

# COMPACTNESS IN COUNTABLE TYCHONOFF PRODUCTS AND CHOICE

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ABSTRACT. We study the relationship between the countable axiom of choice and the Tychonoff product theorem for countable families of topological spaces.

## 1. INTRODUCTION AND DEFINITIONS

Our goal is to study the role played by the countable axiom of choice in the proof of the Tychonoff compactness theorem for countable families of topological spaces. (We will denote these two statements by CAC and  $\Pi_{\mathcal{C}}^{\mathcal{C}}$  respectively. Complete definitions are given below.)

Kelley [kel] proved that the full Tychonoff compactness theorem, TCT, (which asserts that the product of any family of compact topological spaces is compact in the Tychonoff topology) is equivalent to the full axiom of choice, AC. In fact, he proved that TCT restricted to the class of  $T_1$  spaces is equivalent to AC. TCT restricted to  $T_1$  spaces implies the full TCT because for every compact topology, there is a definable, compact,  $T_1$  extension. (The details of the argument are given in Theorem 4.)

A similar proof is not possible for TCT restricted to families of  $T_2$  spaces because, in general, given a compact space  $(X, T)$  one cannot find a compact  $T_2$  topology on  $X$  extending  $T$ . In fact, H. Rubin and D. Scott [rub] and J. Łoś and C. Ryll-Nardzewski [jn] have shown independently that TCT restricted to the class of all  $T_2$  spaces is equivalent to the Boolean prime ideal theorem BPI (form 14 in [hr]) which is known to be strictly weaker than the AC. (See [hl].)

What happens if we restrict TCT to *countable* families of arbitrary compact topological spaces to get  $\Pi_{\mathcal{C}}^{\mathcal{C}}$ ? A straightforward modification of the proof in [kel] yields a proof that  $\Pi_{\mathcal{C}}^{\mathcal{C}}$  implies CAC. We will be concerned with

**Question 1.** Does CAC imply  $\Pi_{\mathcal{C}}^{\mathcal{C}}$ ?

This was (part of) question 7.3 on page 89 of [gt]. In what follows we will give several partial answers. Question 1 in its full generality remains unanswered.

We will use the following terminology from topology and set theory:

**Definition 1.** Assume that  $(X, T)$  is a topological space and  $A \subseteq X$ .

$x$  is a *complete accumulation point* of  $A$  iff  $|O \cap A| = |A|$  for every neighborhood  $O$  of  $x$ .

$x$  is a *cluster point* of the sequence  $(x_n)_{n \in \omega}$  iff for every neighborhood  $O_x$  of  $x$ ,

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$x_n \in O_x$  for infinitely many values of  $n$ .

$(X, T)$  is *compact* ( $C$  for abbreviation) iff every open cover of  $X$  has a finite subcover.

$(X, T)$  is *countably compact* ( $cc$  for abbreviation) iff each countable open cover has a finite subcover.

$(X, T)$  is *sequentially compact* ( $sc$  for abbreviation) iff each sequence  $(x_n)_{n \in \omega}$  of  $X$  has a convergent subsequence.

$(X, T)$  is *limit point compact* ( $slc$  for abbreviation) iff every sequence  $(x_n)_{n \in \omega}$  of  $X$  has a cluster point.

$(X, T)$  is Lindelöf ( $lin$  for abbreviation) iff every open cover of  $X$  has a countable subcover.

For product spaces, we shall also use the abbreviation *proj* to mean "The projection of closed sets is closed."

Following is a list of the abbreviations we will be using for topological and set theoretic statements. All products are assumed to have the Tychonoff topology. Each of these statements is a consequence of AC.

$\Pi_C^C$  (form 113 in [hr]) is the assertion "The product of a countable family of compact topological spaces is compact."

$\Pi_{proj}^C$  is the assertion "For a product of a countable family of compact spaces projections are closed."

We will also consider topological spaces with some combination of the following topological properties:  $T_1$ ,  $T_2$ , first countable (abbreviated 1C), second countable (abbreviated 2C), and separable (abbreviated Sep). If  $S$  is a subset of  $\{T_1, T_2, 1C, 2C, Sep\}$ , then  $\Pi_C^C(S)$  is the assertion "The product of a countable family of compact spaces, all of which satisfy all of the properties in  $S$ , is compact." Similarly,  $\Pi_{proj}^C(S)$  is the assertion "For a product of a countable family of compact spaces, all of which satisfy all of the properties in  $S$ , projections are closed." If  $A$  and  $B$  are in  $S$  we will denote  $\Pi_C^C(\{A, B\})$  by  $\Pi_C^C(A, B)$ . Similarly for other subsets of  $S$  and for the other statements.

$\Pi_{sc}^{sc}$  abbreviates "The product of a countable family of  $sc$  spaces is an  $sc$  space."

$\Pi_{sc}^C$  abbreviates "The product of a countable family of compact  $sc$  spaces is an  $sc$  space."

$\Pi_{slc}^C$  abbreviates "The product of a countable family of compact spaces is an  $slc$  space."

$\Pi_{cc}^C$  abbreviates "The product of a countable family of compact spaces is a  $cc$  space."

$\Pi_{lin}^C$  abbreviates "The product of a countable family of compact spaces is Lindelöf."

$UF_\omega$  is our short hand for "There exists a non-principal ultrafilter on  $\omega$  (form 70 in [hr])."

The axiom of *dependent choices* DC (form 43 in [hr]) is the statement:

*If  $R$  is a non-empty relation on a non-empty set  $X$  such that  $\forall x \exists y xRy$ , then there exists a function  $f : \omega \rightarrow X$  such that  $f(n)Rf(n+1)$  for all  $n \in \omega$ .*

The *countable axiom of choice* CAC (form 8 in [hr]) is the assertion:

*For every set  $\mathcal{A} = \{A_i : i \in \omega\}$  of non-empty disjoint sets there exists a set  $C$  consisting of one and only one element from each element of  $\mathcal{A}$ .*

The *axiom of choice restricted to subsets of the reals*,  $AC\mathbb{R}$ , (form 79 in [hr]) is the assertion:

*Every set whose elements are non-empty subsets of  $\mathbb{R}$  has a choice function.*

$CAC_\omega$  (form 32 in [hr]) is CAC with the additional requirement that the members of  $\mathcal{A}$  are countable sets.

$CAC_{\text{fin}}$  (form 10 in [hr]) is CAC restricted to families  $\mathcal{A}$  of finite non-empty sets.

## 2. SUMMARY OF RESULTS

It is known that DC implies  $\Pi_C^C$ . This was proved in an unpublished paper by D. Pincus. Wright [wr] published recently a proof, well known in Madison, Wisconsin as he points out, that AC implies TCT and this proof can be easily adapted to show that DC implies  $\Pi_C^C$ . We will show that DC implies  $\Pi_{sc}^{sc}$  (Theorem 3). The question ‘‘Does  $\Pi_C^C$  imply DC?’’ was asked in [gt] (question 7.3 on page 89). It was answered negatively by P. Howard and J. E. Rubin in [hr1]. (Also see [hr], model  $\mathcal{N}38$ .)

With regard to the question 1, (Does CAC imply  $\Pi_C^C$ ) we will show that  $\Pi_C^C$  is equivalent to  $\Pi_{proj}^C$  (Corollary 20) and that, under the assumption of CAC,  $\Pi_C^C$  and  $\Pi_{slc}^C$  are equivalent (Corollary 7). Therefore, to show CAC implies  $\Pi_C^C$  it would be sufficient to show either that CAC implies  $\Pi_{proj}^C$  or that CAC implies  $\Pi_{slc}^C$ . We will also show that CAC is equivalent to the countable compactness theorem  $\Pi_C^C(2C)$  and also to  $\Pi_{proj}^C(2C)$  (Corollary 23). Finally, we will show that  $CAC + UF_\omega$  implies  $\Pi_C^C$  (Corollary 13),  $CAC + AC\mathbb{R}$  implies  $\Pi_C^C$  (Corollary 14), and that  $CAC + \Pi_{lin}^C$  implies  $\Pi_C^C$  (Corollary 18). Therefore, since  $UF_\omega$  is true in every Fraenkel-Mostowski model, the independence of  $\Pi_C^C$  from CAC cannot be shown by means of such a model. A summary of our results is given by the diagram below.

