^{W0}_{<\aleph_0} \arrow[r] \arrow[u] & (\forall n)AC^{W0}_{n} \\
 & KW \arrow[u] & & (\forall n)AC_{n} \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
AC \arrow[r] \arrow[d] & AC^{W0} \arrow[r] \arrow[d] & AC^{W0}_{W0} \arrow[d] & AC_{W0} \arrow[l] \arrow[d] & AC \arrow[l] \arrow[d] \\
MC \arrow[r] & MC^{W0} \arrow[r] & MC^{W0}_{W0} & MC_{W0} \arrow[l] & MC \arrow[l]
\end{tikzcd}$$

$$AC_{<\aleph_0} + MC \leftrightarrow AC$$

$$AC^{WO}_{<\aleph_0} + MC^{WO} \leftrightarrow AC^{WO}$$

$$KW^{WO}_{<\aleph_0} + MC^{WO} \to KW^{WO}$$

$$AC^{WO}_{<\aleph_0} + MC^{WO}_{WO} \leftrightarrow AC^{WO}_{WO}$$

$$AC^{WO}_{<\aleph_0} + MC \leftrightarrow AC$$

$$KW^{WO}_{<\aleph_0} + MC^{WO}_{WO} \to KW^{WO}_{WO}$$

$$AC_{<\aleph_0} + MC_{WO} \leftrightarrow AC_{WO}$$

$$AC \to MC_{\text{odd}} \to MC$$

$$AC \to MC_{\text{even}} \to MC$$

The only implications which may not be clear are $KW_{WO} \to AC_{<\aleph_0}$ and $AC^{WO}_{<\aleph_0} + MC \to AC$. The latter implication follows from the fact that MC implies the statement: Each set is a union of a well ordered family of finite sets. (See Levy [21]. We would like to thank Kyriakos Keremedis for bringing this to our attention.)

Lemma 1. $KW_{WO} \to AC_{<\aleph_0}$.

Proof. We shall show that $KW_{<\aleph_0} \to AC_{<\aleph_0}$. Let x be a set of non-empty finite sets and let $y = \{u : (\exists v \in x) u \subseteq v\}$. By $KW_{<\aleph_0}$, there is a Kinna-Wagner selection function f on y. We define a choice function g on x as follows: Let $u \in x$ be such that $|u| = n$. There exists an $m \in \omega$, $m < n$, such that $f^m(u)$ is a singleton. Define $g(u)$ to be the single element of $f^m(u)$. □

In addition, we also have the following result:

Lemma 2. MC^{WO} *implies*

(a) The power set of an infinite set is Dedekind infinite.
(b) Every Dedekind finite, linearly ordered set is finite.

Proof. Let X be an infinite set and for each $n \in \omega$, let $X_n = \{\, Y \subseteq X : |Y| = n \,\}$. By MC^{WO}, there is a function f such that $\forall n \in \omega$, $f(X_n)$ is a finite non-empty subset of X_n. We define the sequence $(Y_n)_{n\in\omega}$ by induction as follows: $Y_0 = \emptyset$. Assume that Y_n is defined and let k be the least natural number such that $|Y_n| < k$. We let $Y_{n+1} = (\bigcup f(X_k)) \cup Y_n$. Since f$(X_k)$ is a set of k element sets $|Y_n| < k \leq |\bigcup f(X_k)| \leq |Y_{n+1}|$.

Part (a) follows immediately. To prove (b), let X be a Dedekind finite set which is linearly ordered by $\leq$. The assumption that X is infinite leads to a contradiction as follows: Using the construction above, we get a sequence $(Y_n)_{n\in\omega}$ of finite subsets of X such that $Y_n \subsetneq Y_{n+1}$ for all $n \in \omega$. If we define t_n to be the $\leq$-least element of $Y_{n+1} - Y_n$ then $(t_n)_{n\in\omega}$ is an infinite sequence of distinct elements of X and therefore, X is Dedekind infinite. $\square$

Let $MC_{/n}$ be the statement: For every set x of non-empty sets there is a function f such that for each $u \in x$, $f(u)$ is a non-empty finite subset of u such that $|u|$ and n are relatively prime. ($MC_{/1}$ is MC and $MC_{/2}$ is MC_{odd}.) Let MC_{ω^+} be the statement: $(\forall n \geq 1)MC_{/n}$. The following lemma was proved in Bleicher [3].

