THE AXIOM OF CHOICE FOR WELL ORDERED FAMILIES AND FOR FAMILIES OF WELL ORDERABLE SETS

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Abstract. We show that it is not possible to construct a Fraenkel-Mostowski model in which the axiom of choice for well ordered families of sets and the axiom of choice for sets for well orderable sets are both true, but the axiom of choice is false.

We are concerned with the following two consequences of the axiom of choice:

$C(WO, \infty)$: Every well ordered collection of sets has a choice function.

$C(\infty, WO)$: Every collection of well orderable sets has a choice function.

It is known that $C(WO, \infty)$ does not imply the Axiom of Choice (AC) in Zermelo-Fraenkel set theory (ZF) ([4, p 127] and [8]) nor is $C(WO, \infty)$ a theorem of ZF (without AC) ([4, p 123 thm 8.3], [5] and [7]). Similarly $C(\infty, WO)$ does not imply AC ([6] and [4, p 82, prob. 5.22]) nor is $C(\infty, WO)$ a theorem of ZF ([4, prob 7.12]).

(In Cohen's original model of ZF – AC ([1]), in which there is a countable set of generic reals along with a set collecting them, $C(\infty, WO)$ is true and $C(WO, \infty)$ is false. Mostowski's linearly ordered Fraenkel-Mostowski (FM) model has this same property. Feferman's model of ZF – AC ([2]) in which there is a countable set of generic reals, but no set to collect them, has the property that $C(WO, \infty)$ is true, but $C(\infty, WO)$ is false. An FM model that has this same property is a variation of Fraenkel's basic model in which the set of atoms, A, is uncountable, the group, G, is the group of all permutations of A, and the filter of subgroups of G is the set of subgroups of G that leave a countable subset of A pointwise fixed.)

The question of whether or not $C(WO, \infty) \wedge C(\infty, WO)$ implies AC is open. Our purpose is to show that there is no FM model in which both $C(WO, \infty)$ and $C(\infty, WO)$ are true, but AC is false. It has been shown by Howard ([3]) that in every FM model, $C(\infty, WO)$ is equivalent to $C(\infty, < \aleph_0)$, AC for a family of finite sets. In the proof below we use $C(\infty, < \aleph_0)$ rather than $C(\infty, WO)$.

Assume that $\mathcal{N}$ is an FM model determined by the the set of atoms A, the group G of permutations of A, and the filter Γ of subgroups of G. Assume that $C(WO, \infty)$ and $C(\infty, WO)$ are true in $\mathcal{N}$ and that AC is false. Let $X = \{ f \in \mathcal{N} : f$

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is a one to one function from ω into $A\,\}$ and define an equivalence relation $\sim$ on X by $f \sim g$ if and only if $(\exists n \in \omega)(\exists k \in \mathbb{Z}, k \geq -n)(\forall j > n)(f(j) = g(j+k)\,)$. Let $\mathfrak{X} = \{\,[f]_\sim : f \in X\,\}$, where $[f]_\sim$ is the $\sim$-equivalence class of f, and let $\mathcal{Y} = \{\,\mathfrak{W} : \mathfrak{W} \subseteq \mathfrak{X} \wedge \mathfrak{W} \text{ is finite }\}$. By $C(\infty, WO)$, there is a $G_1 \in \Gamma$ which fixes a choice function Ch_1 on $\mathcal{Y}$. Also by $C(\infty, WO)$, there is a G_2 in Γ which fixes a choice function Ch_2 on the family of non-empty, finite subsets of A. Let $G_3 = G_1 \cap G_2 \in \Gamma$.

Let W be the set of G_3 orbits of pairs of well orderings of subsets of A, that is,

$$W = \big\{\,\{\,\phi(\langle R_1, R_2\rangle) \mid \phi \in G_3\,\} \mid R_1, R_2 \in WA\,\big\}$$

where $WA = \{\,R \in \mathcal{N} \mid \exists B \subseteq A \text{ such that } R \text{ well orders } B\,\}$. W is well ordered in $\mathcal{N}$ and therefore, by $C(WO, \infty)$, W has a choice function F in $\mathcal{N}$. F is fixed by some G_4 in Γ such that $G_4 \subseteq G_3$. Let $E' = \{\,a \in A \mid \forall \phi \in G_4, \phi(a) = a\,\}$ and let R' be a well ordering of E' fixed by G_4. By the assumption that AC fails in $\mathcal{N}$, $A \neq E'$ so we may choose $a_0 \in A - E'$. $\{\langle a_0, a_0\rangle\}$ and R' are in WA and hence

$$F(\{\,\phi(\langle\{\langle a_0, a_0\rangle\}, R'\rangle \mid \phi \in G_3\,\}) \in \{\,\phi(\langle\{\langle a_0, a_0\rangle\}, R'\rangle \mid \phi \in G_3\,\}$$

$F(\{\,\phi(\langle\{\langle a_0, a_0\rangle\}, R'\rangle \mid \phi \in G_3\,\})$ has the following properties:

1. It is equal to $\phi_0(\langle\{\langle a_0, a_0\rangle\}, R'\rangle) = \langle\{\langle\phi_0(a_0), \phi_0(a_0)\rangle\}, \phi_0(R')\rangle$ for some $\phi_0 \in G_3$.
2. It is fixed by every $\phi \in G_4$.

It follows from 1 and 2 above that $(\forall \phi \in G_4)(\phi(\phi_0(a_0)) = \phi_0(a_0))$ and it therefore follows that $\phi_0(a_0) \in E'$. Similarly, every $\phi \in G_4$ fixes $\phi_0(R')$ and therefore fixes $\phi_0(E')$ pointwise because $\phi_0(R')$ is a well ordering of $\phi_0(E')$. It follows that $\phi_0(E') \subseteq E'$.

The permutation ϕ_0 is in G_2 and therefore has no finite cycles of length greater than one because if $t \in A$ and $(t, \phi_0(t), \phi_0^2(t), \ldots, \phi_0^n(t))$ were such a cycle (where $\phi_0^{n+1}(t) = t$, but $\phi_0^k(t) \neq t$ for any $k < n+1$), then ϕ_0 fixes the finite set

$$C = \{t, \phi_0(t), \phi_0^2(t), \ldots, \phi_0^n(t)\}.$$

However, ϕ_0 does not fix any element of C and therefore cannot fix the choice function Ch_2. Since $a_0 \notin E'$ and $\phi_0(a_0) \in E'$, $a_0 \neq \phi_0(a_0)$ so the elements $\phi_0(a_0)$, $\phi_0^2(a_0)$, $\phi_0^3(a_0)$, $\ldots$ are pairwise distinct. Further (since $\phi_0(E') \subseteq E'$) they are all in E' and therefore the set $\{\phi_0(a_0), \phi_0^2(a_0), \phi_0^3(a_0), \ldots\}$ is well orderable in $\mathcal{N}$. Consequently, the two functions $f, g : \omega \to A$ defined by

$$f(n) = \phi_0^{2(n+1)}(a_0) \text{ and } g(n) = \phi_0^{2n+1}(a_0)$$

are in $\mathcal{N}$ and hence in X. Clearly $[f]_\sim \neq [g]_\sim$, $\phi_0([f]_\sim) = [g]_\sim$ and $\phi_0([g]_\sim) = [f]_\sim$. But this is not possible because $\phi_0 \in G_1$ and therefore fixes the choice function Ch_1.

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