Some of the implications necessary to establish the diagram follow immediately from the definitions:

Let

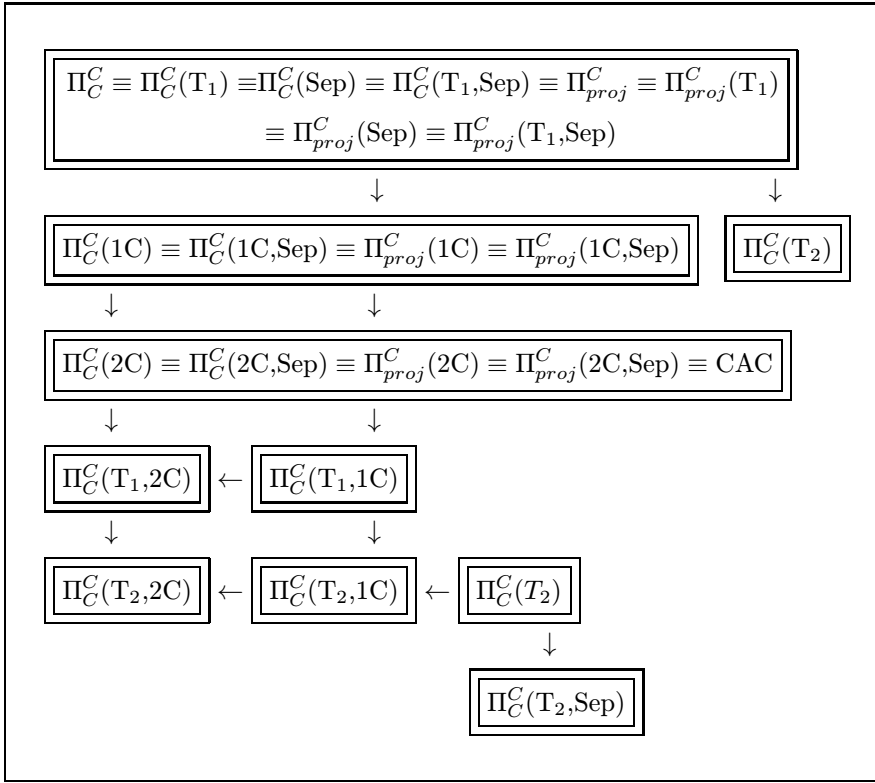
$$P = \{T_1, T_2, 1C, 2C, \text{Sep}\}$$

If  $S_1, S_2 \subseteq P$  and  $S_1 \subseteq S_2$ , then  $\Pi_C^C(S_1) \rightarrow \Pi_C^C(S_2)$  and  $\Pi_{proj}^C(S_1) \rightarrow \Pi_{proj}^C(S_2)$ . Also, since every  $T_2$  space is  $T_1$  and every  $2C$  space is  $1C$ , it follows that if  $S \subseteq P$ ,  $\Pi_C^C(S \cup \{T_1\}) \rightarrow \Pi_C^C(S \cup \{T_2\})$ ,  $\Pi_{proj}^C(S \cup \{T_1\}) \rightarrow \Pi_{proj}^C(S \cup \{T_2\})$ ,  $\Pi_C^C(S \cup \{1C\}) \rightarrow \Pi_C^C(S \cup \{2C\})$ , and  $\Pi_C^C(S \cup \{1C\}) \rightarrow \Pi_C^C(S \cup \{2C\})$ .

We will prove  $\Pi_C^C(S)$  implies  $\Pi_{proj}^C(S)$  for any subset  $S$  of  $P$  (Theorem 5);  $\Pi_C^C(T_1) \rightarrow \Pi_C^C$  (Theorem 4); if  $S \subseteq \{T_1, \text{Sep}\}$  or  $S \subseteq \{1C, 2C, \text{Sep}\}$ ,  $\Pi_{proj}^C(S) \rightarrow CAC$  (Theorem 19) and  $\Pi_{proj}^C(S) \rightarrow \Pi_C^C(S)$  (Corollary 20); if  $S \subseteq \{1C, 2C\}$ ,  $\Pi_C^C(S \cup \{\text{Sep}\}) \rightarrow \Pi_C^C(S)$  (Corollary 21); if  $S \subseteq P$ ,  $CAC + \Pi_C^C(S) \rightarrow \Pi_C^C(S \cup \{\text{Sep}\})$  (Theorem 9);  $\Pi_{proj}^C(2C) \leftrightarrow \Pi_C^C(2C) \leftrightarrow CAC$  (Corollary 23); and  $\Pi_C^C(1C) \leftrightarrow \Pi_C^C(1C, \text{Sep}) \leftrightarrow \Pi_{proj}^C(1C) \leftrightarrow \Pi_{proj}^C(1C, \text{Sep})$ , (Corollary 27).

In addition to the results that follow from the diagram, we note the following:

1. If  $S \subseteq P$ ,  $CAC + \Pi_{proj}^C(S \cup \{T_1\}) \rightarrow \Pi_C^C(S \cup \{T_1\})$  (Corollary 11).
2. If  $S \subseteq P$ ,  $\Pi_C^C(S) \leftrightarrow \Pi_{cc}^C(S)$  (Theorem 6).
3. If  $S \subseteq P$ ,  $\Pi_{proj}^C(S) \rightarrow CAC_{\text{fin}}$  (Theorem 28).



Although we have constructed no new models for set theory, we note that the following independence results follow from existing models.

4. In Cohen's original model, model  $\mathcal{M}1$  in [hr],  $\Pi_C^C(T_2)$  (form 154 of [hr]) holds but CAC (form 8 of [hr]) is false. Therefore, the implication of 1 above is not reversible. It also follows that  $\Pi_C^C(T_2)$  implies neither  $\Pi_C^C(2C)$  nor anything above  $\Pi_C^C(2C)$  in the diagram. Similarly,  $\Pi_{proj}^C(T_2)$  does not imply CAC.

In addition to question 1, we ask the following:

**Question 2.** Which implications in diagram are reversible (other than those which are not reversible by virtue of the fact that  $\Pi_C^C(T_2)$  does not imply CAC).

**Question 3.** What arrows can be added to the diagram? In particular does  $\Pi_C^C(1C)$  imply  $\Pi_C^C(T_2)$  or does  $\Pi_C^C(T_1, 1C)$  imply  $\Pi_C^C(2C)$ ?

**Question 4.** Does CAC imply either "The countable product of compact separable spaces is an slc space." or "The countable product of compact  $T_1$  spaces does not have a countable closed discrete subspace."

**Question 5.** Is the statement "The countable product of sc metric spaces is an sc space." provable in  $ZF^0$ ?

**Question 6.** For what properties  $A$  and  $B$  does  $\Pi_{proj}^C(A, B) \rightarrow \Pi_C^C(A, B)$  or  $\Pi_{proj}^C(A) \rightarrow \Pi_C^C(A)$  (other than those shown in the diagram)?

## 3. PRELIMINARY RESULTS

**3.1 cc, sc, and slc.** In this section we consider briefly the relationships between the properties cc, sc and slc. It is clear that every sc space is slc. It is also true that every cc space is slc. For the proof, let  $(x_n)_{n \in \omega}$  be a sequence in a cc space  $X$ . Then  $G = \{\overline{\{x_m : m \geq n\}} : n \in \omega\}$  is a countable family of closed sets with the finite intersection property. Therefore  $\bigcap G \neq \emptyset$ . Any point in  $\bigcap G$  is a cluster of  $(x_n)_{n \in \omega}$ . These are the only relations between cc, sc, and slc known to us without additional assumptions on the spaces and without any form of the axiom of choice. However, under the assumption of CAC it is easy to prove every slc space is cc.