Lemma 3. $(\forall n)AC_n + MC \leftrightarrow MC_{\omega^+}$.

Proof. It is clear that $MC_{\omega^+} \to (\forall n)AC_n + MC$. We'll indicate how the proof goes the other way by an example. Suppose we want to derive $MC_{/p}$ where p is a prime. Suppose X is a set of non-empty pair-wise disjoint sets. MC implies that we may assume all the sets in X are finite. Suppose $u \in X$ is a set such that $|u| = np^m$, where n and m are positive integers. AC_p implies that there is a choice function f for $\{v \subseteq u : |v| = p\}$. For each $a \in u$ let s_a be number of times that f chooses a. Let s be the largest s_a that is not divisible by p^m. Then, $|\{a \in u : s_a = s\}| < |u|$. $\square$

Lemma 4. $MC_{\text{odd}} + MC_{\text{even}} \leftrightarrow AC$.

The proof is given in Keremedis [17].

Let $MC(2)$ be the statement: If x is a set of non-empty sets, then there is a multiple choice function f such that for each $u \in x$, $|f(u)|$ is a power of two.

Lemma 5. $MC(2) \to MC_{\text{even}}$.

Proof. Let x be a set of sets, each of whose members has at least two elements and let f be a multiple choice function on $y = \{z : (\exists u \in x) z \subseteq u\}$ that satisfies $MC(2)$. Then we define a multiple choice function g satisfying MC_{even} as follows. Let $u \in x$. If $|f(u)| > 1$, then $g(u) = f(u)$. Suppose $|f(u)| = 1$. If $|f(u - f(u))| = 1$, then $g(u) = f(u) \cup f(u - f(u))$. If $|f(u - f(u))| > 1$, then $g(u) = f(u - f(u))$. $\square$

Independence Results. We shall give 13 models, 5 Cohen models and 8 FM models.

$\mathcal{M}1$: This is Cohen's original model in which there are a countable number of generic reals, $a_1, a_2, \cdots$, along with the set b containing them. (Cohen [7].)

$\mathcal{M}2$: Feferman's model in which there is again a countable number of generic reals, but no set collecting them. (Feferman [8].)

$\mathcal{M}3$: Sageev's Model. Using iterated forcing, Sageev constructs $\mathcal{M}3$ by adding a countable number of generic tree-like structures to the ground model, a model of ZF + V = L. (Sageev [29].)

$\mathcal{M}4$: Pincus' Model I. Pincus constructs a generic extension $M[I]$ of a model M of ZF + class choice + GCH in which $I = \bigcup_{n\in\omega} I_n$, $I_{-1} = 2$ and I_{n+1} is a countable set of independent functions from ω onto I_n. (Pincus [25, 27].)

$\mathcal{M}5$: Pincus' Model II. Pincus proves that Cohen's model $\mathcal{M}1$ can be extended by adding $\aleph_1$ generic sets along with the set b containing them and well orderings of all countable subsets of b. (Pincus [26].)

We shall describe the FM models by giving the set of atoms A, the group of permutations of A, $\mathcal{G}$, and the ideal of supports, S.

$\mathcal{N}1$: The Basic Fraenkel Model. The set of atoms, A is countably infinite; $\mathcal{G}$ is the

$\mathcal{N}2$: The Second Fraenkel Model. $A = \{a_i : i \in \omega\}$. Let $B = \{\{a_{2i}, a_{2i+1}\} : i \in \omega\}$. $\mathcal{G}$ is the group of permutations on A that leaves B point-wise fixed. S is the set of all finite subsets of A.