**Lemma 1.** *CAC implies that every slc space is cc.*

*Proof.* Let  $G = \{G_n : n \in \omega\}$  be a strictly increasing open cover of an slc space. If  $G$  has no finite subcover then choosing  $x_n \in G_{n+1} \setminus G_n$  gives a sequence with no cluster points.  $\square$

In fact, the assertion that every slc space is cc implies CAC.

**Lemma 2.** *If every slc space is cc then CAC holds.*

*Proof.* Let  $X = \{X_i : i \in \omega\}$  be a countable set of non-empty, pairwise disjoint sets. It suffices by [ker] to prove that there is an infinite subset of  $X$  with a choice function. Let  $T = \{U \subseteq \bigcup X : \forall i \in \omega, (U \cap X_i \neq \emptyset \rightarrow U \cap X_i \text{ is cofinite in } X_i)\}$ . The topology  $T$  on  $\bigcup X$  is not cc. Under the assumption that every slc space is cc we may conclude that  $T$  is not slc. There is, therefore, a sequence in  $\bigcup X$  which does not have a cluster point. From such a sequence we can construct a choice function for an infinite subset of  $X$ .  $\square$

We also note that even under the assumption of AC a cc space need not be sc. An example is provided by  $2^{\mathbb{R}}$ .

Under the assumption that the spaces are first countable (and without any “choice”) we have, in addition to  $sc \rightarrow slc$  and  $cc \rightarrow slc$ , that every slc space is sc. (Let  $(x_n)_{n \in \omega}$  be a sequence with cluster point  $y$  in an slc space. Let  $(U_n)_{n \in \omega}$  be a countable decreasing neighborhood base for  $y$ . Define the sequence  $(n_j)_{j \in \omega}$  by induction:  $n_{k+1} =$  the smallest natural number  $j$  such that  $j > n_k$  and  $x_j \in U_j$ . Then  $(x_{n_j})_{j \in \omega}$  is a convergent subsequence of  $(x_n)_{n \in \omega}$ .) It follows that for first countable spaces, under the assumption of CAC, all three properties cc, sc and slc are equivalent.

If we assume that the spaces are countable (and do not assume any “choice”) we can prove (in addition to  $sc \rightarrow slc$  and  $cc \rightarrow slc$ ) that every slc space is cc. The argument is almost identical to the proof that CAC implies  $slc \rightarrow cc$ . The only difference is that rather than appealing to CAC to choose  $x_n$  we use the well ordering of the countable slc space. As far as we can see CAC gives no other implications for countable spaces. However, with the assumption of DC we are able to prove that every countable slc space is sc. This follows from the fact that DC implies that if  $X$  is an slc topological space and  $\mathbf{t} = (t_n)_{n \in \omega}$  is a sequence in  $X$  with no convergent subsequence, then  $\mathbf{t}$  has uncountably many cluster points. (The proof is straight forward.)

### 3.2 DC implies $\Pi_{sc}^{sc}$ .

**Theorem 3.** *DC implies  $\Pi_{sc}^{sc}$ .*

*Proof.* Fix  $A = \{(X_i, T_i) : i \in \omega\}$  a family of sc spaces and let  $X = \prod_{i \in \omega} X_i$  be the Tychonoff product of the family  $A$ . Let  $(\mathbf{x}_n)_{n \in \omega}$  be a sequence in  $X$  and let  $S$  be the set of all subsequences of  $(\mathbf{x}_n)_{n \in \omega}$ . We say that an element  $\vec{y} = (\mathbf{y}_n)_{n \in \omega}$  converges at level  $i$  if the projection of  $\vec{y}$  onto the  $i$ th coordinate,  $(\pi_i(\mathbf{y}_n))_{n=0}^\infty$ , converges in  $X_i$ . The convergence height of  $\vec{y}$  is the least  $j$  such that  $\vec{y}$  does not converge at level  $j$ .

If there is a  $\vec{y} \in S$  for which the convergence height is not defined, then  $\vec{y}$  converges at every level. Say  $(\pi_i(\mathbf{y}_n))_{n=0}^\infty$  converges to  $t_i$  for each  $i \in \omega$ . In this case  $\vec{y}$  converges to  $\mathbf{t} = (t_i)_{i \in \omega}$  and therefore  $(\mathbf{x}_n)_{n \in \omega}$  has a convergent subsequence.

We may therefore assume that the every  $\vec{y} \in S$  has a convergence height. We define an ordering  $\prec$  on  $S$  as follows:  $\vec{z} \prec \vec{y}$  if  $\vec{y}$  is a subsequence of  $\vec{z}$  and the convergence height of  $\vec{y}$  is strictly greater than that of  $\vec{z}$ . (Note that the convergence height of a subsequence of  $\vec{z}$  must be greater than or equal to the convergence height of  $\vec{z}$ .)

We now verify that the hypothesis of DC is satisfied for the ordering  $\prec$ . Assume that  $\vec{z} \in S$  has convergence height  $j$ . It suffices to find a subsequence of  $\vec{z}$  which converges at level  $j$ . This can be done by using the fact that  $X_j$  is an sc space.

Let  $\vec{y}_0$  be an element of  $S$  that converges at level 0 (that is, it has convergence height 1). Applying DC we obtain a sequence  $(\vec{y}_k)_{k \in \omega}$  such that for all  $k \in \omega$ ,  $\vec{y}_k \prec \vec{y}_{k+1}$ . It is clear that for every  $k \in \omega$ ,  $\vec{y}_k$  converges at level  $k$ . Say  $(\pi_k(\vec{y}_k)_n)_{n=0}^\infty$  converges to  $t_k$ . Then for  $r \in \omega$ ,  $r > k$ ,  $\vec{y}_r$  also converges to  $t_k$  at level  $k$ .

We define the sequence  $\vec{s} = (\mathbf{s}_n)_{n \in \omega}$  in  $S$  by  $s_n = (\vec{y}_n)_n$ . We leave to the reader the proof that  $\vec{s}$  converges to  $\mathbf{t}$ .  $\square$

#### 4. $\Pi_C^C(S)$ AND $\Pi_{proj}^C(S)$

In this section and the following sections we will let  $S$  denote an arbitrary subset of the set  $\{T_1, T_2, 1C, 2C, \text{Sep}\}$  of properties. We will see what can be said in general about  $\Pi_C^C(S)$  and  $\Pi_{proj}^C(S)$ . We first note that it is clear that  $\Pi_C^C(S)$  implies  $\Pi_C^C(S')$  and  $\Pi_{proj}^C(S)$  implies  $\Pi_{proj}^C(S')$  if  $S' \subseteq S$  or if every space with the properties in  $S$  has all of the properties in  $S'$ . In addition we have:

**Theorem 4.**  $\Pi_C^C(T_1)$  implies  $\Pi_C^C$ .

*Proof.* Let  $\{(X_i, T_i) : i \in \omega\}$  be a countable family of compact spaces. Let  $(X, T)$  be the Tychonoff product. For each  $i \in \omega$ , let  $C_i$  be the  $T_1$  topology generated by the cofinite subsets of  $X_i$  and let  $R_i$  be the topology generated by  $T_i \cup C_i$ . Then  $(X_i, R_i)$  is  $T_1$  and compact. (The union of two compact topologies is compact.) By  $\Pi_C^C(T_1)$ , the product  $(X, R)$  of the family  $\{(X_i, R_i) : i \in \omega\}$  is compact. Since the original topology  $T$  is weaker than  $R$ , it follows that  $(X, T)$  is also compact.  $\square$

We note that if  $\emptyset \neq S \subseteq \{1C, 2C, \text{Sep}\}$  and  $(X_i, T_i)$  has the properties in  $S$ , then  $(X_i, T_i \cup C_i)$ , where  $C_i$  is the cofinite topology, need not have the properties in  $S$ .