$\mathcal{N}3$: Mostowski's linearly Ordered Model. A is countably infinite; $\precsim$ is a dense linear ordering on A without first or last elements $((A, \precsim) \cong (\mathbb{Q}, \leq))$; $\mathcal{G}$ is the group of all order automorphisms on $(A, \precsim)$; and S is the set of all finite subsets of A. (Mostowski [22].)

$\mathcal{N}4$: Levy's Model. $A = \{a_n : n \in \omega\}$ and $A = \bigcup\{P_n : n \in \omega\}$, where $P_0 = \{a_0\}$, $P_1 = \{a_1, a_2\}$, $P_2 = \{a_3, a_4, a_5\}$, $P_3 = \{a_6, a_7, a_8, a_9, a_{10}\}, \cdots$; in general for $n > 0$, $|P_n| = p_n$, where p_n is the nth prime. $\mathcal{G}$ is the group generated by $\{\pi_n : n \in \omega\}$, where, if $P_n = \{a_{m+1}, a_{m+2}, \cdots, a_{m+p_n}\}$, then

$$\pi_n : a_{m+1} \mapsto a_{m+2} \mapsto \cdots a_{m+p_n} \mapsto a_{m+1} \text{ and } \pi_n(x) = x, \text{ for all } x \notin P_n.$$

S is the set of all finite subsets of A. (Levy [21].)

$\mathcal{N}5$: A variation of $\mathcal{N}3$. A is countably infinite; $\precsim$ is a dense linear ordering on A without first or last elements $((A, \precsim) \cong (\mathbb{Q}, \leq))$; $\mathcal{G}$ is the group of all order automorphisms on $(A, \precsim)$; and S is the set of all bounded subsets of A. (Howard/Rubin/Rubin [13].)

$\mathcal{N}6$: A variation of $\mathcal{N}2$. $A = \bigcup B$, where B is a set of pairwise disjoint three element sets, $T_i = \{a_i, b_i, c_i\}$. For each $i \in \omega$ define a function $\eta_i : T_i \to T_i$ such that $\eta_i : a_i \mapsto b_i \mapsto c_i$. $\mathcal{G}$ is the group of all permutations ϕ of A such that for each $i \in \omega$, $\phi | T_i$ is either the identity, η_i, or η_i^2. S is the set of all finite subsets of A. (Howard [12].)

$\mathcal{N}7$: Another variation of $\mathcal{N}3$. $A = \bigcup\{B_n : n \in \omega\}$, where each B_n is countable and ordered like the rationals by $\leq_n$. Thus, for each $n \in \omega$, $(B_n, \leq_n) \cong (\mathbb{Q}, \leq)$. $\mathcal{G}$ is the group of all permutations ϕ on A such that for all $n \in \omega$ and for all $\phi \in \mathcal{G}$, ϕ is an order automorphism on $(B_n, \leq_n)$. S is the set of all finite subsets of A.

$\mathcal{N}8$: Brunner's Model. The set A is the disjoint union $A = \bigcup_{n\in\omega} B_n$ where $|B_n| = 2^n$ for all $n \in \omega$. For each $n \in \omega$, we let η_n be a permutation of B_n consisting of a single cycle of lenght 2^n and let η be the permutation of A that agrees with η_n on B_n for each $n \in \omega$. $\mathcal{G}$ is the group generated by η. S is the set of all finite subsets of A. This model is a special case of a general construction described by Brunner. The construction uses topological groups and is given in the lemma on page 72 of [6]. Our model is the special case where $E = \{2\}$.

Theorem 1. *$\mathcal{M}1 \models KW \wedge AC_{WO} \wedge \neg MC^{WO}$.*

Proof. Halpern/Levy [10] proved that KW is true in $\mathcal{M}1$. Jech gave a proof that AC_{WO} is also true. (See Jech [15, p82].) Cohen [7] proved that there is an infinite Dedekind finite subset of the reals in $\mathcal{M}1$. Thus, the fact that MC^{WO} is false follows from Lemma 2(b). □

Theorem 2. *$\mathcal{M}2 \models AC^{WO} \wedge \neg(\forall n)AC_n \wedge \neg MC$.*

Proof. Truss [30] has shown that AC^{WO} is true. Feferman [8] proved that there is an infinite set of pairs with no choice function so $(\forall n)AC_n$ is false and of course MC is false in every Cohen model in which AC is false. □

Theorem 3. $\mathcal{M}3 \models AC^{WO}_{<\aleph_0} \wedge \neg AC^{WO}_{WO} \wedge \neg MC$.

Proof. Sageev [29] proves that every set can be linearly ordered which implies $AC^{WO}_{<\aleph_0}$. Sageev also proves that there is a countable set of countable subsets of the reals with no choice function so AC^{WO}_{WO} is false. $\square$

Theorem 4. $\mathcal{M}4 \models AC_{<\aleph_0} \wedge \neg MC^{WO}_{WO} \wedge \neg KW^{WO}_{WO}$.

Proof. Pincus [25] proves that every set can be linearly ordered in $\mathcal{M}4$ which implies $AC_{<\aleph_0}$. It also follows that the set $\{I_n : n \in \omega\}$ is a countable set of countable sets that has neither a multiple choice function (MC^{WO}_{WO} is false) nor a Kinna-Wagner selection function (KW^{WO}_{WO} is false). $\square$

Theorem 5. $\mathcal{M}5 \models KW \wedge \neg MC^{WO}_{WO}$.

Proof. Pincus [26] proves that KW is true and KW implies $AC^{WO}_{<\aleph_0}$ so it is also true. Pincus also proves that there is a countable set of sets of cardinality $\aleph_1$ that does not have a choice function so AC^{WO}_{WO} is false. However, $AC^{WO}_{<\aleph_0} + MC^{WO}_{WO} \leftrightarrow AC^{WO}_{WO}$. Thus, MC^{WO}_{WO} is false. $\square$

Theorem 6. $\mathcal{N}1 \models AC^{WO}_{WO} \wedge KW^{WO} \wedge \neg(\forall n)AC_n \wedge \neg MC^{WO} \wedge \neg KW_{WO}$.

Proof. Since, in $\mathcal{N}1$ each element has a least support, it follows from Brunner [4 Proposition 4.5] that KW^{WO} is true. KW_{WO} is false because the set of finite subsets of A has no Kinna-Wagner selection function. To show that AC^{WO} is false, for each $n \in \omega$, $n > 0$, let B_n be the set of all 1-1 functions from n into A, and let $B = \{B_n : 0 < n \in \omega\}$. A choice function on B would lead to the contradiction that A has a countably infinite subset. Moreover, it is shown in Blass [1] that every set in $\mathcal{N}1$ that cannot be well ordered has an infinite amorphous (not the union of two disjoint infinite sets) subset and Brunner [4] shows that if this is the case then AC^{WO}_{WO} is true. Since $AC^{WO}_{WO} \rightarrow AC^{WO}_{<\aleph_0}$ and $AC^{WO}_{<\aleph_0} + MC^{WO} \leftrightarrow AC^{WO}$, it follows that MC^{WO} is false. The set of all two-element subsets of A has no choice function, so $(\forall n)AC_n$ is false. $\square$

Theorem 7. $\mathcal{N}2 \models MC_{\text{even}} \wedge \neg(\forall n)AC^{WO}_n \wedge \neg MC_{\text{odd}}$.

Proof. The set $B = \{\{a_{2i}, a_{2i+1}\} : i \in \omega\}$ is a well ordered set of two element sets that has no choice function so $(\forall n)AC^{WO}_n$ is false.