**Theorem 5.**  $\Pi_C^C(S)$  implies  $\Pi_{proj}^C(S)$ .

*Proof.* Let  $X = \prod_{i \in \omega} X_i$  be a product of compact topological spaces. Assume that  $X$  is compact. It suffices to show that for each  $i \in \omega$ , the projection  $\pi_i$  on to  $X_i$  is closed. We will show that  $\pi_0$  is a closed map. The argument for any other  $i \in \omega$  is similar. Assume that  $C$  is a closed set in  $X$ . To show that  $C_0 = \pi_0(C)$  is closed, let  $x \in X_0 \setminus C_0$ . We will find a neighborhood of  $x$  which is disjoint from  $C_0$ . For each  $\mathbf{y} = (x, y_1, y_2, \dots) \in \pi_0^{-1}(x)$ ,  $\mathbf{y} \notin C$  and therefore there is a neighborhood  $N \times \mathbf{M}$  of  $\mathbf{y}$  where  $N$  is a neighborhood of  $x$  in  $X_0$  and  $\mathbf{M}$  is a neighborhood of  $(y_1, y_2, \dots)$  in  $Y = \prod_{i > 0} X_i$ . Hence, we obtain an open cover of  $Y$  if we take all opens sets  $\mathbf{M}$  such that for some neighborhood  $N$  of  $x$  in  $X_0$ ,  $N \times \mathbf{M}$  is disjoint from  $C$ .  $Y$  is compact because  $X$  is compact. We may therefore choose a finite subcover  $\mathbf{M}_1, \dots, \mathbf{M}_n$  of  $Y$  and corresponding neighborhoods  $N_1, \dots, N_n$  of  $x$  such that for  $1 \leq j \leq n$ ,  $N_j \times \mathbf{M}_j$  is disjoint from  $C$ . Then  $N_1 \cap \dots \cap N_n$  is a neighborhood of  $x$  which is disjoint from  $C_0$ .  $\square$

**Theorem 6.**  $\Pi_C^C(S)$  if and only if  $\Pi_{cc}^C(S)$ .

*Proof.* The implication from left to right is clear. For the proof in the other direction fix  $A = \{(X_i, T_i) : i \in \omega\}$  a countable family of compact spaces each of which has the properties in  $S$  and let  $X = \prod_{i \in \omega} X_i$ , their Tychonoff product, be countably compact. Let  $F = \{F_j : j \in J\}$  be a family of closed sets having the finite intersection property. For every  $n \in \omega$ , put  $Y_n = \prod_{i \in n} X_i$  and let  $\pi_n$  denote the projection of  $X$  onto  $Y_n$ . For every  $n \in \omega$  set

$$F^n = \{\overline{\pi_n(F_j)} : j \in J\}, \quad H^n = F^n \times \prod_{i \geq n} X_i.$$

As each  $Y_n$  is compact, it follows that each  $F^n \neq \emptyset$  and consequently  $H = \{H^n : n \in \omega\}$  is a countable descending family of non-empty closed sets in  $X$ . Hence,  $\bigcap H \neq \emptyset$ . Fix  $x \in \bigcap H$ . We claim that  $x \in \bigcap F$ . Fix

$$O_x = O_{i_0} \times O_{i_1} \times O_{i_2} \times \dots \times O_{i_n} \times \prod_{i \neq i_0, i_1, \dots, i_n} X_i, \quad O_{i_j} \in T_{i_j}, \quad j = 0, \dots, n$$

a neighborhood of  $x$ . Without loss of generality we may assume that  $i_0 = 0, i_1 = 1, \dots, i_n = n$ . Then  $O_0 \times O_1 \times \dots \times O_n$  is a neighborhood of  $x|(n+1)$  in  $\prod_{i \in n+1} X_i$ .

Hence,

$$\pi_{n+1}(F_j) \cap (O_0 \times O_1 \times \dots \times O_n) \neq \emptyset$$

and, consequently,  $O_x \cap F_j \neq \emptyset$ . As  $O_x$  was arbitrary, we see that  $x \in F_j$  and this holds for every  $j \in J$ . Thus,  $x \in \bigcap F$  as required.  $\square$

**Corollary 7.** *CAC* implies  $(\Pi_C^C(S)$  if and only if  $\Pi_{slc}^C(S)$ ).

*Proof.* It is clear that  $\Pi_C^C(S)$  implies  $\Pi_{slc}^C(S)$ . By Lemma 1, *CAC* implies that slc spaces are cc. Therefore, by Theorem 6, *CAC* +  $\Pi_{slc}^C(S)$  implies  $\Pi_C^C(S)$ .  $\square$

Restating Corollary 7, we get the result:

*CAC* implies  $(\Pi_C^C(S)$  if and only if every countably infinite sequence in a countable product of compact spaces, each of which has the properties in  $S$ , has a cluster point).

If the spaces in Corollary 7 are  $T_1$ , we have the following consequence.

**Corollary 8.** *CAC implies  $(\Pi_C^C(S \cup \{T_1\}))$  if and only if a countable product of compact spaces each of which has the properties in  $S \cup \{T_1\}$  has no countably infinite, closed, discrete subsets).*

**Theorem 9.** *CAC implies  $(\Pi_C^C(S \cup \{\text{Sep}\}))$  if and only if  $\Pi_C^C(S)$ .*

*Proof.* We may assume that  $\text{Sep} \notin S$ . It is clear that  $\Pi_C^C(S)$  implies  $\Pi_C^C(S \cup \{\text{Sep}\})$ . For the other implication, assume CAC and  $\Pi_C^C(S \cup \{\text{Sep}\})$ . Let  $A = \{A_i : i \in \omega\}$  be a family of compact spaces each of which has the properties in  $S$ . By Corollary 7, it suffices to show that every countably infinite sequence in the product  $X = \prod_{i \in \omega} A_i$ , has a cluster point. Let  $\{x_n : n \in \omega\}$  be a countably infinite sequence in  $X$ . For  $i \in \omega$ , let  $B_i = \overline{\pi_i(\{x_n : n \in \omega\})}$  ( $\pi_i$  is the projection on  $A_i$ ). Since  $\text{Sep} \notin S$ , all of the properties in  $S$  are hereditary and, therefore, each of the spaces  $B_i$  (with the subset topology) has all of the properties in  $S$ . Further, each  $B_i$  is separable. Let  $Y = \prod_{i \in \omega} B_i$ . By  $\Pi_C^C(S \cup \{\text{Sep}\})$ ,  $Y$  is compact. By Corollary 7, the set  $\{x_n : n \in \omega\}$  has a cluster point in  $Y \subseteq X$ .  $\square$

**Theorem 10.**  *$\Pi_{proj}^C(S)$  implies a countable product of compact spaces, each of which has the properties in  $S$ , has no countably infinite, closed, discrete subsets.*

*Proof.* Assume  $\Pi_{proj}^C(S)$ , let  $A = \{(X_i, T_i) : i \in \omega\}$  be a countable family of compact topological spaces each of which has the properties in  $S$ , and let  $G$  be a countably infinite closed subset of the Tychonoff product  $X = \prod_{i \in \omega} X_i$ . Let  $G = \{g_n : n \in \omega\}$ , where the function  $n \mapsto g_n$  is one to one. We will construct a cluster point for the sequence  $G$ . For  $i \in \omega$ , let  $\pi_i$  be the projection function of  $X$  onto  $\prod_{k \leq i} X_k$ . We claim that for each  $i \in \omega$ ,  $\pi_i$  is a closed map. We show this using  $\Pi_{proj}^C(S)$  as follows. It is easy to verify that a finite product of topological spaces, each of which has property  $A$ , also has property  $A$ , where  $A$  is any of the properties  $T_1, T_2, 1C, 2C$ , or  $\text{Sep}$ . Therefore,  $(X_1 \times \cdots \times X_i) \times X_{i+1} \times X_{i+2} \cdots$  is a product of the compact spaces  $(X_1 \times \cdots \times X_i), X_{i+1}, X_{i+2}, \dots$  where each of the factors satisfies all of the properties in  $S$ . Consequently, a projection onto the first factor  $(X_1 \times \cdots \times X_i)$  is a closed map. This projection is  $\pi_i$ .

We are also going to use the fact that if  $Y$  and  $Z$  are topological spaces,  $Z$  is compact and if  $t = (t_n)_{n \in \omega}$  is a sequence in  $Y \times Z$ , then for every cluster point  $p \in Y$  of the projection of  $t$  onto  $Y$  there is a  $q \in Z$  such that  $(p, q)$  is a cluster point of  $t$ . (Assume  $p$  is a cluster point of the projection of  $t$  onto  $Y$  and for no  $q \in Z$  is  $(p, q)$  a cluster point of  $t$ . Then the collection of all open sets  $U$  in  $Z$  such that for some neighborhood  $V$  of  $p$  in  $Y$ ,  $V \times U \cap \{t_n : n \in \omega\} = \emptyset$  covers  $Z$ . By the compactness of  $Z$ , there is a finite subcover  $U_1, \dots, U_n$  of  $Z$  and corresponding neighborhoods  $V_1, \dots, V_n$  of  $p$  such that  $V_i \times U_i \cap \{t_n : n \in \omega\} = \emptyset$  for  $1 \leq i \leq n$ . Then  $V_1 \cap \cdots \cap V_n$  is a neighborhood of  $p$  disjoint from the projection of  $t$  onto  $Y$ , a contradiction.)

We now choose a sequence  $q = (q_k)_{k \in \omega} \in X$  so that for any  $k \in \omega$ ,  $(q_0, q_1, \dots, q_k)$  is a cluster point of the sequence  $(\pi_k(g_n))_{n \in \omega}$ . Since any open neighborhood of  $q$  of the form  $U_1 \times \cdots \times U_k \times X_{k+1} \times X_{k+2} \cdots$  must contain a point of  $G$ ,  $q$  will be the desired cluster point of  $G$ .

To choose  $q_0$ , we note that the sequence  $(\pi_0(g_n))_{n \in \omega}$  must have a cluster point since  $X_0$  is compact. Further, since  $G$  is closed and  $\pi_0$  is a closed map all such cluster points are in  $\pi_0(G)$ . Let  $q_0$  be  $\pi_0(g_n)$  where  $n$  is the smallest natural number such that  $\pi_0(g_n)$  is a cluster point of  $(\pi_0(g_n))_{n \in \omega}$ . Assume that  $q_0, q_1, \dots, q_{k-1}$  have been chosen so that  $(q_0, q_1, \dots, q_{k-1})$  is a cluster point of  $(\pi_{k-1}(g_n))_{n \in \omega}$ .

Let  $Y = \prod_{i=0}^{k-1} X_i$ , let  $Z = X_k$ , and let  $t = (\pi_k(g_n))_{n \in \omega}$ .  $t$  is a sequence in  $Y \times Z$  and  $p = (q_0, q_1, \dots, q_{k-1})$  is a cluster point of the projection of  $t$  onto  $Y$ . Therefore, (using the fact proved above), there is a  $q \in Z = X_k$  such that  $(p, q) = (q_0, q_1, \dots, q_{k-1}, q)$  is a cluster point of  $t$ . Since  $\pi_k$  is a closed map and  $G$  is closed, every such  $(p, q)$  must be in  $\pi_k(G)$ . Let  $q_k = \pi_k(g_n)$  where  $n$  is the least natural number such that  $\pi_k(g_n) = (p, q)$  and  $\pi_k(g_n) = q_k$  is a cluster point of  $t = (\pi_k(g_n))_{n \in \omega}$ .  $\square$

As a consequence of Corollary 8 and Theorem 10 we have the corollary:

**Corollary 11.** *CAC implies  $(\Pi_C^C(S \cup \{T_1\}) \leftrightarrow \Pi_{proj}^C(S \cup \{T_1\}))$ .*

**Theorem 12.** *CAC +  $UF_\omega$  implies  $\Pi_{slc}^C$ .*

*Proof.* Let  $\{(X_i, T_i) : i \in \omega\}$  be a family of compact spaces and let  $X$  be their Tychonoff product. Let  $\mathbf{s} = (x_n)_{n \in \omega}$  be a sequence in  $X$ . Without loss of generality we may assume that no two terms of  $\mathbf{s}$  are equal. Let  $\mathcal{F}$  be a non-principal ultrafilter on  $\mathbf{s}$ . (Note that a non-principal ultrafilter on  $\mathbf{s}$  must contain all the cofinite subsets of  $\mathbf{s}$ .) For every  $m \in \omega$ , let  $B_m = \{x \in X_m : \pi_m^{-1}(O_x) \cap \mathbf{s} \in \mathcal{F}, \text{ for every neighborhood, } O_x \text{ of } x\}$ . As  $X_m$  is compact, it follows that  $B_m \neq \emptyset$  for all  $m \in \omega$ .

It is easy to see that any choice function  $b$  on  $B = \{B_m : m \in \omega\}$  is a cluster point of  $\mathbf{s}$  and, therefore,  $X$  is slc.  $\square$

**Corollary 13.** *CAC +  $UF_\omega$  implies  $\Pi_C^C$ .*

*Proof.* Let  $\{(X_i, T_i) : i \in \omega\}$  be a family of compact spaces and let  $X$  be their Tychonoff product. By Theorem 12,  $X$  is an slc space. It follows from Lemma 1, that, under CAC, every slc space is cc. Consequently, by Theorem 6,  $X$  is compact.  $\square$

**Corollary 14.** *CAC +  $AC\mathbb{R}$  implies  $\Pi_C^C$ .*

*Proof.*  $AC\mathbb{R}$  implies  $UF_\omega$ .  $\square$

**Lemma 15.**  *$AC\mathbb{R}$  implies that the countable product of sc  $T_2$  spaces is sc.*

*Proof.* Let  $A = \{(X_i, T_i) : i \in \omega\}$  be a family of sc  $T_2$  spaces and let  $X = \prod_{i \in \omega} X_i$  be their Tychonoff product. Let  $(x_n)_{n \in \omega}$  be a sequence in  $X$  and  $S$  be the set of all subsequences of  $(x_n)_{n \in \omega}$ . Clearly we may identify  $(x_n)_{n \in \omega}$  with  $\omega$  and  $S$  with  $\mathcal{P}(\omega)$ , and, hence, with the real line  $\mathbb{R}$ . Let  $f$  be a choice function on all non-empty sets of  $\mathcal{P}(\mathcal{P}(\omega))$ . Using an easy induction we construct a convergent subsequence of  $(x_n)_{n \in \omega}$ .

For  $n = 0$ ,  $X_0$  is an sc space. Hence, the set of all subsequences  $S_0$  of  $S$  such that  $\pi_0(s)$  converges in  $X_0$  for every  $s \in S_0$  is non empty. Put  $g_0 = f(S_0)$  and let  $x(0) \in X_0$  be the limit of  $\pi_0(g_0)$ . (Since the space is  $T_2$ , if a sequence converges, it converges to exactly one point.)

Assume that  $g_0, g_1, \dots, g_n \in S$  have been chosen so that  $g_j$  is a subsequence of  $g_{j-1}$  and  $\pi_j(g_j)$  converges to  $x(j) \in X_j$ , for all  $j = 0, 1, 2, 3, \dots, n$ .  $X_{n+1}$  is an sc space. Hence, the set of all subsequences  $S_{n+1}$  of  $g_n$  such  $\pi_{n+1}(s)$  converges in  $X_{n+1}$  for every  $s \in S_{n+1}$  is non-empty. Put  $g_{n+1} = f(S_{n+1})$  and let  $x(n+1) \in X_{n+1}$  be the limit of  $\pi_{n+1}(g_{n+1})$ . It can be readily verified that the diagonal  $(y_n)_{n \in \omega}$ , where  $y_n$  is the  $n$ th term of  $g_n$ , is a subsequence of  $(x_n)_{n \in \omega}$  converging to  $y$ , where  $y(n) = x(n)$  for all  $n \in \omega$ .  $\square$

Finally, in this section we shall show that  $CAC + \Pi_{lin}^C(S)$  implies  $\Pi_C^C(S)$  (Corollary 18). First we give some preliminary results.