To prove MC_{even}, let M be a non-empty set in $\mathcal{N}2$, each of whose elements has at least two elements. The group $\mathcal{G}$ in $\mathcal{N}2$ is Abelian and each of its elements has order two. Let $\mathcal{F}$ be the filter determined by S, the set of finite subsets of the set of atoms, A. ($\mathcal{F}$ is generated by the set of all subgroups H of $\mathcal{G}$ such that there is an $E \in S$ such that H fixes E pointwise.) The filter $\mathcal{F}$ is in the model because each element in $\mathcal{G}$ has order two so $\mathcal{F}$ has null support. In fact,

$$(\forall H \in \mathcal{F})(\forall \sigma \in H)\sigma(H) = H. \tag{1}$$

Define a linear ordering $\leq$ on $\mathcal{F}$ which extends $\supseteq$, that is, if $H_1, H_2 \in \mathcal{F}$ and $H_1 \supseteq H_2$, then $H_1 \leq H_2$. The relation $\leq$ is in the model because of (1). (In fact, $\leq$ is a well ordering, but we do not need this.) Let W be the set of all sets in the model that have null support. Every permutation in $\mathcal{G}$ leaves W pointwise fixed so W can be well ordered by a relation R.

Suppose $X \in M$. We shall give an algorithm to construct a subset of X whose cardinality is a power of two. Let $Y = \{\mathcal{G}``\{x\} : x \in X\}$. Each element of Y

has null support so Y can be well ordered by R. Moreover, each element of Y is finite. (In fact, the cardinality of each element of Y is a power of two because $|\mathcal{G}``\{x\}| = |\mathcal{G}/\mathrm{sym}_{\mathcal{G}}(x)|$, where $\mathrm{sym}_{\mathcal{G}}(x) = \{\sigma \in \mathcal{G} : \sigma(x) = x\}$.) Let y be the R-first element of Y and let $x = y \cap X$. Let $\mathrm{fix}_{\mathcal{G}}(x)$ be the group of all $\sigma \in \mathcal{G}$ that fixes x pointwise and let

$$J = \{H \in \mathcal{F} : \mathrm{fix}_{\mathcal{G}}(x) \subseteq H \ \ \& \ \ (\exists z \in x) H``\{z\} \subseteq x\}.$$

Since $\mathrm{fix}_{\mathcal{G}}(x) \in J$, $J \neq \emptyset$. Moreover, J is finite and is linearly ordered by $\leq$. Let H_0 be the $\leq$-first element of J. We claim that if $z, w \in x$ and, $H_0``\{z\}$ and $H_0``\{w\}$ are both subsets of x, then $H_0``\{z\} = H_0``\{w\}$. The reason for this is as follows. By the definition of y ($x \subseteq y$), there is a $\sigma \in \mathcal{G}$ such that $\sigma(z) = w$. Let $K = H_0 \cup H_0\sigma$. Then $K``\{z\} = H_0``\{z\} \cup H_0``\{w\} \subseteq x$. We have $\mathrm{fix}_{\mathcal{G}}(x) \subseteq H_0 \subseteq K$ so $K \in J$, and $K \leq H_0$. However, $H_0$ is the $\leq$-first element of $J$, so $K = H_0$. Consequently, $H_0``\{z\} = H_0``\{w\}$.

Then, for each $X \in M$, we choose the unique $H_0``\{z\}$ which is a subset of $x \subseteq X$. $H_0``\{z\}$ has the same cardinality as the factor group, $H_0/\mathrm{sym}_{H_0}(z)$, which is a power of two.

It follows from Lemma 5 that MC_{even} is true and from Lemma 4 that MC_{odd} is false. $\square$

Theorem 8. $\mathcal{N}3 \models AC_{WO} \wedge \neg MC^{WO}$.

Proof. For each $i \in \omega - \{0\}$, let c_i be the set of all i-element subsets of A. Then the set $\{c_i : i \in \omega - \{0\}\}$ has no multiple choice function (MC^{WO} is false). For a proof that AC_{WO} is true see either Läuchli [20] or Jech [15, p53]. $\square$

Theorem 9. $\mathcal{N}4 \models (\forall n) AC_n \wedge MC_{\mathrm{odd}} \wedge \neg KW^{WO}_{<\aleph_0} \wedge \neg MC_{\mathrm{even}}$.