**Theorem 16.** *Let  $\{A_i : i \in \omega\}$  be a family of topological spaces. If  $\{U_n : n \in \omega\}$  is a family of basic open sets in  $X = \prod_{i \in \omega} A_i$  such that for every  $a \in [\omega]^{<\omega}$ ,  $\bigcup_{n \in \omega} U_n \setminus \bigcup_{n \in a} U_n \neq \emptyset$ , then there exists a family of basic open sets  $\{V_n : n \in \omega\}$  satisfying the following properties:*

- (1)  $\forall n \in \omega, |\{i \in \omega : \pi_i(V_n) \neq A_i\}| = 1$ .
- (2)  $\forall i, n \in \omega, \pi_i(V_n) = \pi_i(U_n)$  or  $\pi_i(V_n) = A_i$ .
- (3)  $\forall a \in [\omega]^{<\omega}, \bigcup_{n \in \omega} V_n \setminus \bigcup_{n \in a} V_n \neq \emptyset$ .

*Proof.* For each basic open set  $U \subseteq X$ , let  $r(U) = \{i \in \omega : \pi_i(U) \neq A_i\}$ . (An element in the subbase of the product topology of the product  $X$  is a product  $\prod_{i \in \omega} B_i$  such that  $B_i = A_i$  for all but one value of  $i$ . A basic element is a finite intersection of subbase elements so that  $r(U)$  is finite. See, for example, [kel] page 90.) We shall define by induction on  $n \in \omega$ ,  $\{i(n) : n \in \omega\} \subseteq \omega$  and a family of basic open sets  $\{V_n : n \in \omega\}$  in  $X$  which satisfy the following conditions for each  $n \in \omega$ :

- (a)  $r(V_n) = \{i(n)\}$ .
- (b)  $\pi_{i(n)}(V_n) = \pi_{i(n)}(U_n)$ .
- (c)  $\forall a \in [\omega]^{<\omega}, \bigcup_{m \in \omega} U_m \not\subseteq \bigcup_{m \in a} U_m \cup \bigcup_{m \leq n} V_m$ .

Fix  $n \in \omega$ . Suppose that we have defined  $\{i(m) : m < n\}$  and  $\{V_m : m < n\}$  which satisfy the induction hypothesis. For each  $a \in [\omega]^{<\omega}$  and  $i \in \omega$ , let

$$W(n, a, i) = \bigcup_{m \in a} U_m \cup \bigcup_{m < n} V_m \cup \left( \prod_{j \neq i} A_j \times \pi_i(U_n) \right).$$

Now we define

$$C(n) = \{i \in r(U_n) : \{a \in [\omega]^{<\omega} : \bigcup_{m \in \omega} U_m \subseteq W(n, a, i)\} = \emptyset\}.$$

Suppose  $C(n) = \emptyset$ . For each  $i \in r(U_n)$  choose  $a_i \in [\omega]^{<\omega}$  such that

$$\bigcup_{m \in \omega} U_m \subseteq W(n, a_i, i).$$

Define  $a = \{n\} \cup \bigcup_{i \in r(U_n)} a_i$ . Let  $x \in \bigcup_{m \in \omega} U_m \setminus [\bigcup_{m \in a} U_m \cup \bigcup_{m < n} V_m]$ . Then  $x \in \prod_{j \neq i} A_j \times \pi_i(U_n)$  for every  $i \in r(U_n)$ . Since  $U_n$  is a basic open set,  $x \in U_n$ . Thus,  $\bigcup_{m \in \omega} U_m \subseteq \bigcup_{m \in a} U_m \cup \bigcup_{m < n} V_m$ . This is a contradiction so  $C(n) \neq \emptyset$ .

Let  $i(n) \in \omega$  be the least element in  $C(n)$ . Define  $V_n = \prod_{j \neq i(n)} A_j \times \pi_{i(n)}(U_n)$ . Clearly,  $i(n)$  and  $V_n$  satisfy (a) and (b). Let  $a \in [\omega]^{<\omega}$ . By our choice of  $i(n) \in C(n)$ ,  $\bigcup_{m \in \omega} U_m \not\subseteq \bigcup_{m \in a} U_m \cup \bigcup_{m \leq n} V_m$ . Therefore, (c) is satisfied.

To finish the proof we only need to show that if  $a \in [\omega]^{<\omega}$ , then

$$\bigcup_{m \in \omega} V_m \setminus \bigcup_{m \in a} V_m \neq \emptyset.$$

Suppose, to the contrary, that there is an  $a \in [\omega]^{<\omega}$  such that  $\bigcup_{m \in a} V_m \supseteq \bigcup_{m \in \omega} V_m$ . Let  $n$  be the smallest element of  $\omega$  such that  $a \subseteq n$ . By condition (c),

$$\bigcup_{m \in \omega} U_m \not\subseteq \bigcup_{m \in a} U_m \cup \bigcup_{m \leq n} V_m.$$

But

$$\bigcup_{m \in \omega} U_m \subseteq \bigcup_{m \in \omega} V_m \subseteq \bigcup_{m \in a} V_m \subseteq \bigcup_{m \leq n} V_m \subseteq \bigcup_{m \in a} U_m \cup \bigcup_{m \leq n} V_m.$$

So no such  $a$  exists.  $\square$

**Theorem 17.** *Let  $\{A_i : i \in \omega\}$  be a family of compact topological spaces. CAC implies that if  $\mathcal{U}$  is a countable open cover of  $X = \prod_{i \in \omega} A_i$  and  $\mathcal{U}$  only consists of basic open sets then  $\mathcal{U}$  has a finite subcover.*

*Proof.* Let  $\mathcal{U} = \{U_n : n \in \omega\}$  be a countable collection of basic open subsets of  $X$  such that for every  $a \in [\omega]^{<\omega}$ ,  $\bigcup_{n \in \omega} U_n \setminus \bigcup_{n \in a} U_n \neq \emptyset$ . It is sufficient to show that  $\mathcal{U}$  does not cover  $X$ . Let  $\{V_n : n \in \omega\}$  be a collection of basic open sets defined as in Theorem 16. Notice that  $\bigcup_{n \in \omega} U_n \subseteq \bigcup_{n \in \omega} V_n$ .

For each  $i \in \omega$ , let  $\mathcal{F}_i = \{A_i \setminus \pi_i(V_n) : n \in \omega \wedge \pi_i(V_n) \neq A_i\}$ . Clearly  $\mathcal{F}_i$  is a family of closed sets. Further, since each  $V_n$  is different from  $A_i$  on one coordinate and  $\{V_n : n \in \omega\}$  does not contain a finite subcover of  $X$ ,  $\mathcal{F}_i$  has the finite intersection property. Since  $A_i$  is compact, there exists  $x_i \in \bigcap \mathcal{F}_i$ .

Define  $x \in X$  as  $x = (x_i)_{i \in \omega}$ . Let  $n \in \omega$  and let  $j \in \omega$  be such that  $\pi_j(V_n) \neq A_j$ . So  $\pi_j(x) = x_j \in \bigcap \mathcal{F}_j$ . Thus,  $x_j \notin \pi_j(V_n)$ . Therefore,  $\bigcup_{n \in \omega} V_n$  does not cover  $X$  and  $\bigcup_{n \in \omega} U_n$  does not cover  $X$ . This completes the proof.  $\square$

The following corollary follows immediately from Theorem 17.

**Corollary 18.** *CAC +  $\Pi_{lin}^C(S)$  implies  $\Pi_C^C(S)$ .*

## 5. WHEN DOES $\Pi_C^C(S)$ OR $\Pi_{proj}^C(S)$ IMPLY CAC?

Several of our results in the previous two sections had a hypothesis of CAC, notably Theorem 9 and Corollary 11. In this section we shall prove that in some cases this hypothesis can be eliminated.

**Theorem 19.** *If  $S \subseteq \{T_1, Sep\}$  or  $S \subseteq \{1C, 2C, Sep\}$ , then  $\Pi_{proj}^C(S)$  implies CAC.*

*Proof.* We assume that CAC fails and prove that  $\Pi_{proj}^C(S)$  fails for all of the sets  $S$  mentioned in the hypothesis of the theorem. First assume  $S = \{T_1, Sep\}$  and fix  $A = \{A_i : i \in \omega\}$ , a family of infinite sets without a choice set. We may assume without loss of generality that  $(\bigcup_{i \in \omega} A_i) \cap \omega = \emptyset$ . For each  $i \in \omega$ , let  $X_i = A_i \cup \omega$  and let  $T_i$  be the topology consisting of all cofinite subsets of  $X_i$  together with all cofinite subsets of  $\omega$ . This topology is  $T_1$  and Sep. Let  $X = \omega \times X_0 \times X_1 \dots$  where we use the cofinite topology on  $\omega$  which is  $T_1$  and Sep (and 1C and 2C). For  $n \in \omega$ , let  $C_n = \{2n\} \times A_0 \times A_1 \times \dots \times A_n \times \{0\} \times \{0\} \times \dots$  and let  $C = \bigcup_{n \in \omega} C_n$ . It can be readily verified that each  $C_n$  is closed in  $X$ . We will argue, based on the fact that  $A$  has no choice function, that the family  $\{C_n : n \in \omega\}$  is locally finite. (That is, each point in  $X$  has a neighborhood which meets only finitely many of the  $C_n$ 's.) Since the union of a locally finite family of closed sets is closed it follows that  $C$  is closed. Assume  $x \in X$ . Since  $x$  is not a choice function for  $A$ , there is an  $i \in \omega$  such that  $\pi_i(x) \in \omega$ . It follows that the set  $\omega \times X_0 \times X_1 \times \dots \times X_{i-1} \times \omega \times X_{i+1} \times X_{i+2} \times \dots$  is a neighborhood of  $x$  disjoint from  $C_n$ , for all  $n \geq i$ . Hence,  $C$  is closed, but the projection of  $C$  onto  $\omega$ , which is the first component of  $X$ , is the set of even natural numbers which is not closed.

Now assume  $S = \{1C, 2C, Sep\}$ . Let  $A = \{A_i : i \in \omega\}$  be a countable family of sets without a choice function. We may assume without loss of generality that  $0 \notin A_i$  for all  $i \in \omega$ . Let  $X_i = A_i \cup \{0\}$  and let  $T_i$  be the topology  $\{\emptyset, A_i, \{0\}, X_i\}$ . This topology is 1C, 2C, and Sep. The cofinite topology on  $\omega$  is also 1C, 2C and Sep. Therefore, to complete the proof it will suffice to show that projections are not closed in  $X = \omega \times X_0 \times X_1 \dots$ . The proof is now almost identical to the proof

in the paragraph above. The sets  $C_n$ ,  $n \in \omega$ , and  $C$  are defined in exactly the same way. If  $x \in X$ , then since  $x$  is not a choice function for  $A$ , there is some  $i \in \omega$  such that  $\pi_i(x) = 0$ . Then the set  $\omega \times X_0 \times X_1 \times \cdots \times X_{i-1} \times \{0\} \times X_{i+1} \times X_{i+2} \times \cdots$  is a neighborhood of  $x$  disjoint from  $C_n$  for  $n \geq i$ . Therefore, the family  $\{C_n : n \in \omega\}$  is locally finite. Hence, its union  $C$  is closed. But the projection of  $C$  onto the first component of  $X$  is not.  $\square$

Using Theorem 19 along with Theorem 5 and Corollary 11, we obtain the following corollary:

**Corollary 20.** *If  $S \subseteq \{T_1, Sep\}$ , or  $S \subseteq \{1C, 2C, Sep\}$ , then  $\Pi_{proj}^C(S)$  is equivalent to  $\Pi_C^C(S)$ .*

Combining Theorem 19 with Theorem 9 gives us the corollary:

**Corollary 21.** *If  $S \subseteq \{1C, 2C\}$ , then  $\Pi_C^C(S \cup Sep)$  implies  $\Pi_C^C(S)$ .*

**Theorem 22.** *CAC implies  $\Pi_C^C(2C)$ .*

*Proof.* By Corollary 18, it suffices to prove CAC implies  $\Pi_{lin}^C(2C)$ . Let  $\{(X_i, T_i) : i \in \omega\}$  be a family of second countable compact spaces and let  $X$  denote their Tychonoff product. Assuming CAC, for each  $i \in \omega$ , choose a base (and a well ordering of the base of type  $\omega$ )  $B_i = \{B_{im} : m \in \omega\}$  for  $(X_i, T_i)$  and let

$$\mathcal{B} = \left\{ \prod_{i \in \omega} Q_i : (\exists j \in \omega) ((\forall i < j)(Q_i \in B_i) \wedge (\forall i \geq j)(Q_i = X_i)) \right\}$$

It follows from CAC that  $\mathcal{B}$  is a countable base for  $X$  and that  $X$  is Lindelöf.  $\square$

**Corollary 23.**  $\Pi_C^C(2C) \leftrightarrow \Pi_{proj}^C(2C) \leftrightarrow CAC$ .

*Proof.* Use Theorem 5, Theorem 19, and Theorem 22.  $\square$

**Theorem 24.** *If  $S \subseteq \{T_2, Sep\}$ , then  $CAC + \Pi_{proj}^C(S)$  implies  $\Pi_C^C(S)$ .*

*Proof.* We shall prove the theorem for  $S = \{T_2\}$ . Then the remaining parts follow from Theorem 9 ( $S = \{T_2, Sep\}$ ) and Corollary 11 ( $S = \emptyset$ ). It suffices, in view of Corollary 8, to show that if  $A = \{(X_i, T_i) : i \in \omega\}$  is a family of compact  $T_2$  spaces, then no countable closed set  $Q = \{g_n : n \in \omega\}$  is discrete in the Tychonoff product  $X$  of  $A$ . Assume on the contrary that  $Q = \{g_n : n \in \omega\}$  is a closed and discrete subspace of  $X$ . Since projections are closed, it follows that  $Q_i = \pi_i(Q)$  is also closed, hence compact.

**Claim.** CAC implies that compact countable Hausdorff spaces are second countable.

*Proof of the claim.* Fix  $(X, T)$ , a compact countable Hausdorff space. Put  $A = \{A_{x,y} : x, y \in X, x \neq y\}$ , where  $A_{x,y} = \{(O_x, O_y) : O_x \text{ and } O_y \text{ are disjoint open neighborhoods of } x \text{ and } y \text{ respectively}\}$ . Using CAC, let  $C$  be a choice set for  $A$ . Let  $\mathcal{B}$  be the base which is generated by the countable subbase  $\mathcal{D} = \text{Domain}(C) \cup \text{Range}(C)$ . Clearly  $T_{\mathcal{B}}$ , the topology generated by  $\mathcal{B}$ , is  $T_2$ , second countable, and the identity function  $h : (X, T) \rightarrow (X, T_{\mathcal{B}})$  is 1-to-1, continuous, onto, and closed. (It carries closed, and therefore compact, subsets of  $(X, T)$  to compact, therefore closed, subsets of  $(X, T_{\mathcal{B}})$ ). Thus,  $h$  is a homeomorphism and consequently,  $\mathcal{B} = h^{-1}(\mathcal{B})$  is also a countable base of  $(X, T)$ , finishing the proof of the claim.