Proof. Levy [21] proves $(\forall n) AC_n$ and MC are true in $\mathcal{N}4$. The set $\{P_n : n \in \omega\}$ is a well ordered set of finite sets that has no Kinna-Wagner selection function so $KW^{WO}_{<\aleph_0}$ is false. By Lemma 3, $MC_{/2}$ is true in this model and $MC_{/2}$ is the same as MC_{odd}. It follows from Lemma 4 that MC_{even} is false. $\square$

Theorem 10. $\mathcal{N}5 \models AC^{WO} \wedge \neg MC_{WO} \wedge \neg AC_{<\aleph_0}$.

Proof. It is shown in Howard/Rubin/Rubin [13] that AC^{WO} is true. In Howard/Rubin [14], it is shown that in every FM model, $AC^{WO} + AC_{WO} \rightarrow$ AC. Thus, AC_{WO} is false. Howard [11] proves that in every FM model $AC_{<\aleph_0} \rightarrow AC_{WO}$. Therefore, $AC_{<\aleph_0}$ is false. MC_{WO} is also false because the set of all infinite bounded intervals of A is a set of well orderable sets which has no multiple choice function. $\square$

Theorem 11. $\mathcal{N}6 \models MC_{\mathrm{odd}} \wedge \neg(\forall n) AC^{WO}_n$.

Proof. The set B in $\mathcal{N}6$ is a well ordered set of triples that has no choice function so $(\forall n) AC^{WO}_n$ is false. The group $\mathcal{G}$ is an Abelian group in which each element has order three. We shall prove that if M is a set of non-empty sets then there is a multiple choice function f on M such that for each $X \in M$, $|f(X)|$ is a power of three. The proof is the same as the proof of MC_{even} in Theorem 7 up to and including the definition of $x = y \cap X$. Then we proceed as follows. For each $u \in x$, let $Q(u) = \{\sigma \in \mathcal{G} : \sigma(u) \in x\}$. Each $Q(u)$ has null support and is therefore in W. Let Q_0 be the R-first element of $\{Q(u) : u \in x\}$ and let three. Fix $u \in z$ and let $K = \{\sigma \in \mathcal{G} : \sigma(u) \in z\}$. K is a subgroup of $\mathcal{G}$ because $K = \{\sigma \in \mathcal{G} : \sigma^{-1}``Q_0 = Q_0\}$. Therefore, $|z| = |\mathcal{G}/\text{fix}_{\mathcal{G}}(u)|$ which is a power of 3. □

Theorem 12. $\mathcal{N}7 \models AC_{WO} \wedge \neg KW^{WO} \wedge \neg MC^{WO}$.

Proof. We can prove in essentially the same way as in $\mathcal{N}3$ that every set in $\mathcal{N}7$ can be linearly ordered. (See, for example, Jech [15] pp50-1.) Thus, it follows that $AC_{<\aleph_0}$ is true. Howard [11] proves that in every FM model $AC_{<\aleph_0} \rightarrow AC_{WO}$. (Alternatively, see the references in Theorem 8.)

The set $\{B_n : n \in \omega\}$ is a well ordered set that has no Kinna-Wagner selection function (KW^{WO} is false) and no multiple choice function (MC^{WO} is false). □

Theorem 13. $\mathcal{N}8 \models KW^{WO} \wedge \neg AC^{WO}_{<\aleph_0}$.

Proof. We first note that for $k \in \omega$, the subgroup of $\mathcal{G}$, $\{\phi : \phi$ fixes $\bigcup_{i \leq k} B_i$ pointwise$\}$ is the subgroup of $\mathcal{G}$ generated by η^{2^k}. and further that $\eta^{2^k}|B_j$ for $j > k$ has no fixed points. It follows that the set $\{\, B_n : n \in \omega \,\}$ has no choice function in $\mathcal{N}8$. Since this set is countable in $\mathcal{N}8$ and each of its elements is finite, $AC^{WO}_{<\aleph_0}$ is false.

Brunner proves in [6, page 72] that KW^{WO} is true in $\mathcal{N}8$. □

Conclusions. Using the models constructed in the last section, it follows that none of the implications in the diagram above are reversible. We have summarized these results in the tables below. The 19 forms we consider are the row and column headings. If the row form is the same as the column form, we put "↔" in that position in the table; if the row form implies the column form, "→" is in the table; and if the row form does not imply the column form we put the name of one of our models in which the row form is true and the column form is false. (The model may not be unique.) For the cases that we just have an FM model and the results are transferable to ZF, we will add the letter "T" to the FM model. (See Jech/Sochor [16] and Pincus [23,24,26].)

	AC	AC^{W0}	AC_{W0}	AC^{W0}_{W0}	$AC_{<\aleph_0}$	$AC^{W0}_{<\aleph_0}$	$(\forall n)AC_n$	$(\forall n)AC^{W0}_n$
AC	$\leftrightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$
AC^{W0}	$M2$	$\leftrightarrow$	$M2$	$\rightarrow$	$M2$	$\rightarrow$	$M2$	$\rightarrow$
AC_{W0}	$M1$	$M1$	$\leftrightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$
AC^{W0}_{W0}	$M1$	$M1$	$M2$	$\leftrightarrow$	$M2$	$\rightarrow$	$M2$	$\rightarrow$
$AC_{<\aleph_0}$	$M4$	$M4$	$M4$	$M4$	$\leftrightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$
$AC^{W0}_{<\aleph_0}$	$M3$	$M3$	$M3$	$M3$	$M2$	$\leftrightarrow$	$M2$	$\rightarrow$
$(\forall n)AC_n$	$M4$	$M4$	$M4$	$M4$	$N4T$	$N4T$	$\leftrightarrow$	$\rightarrow$
$(\forall n)AC^{W0}_n$	$M1$	$M1$	$M2$	$M3$	$M2$	$N4T$	$M2$	$\leftrightarrow$
MC	$N2$	$N2$	$N2$	$N2$	$N2$	$N2$	$N2$	$N2$
MC^{W0}	$M2$	$N2$	$M2$	$N2$	$N2$	$N2$	$M2$	$N2$
MC_{W0}	$M1$	$M1$	$N2$	$N2$	$N2$	$N2$	$N2$	$N2$
MC^{W0}_{W0}	$M1$	$M1$	$M2$	$N2$	$M2$	$N2$	$M2$	$N2$
MC_{even}	$N2$	$N2$	$N2$	$N2$	$N2$	$N2$	$N2$	$N2$
MC_{odd}	$N6$	$N6$	$N6$	$N6$	$N6$	$N6$	$N6$	$N6$
KW	$M1$	$M1$	$M5$	$M5$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$
KW^{W0}	$M1$	$M1$	$M2$	$M5$	$M2$	$N8$	$M2$	$\rightarrow$
KW_{W0}	$M1$	$M1$	$M5$	$M5$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$
KW^{W0}_{W0}	$M1$	$M1$	$M5$	$M5$	$M2$	$N8$	$M2$	$\rightarrow$
$KW^{W0}_{<\aleph_0}$	$M1$	$M1$	$M5$	$M5$	$M2$	$N8$	$M2$	$\rightarrow$