Since CAC implies  $\Pi_C^C(2C)$  (Theorem 22), it follows from the claim that  $Q$  is an infinite subspace of the compact space  $\prod_{i \in \omega} Q_i$  and, consequently, it has a limit point  $q$ . This is a contradiction.  $\square$

**Theorem 25.**  $\Pi_{proj}^C(1C, Sep)$  implies  $\Pi_{proj}^C(1C)$ .

*Proof.* Let  $X = \prod_{i \in \omega} A_i$ , where each  $A_i$  is compact and first countable. Let  $H \subseteq X$  be closed and suppose  $\pi_j(H)$  is not closed in  $A_j$ , for some  $j \in \omega$ . Let  $z \in \overline{\pi_j(H)} \setminus \pi_j(H)$ . Since  $A_j$  is first countable,  $z$  has a countable neighborhood base  $\{U_n : n \in \omega\}$ . Since  $\Pi_{proj}^C(1C, Sep)$  implies CAC (Theorem 19), we can choose a countable set  $\{x_n : n \in \omega\} \subseteq H$  such that  $z \in \overline{\{\pi_j(x_n) : n \in \omega\}}$  and for each  $n \in \omega$ ,  $\pi_j(x_n) \in U_n$ . (Notice that  $\{x_n : n \in \omega\}$  is not necessarily infinite.)

For each  $i \in \omega$ , let  $B_i = \{\pi_i(x_n) : n \in \omega\}$ . Then each  $B_i$  is compact, separable, and first countable. Let  $Y = \prod_{i \in \omega} B_i$ . By  $\Pi_{proj}^C(1C, Sep)$ ,  $\pi_j(H \cap Y)$  is closed in  $B_j \subseteq A_j$ . Since  $\{\pi_j(x_n) : n \in \omega\} \subseteq \pi_j(H \cap Y)$  and  $\pi_j(H \cap Y)$  is closed, we get the contradiction that  $z \in \pi_j(H \cap Y) \subseteq \pi_j(H)$ .  $\square$

**Theorem 26.**  $\Pi_{proj}^C(1C)$  implies  $\Pi_C^C(1C)$ .

*Proof.* Let  $A = \{(X_i, T_i) : i \in \omega\}$  be a family of compact 1st countable spaces and let  $G = \{G_n : n \in \omega\}$  be a strictly decreasing family of closed sets in the Tychonoff product  $(X, T)$  of  $A$ . We shall show that  $\bigcap G \neq \emptyset$ , then it follows from Theorem 6 that  $X$  is compact. Assume the contrary,  $\bigcap G = \emptyset$ . Since  $\Pi_{proj}^C(1C)$  implies CAC (Theorem 19), choose a set  $Q = \{g_n \in G_n \setminus G_{n+1} : n \in \omega\}$ . Since  $Q$  may not be closed, we shall embed  $(X, T)$  into a compact space where  $Q$  will be closed. We do this as follows: Let  $T_i^*$  be the topology which is generated by the base

$$B_i^* = T_i \cup \{O_x \setminus b : x \in X_i, O_x \text{ is a neighborhood of } x \text{ and } b \in [\{g_n(i) : n \in \omega\} \setminus \{x\}]^{<\omega}\}.$$

Clearly, for all  $i \in \omega$ ,  $(X_i, T_i^*)$  is a first countable space. Moreover, working as in Theorem 4, we can show that  $(X_i, T_i^*)$  is also a compact space.

We claim that  $Q$  is closed in the product space  $(X, T^*)$ . To see this, let  $x \in X \setminus Q$ . Since  $\bigcap G = \emptyset$ , there exists  $n \in \omega$  such that  $x \notin G_n$ . Hence,  $x$  has a neighborhood  $O_x$  of the form  $O_x = O_1 \times O_2 \times \dots \times O_m \times (\prod_{i \in (\omega \setminus m+1)} X_i)$ , in  $(X, T)$  disjoint from  $G_n$  ( $G_n$  is closed in  $(X, T)$ ). It follows that

$$O_x^* = (O_1 \setminus (\{g_1(1), g_2(1), \dots, g_{n-1}(1)\})) \times (O_2 \setminus (\{g_1(2), g_2(2), \dots, g_{n-1}(2)\})) \times \dots \times$$

$$(O_m \setminus (\{g_1(m), g_2(m), \dots, g_{n-1}(m)\})) \times (\prod_{i \in (\omega \setminus m+1)} X_i)$$

is a basic neighborhood of  $x$  in  $(X, T^*)$  disjoint from  $Q$ . Thus  $Q$  is closed in  $(X, T^*)$  as required.

Since  $Q$  is closed and  $(X_i, T_i^*)$  is 1C and compact, it follows from  $\Pi_{proj}^C(1C)$  that  $Q_i = \pi_i(Q)$  is a closed subset of a compact space and, therefore, compact. Since  $Q_i$  is countable and  $(X_i, T_i^*)$  is 1C, it follows that  $Q_i$  is a compact, 2C space. Thus, by Theorem 22,  $\prod_{i \in \omega} Q_i$  compact.  $Q$  is a closed, countably infinite sequence in the compact space  $\prod_{i \in \omega} Q_i$ , consequently, it has a cluster point  $q$ . Clearly,  $q$  is also a cluster point of  $Q$  in  $(X, T^*)$ . This implies the contradiction that  $q \in \bigcap G$ .  $\square$

Using Theorem 5, Theorem 19, Theorem 25, and Theorem 26, we obtain:

**Corollary 27.**  $\Pi_C^C(1C) \leftrightarrow \Pi_C^C(1C, Sep) \leftrightarrow \Pi_{proj}^C(1C) \leftrightarrow \Pi_{proj}^C(1C, Sep)$ .

Using an argument similar to the one given in the proof of Theorem 19 we can show:

**Theorem 28.** (i) *If  $S$  is any subset of  $\{T_1, T_2, 1C, 2C, Sep\}$ , then  $\Pi_{proj}^C(S)$  implies  $CAC_{fin}$ .*

(ii) *If  $S$  is any subset of  $\{T_1, 1C, 2C, Sep\}$ , then  $\Pi_{proj}^C(S)$  implies  $CAC_\omega$ .*

*Proof.* We will prove both parts simultaneously by contradiction. Suppose that  $A = \{A_i : i \in \omega\}$  is a countable collection of sets without a choice function. Our plan is to show that if each of the  $A_i$ 's is finite then  $\Pi_{proj}^C(S)$  fails, where  $S = \{T_1, T_2, 1C, 2C, Sep\}$ . And if each of the  $A_i$ 's is countable then  $\Pi_{proj}^C(S)$  fails, where  $S = \{T_1, 1C, 2C, Sep\}$ . We may assume without loss of generality that for all  $i \in \omega$ ,  $0 \notin A_i$ . For each  $i \in \omega$ , let  $X_i = A_i \cup \{0\}$  and let  $T_i$  be the topology consisting of all cofinite subsets of  $X_i$  and  $\{0\}$ . If  $A_i$  is countable then this topology is compact,  $T_1, 1C, 2C$ , and  $Sep$ . In addition, if  $A_i$  is finite this topology is  $T_2$ . We will also use the topological space  $(\omega + 1, T)$  where  $T$  is the order topology.  $T$  is compact,  $T_1, T_2, 1C, 2C$ , and  $Sep$ . Let  $X = (\omega + 1) \times X_0 \times X_1 \times X_2 \times \dots$ . For  $n \in \omega$ , let  $C_n = \{n\} \times A_0 \times A_1 \times \dots \times A_n \times \{0\} \times \{0\} \times \dots$  and let  $C = \bigcup_{n \in \omega} C_n$ . As in the proof of Theorem 19,  $C$  is closed, but the projection of  $C$  onto the first component of  $X$  is  $\omega$ , which is not closed in  $\omega + 1$ .  $\square$

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