	MC	MC^{W0}	MC_{W0}	MC^{W0}_{W0}	MC_{even}	MC_{odd}	KW	KW^{W0}	KW_{W0}	KW^{W0}_{W0}	$KW^{W0}_{<\aleph_0}$
AC	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$
AC^{W0}	$M2$	$\rightarrow$	$N5T$	$\rightarrow$	$M2$	$M2$	$M2$	$\rightarrow$	$M2$	$\rightarrow$	$\rightarrow$
AC_{W0}	$M1$	$M1$	$\rightarrow$	$\rightarrow$	$M1$	$M1$	$N7T$	$N7T$	$\rightarrow$	$\rightarrow$	$\rightarrow$
AC^{W0}_{W0}	$M1$	$M1$	$N1T$	$\rightarrow$	$M1$	$M1$	$N1T$	$M2$	$N7T$	$\rightarrow$	$\rightarrow$
$AC_{<\aleph_0}$	$M4$	$M4$	$M4$	$M4$	$M4$	$M4$	$M4$	$M4$	$M4$	$M4$	$\rightarrow$
$AC^{W0}_{<\aleph_0}$	$M2$	$M4$	$M4$	$M4$	$M4$	$M4$	$M4$	$M4$	$M4$	$M4$	$\rightarrow$
$(\forall n)AC_n$	$M4$	$M4$	$M4$	$M4$	$M4$	$M4$	$M4$	$M4$	$M4$	$M4$	$N4T$
$(\forall n)AC^{W0}_n$	$M1$	$M1$	$M4$	$M4$	$M4$	$M4$	$M4$	$M4$	$M4$	$M4$	$N4T$
MC	$\leftrightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$N4$	$N2$	$N2$	$N2$	$N2$	$N2$	$N2$
MC^{W0}	$M2$	$\leftrightarrow$	$N5$	$\rightarrow$	$M2$	$M2$	$M2$	$N2$	$M2$	$N2$	$N2$
MC_{W0}	$M1$	$M1$	$\leftrightarrow$	$\rightarrow$	$M1$	$M1$	$N2$	$N2$	$N2$	$N2$	$N2$
MC^{W0}_{W0}	$M1$	$M1$	$N1$	$\leftrightarrow$	$M1$	$M1$	$M2$	$N2$	$M2$	$N2$	$N2$
MC_{even}	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\leftrightarrow$	$N2$	$N2$	$N2$	$N2$	$N2$	$N2$
MC_{odd}	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$N6$	$\leftrightarrow$	$N6$	$N6$	$N6$	$N6$	$N6$
KW	$M5$	$M5$	$M5$	$M5$	$M5$	$M5$	$\leftrightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$	$\rightarrow$
KW^{W0}	$M1$	$M1$	$M5$	$M5$	$M5$	$M5$	$M2$	$\leftrightarrow$	$M2$	$\rightarrow$	$\rightarrow$
KW_{W0}	$M1$	$M1$	$M5$	$M5$	$M5$	$M5$	$N7$	$N7$	$\leftrightarrow$	$\rightarrow$	$\rightarrow$
KW^{W0}_{W0}	$M1$	$M1$	$M5$	$M5$	$M5$	$M5$	$M2$	$N7$	$M2$	$\leftrightarrow$	$\rightarrow$
$KW^{W0}_{<\aleph_0}$	$M1$	$M1$	$M5$	$M5$	$M5$	$M5$	$M2$	$M4$	$M4$	$M4$	$\leftrightarrow$

(We would like to thank our typist, Betty Gick at Purdue University, for typing the diagram on page 2 and the above tables. It was a difficult job and she was very patient.)

Directions for further study. For each of the forms mentioned above we can replace "WO" by "$\aleph_0$" and obtain countable forms of KW, AC and MC. Countable forms of the axiom of choice are well known. Countable forms of the multiple choice axiom are mentioned, for example, in Blass [2], Brunner [5], and Keremedis [18]. Countable forms of KW are natural extensions.

There are obvious implications between the countable forms and between the forms mentioned above and the countable forms. Moreover, the following are easy to verify:

1. $\mathcal{N}3 \models AC_{\aleph_0} \wedge \neg MC^{\aleph_0} \wedge \neg KW^{\aleph_0}$. ($AC_{\aleph_0} \rightarrow MC_{\aleph_0}$ and $AC_{\aleph_0} \rightarrow KW_{\aleph_0}$.)
2. $\mathcal{N}5 \models AC^{\aleph_0} \wedge \neg MC_{\aleph_0} \wedge \neg KW_{\aleph_0}$.
3. $\mathcal{M}4(\mathcal{N}7) \models \neg MC^{\aleph_0}_{\aleph_0} \wedge \neg KW^{\aleph_0}_{\aleph_0}$.

However, there are still some unanswered questions. We do not know, for example, whether MC implies $KW^{\aleph_0}_{\aleph_0}$ or whether KW implies $MC^{\aleph_0}_{\aleph_0}$. We also do not know the relationship between $KW^{\aleph_0}_{\aleph_0}$ and $KW^{\aleph_0}_{<\aleph_0}$.

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