METRIC SPACES AND THE AXIOM OF CHOICE

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Abstract. We study conditions for a topological space to be metrizable, properties of metrizable spaces, and the role the axiom of choice plays in these matters.

1. Introduction and Definitions

We shall start with some definitions from topology. First of all, a *metric space* is a topological space whose topology is determined by a metric. A *metric* on a topological space X is a function d from $X \times X$ to $\mathbb{R}$, the reals, which has the following properties: For all x, y, z in X,

1. $d(x, y) \geq 0$.
2. $d(x, x) = 0$.
3. If $d(x, y) = 0$, then $x = y$.
4. $d(x, y) = d(y, x)$.
5. $d(x, y) + d(y, z) \geq d(x, z)$

Functions which satisfy 1, 2, 4, and 5 are called *pseudometrics*. We let "AC" be the axiom of choice, "ZF", Zermelo-Fraenkel set theory, "ZF^0", ZF without the axiom of foundation and "ZFC", ZF + AC. Additional definitions follow.

Definition 1. Let (X, T) be a topological space.

1. Let $\mathcal{U}, \mathcal{V} \subseteq \mathcal{P}(X)$. We say that $\mathcal{U}$ *covers* X or that $\mathcal{U}$ is a *covering* of X iff $\bigcup \mathcal{U} = X$. We say that $\mathcal{V}$ is a *refinement* of $\mathcal{U}$ iff $\bigcup \mathcal{V} = X$ and every member of $\mathcal{V}$ is included in a member of $\mathcal{U}$.
2. A family $\mathcal{U}$ of subsets of X is *locally finite* (respectively *locally countable*) iff each point of X has a neighborhood meeting only a finite number (respectively countable number) of elements of $\mathcal{U}$.

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3. (X,T) is *paracompact* (respectively *para-Lindelöf*) iff (X,T) is T_2 and every open cover $\mathcal{U}$ of X has an open locally finite (respectively locally countable) refinement.
4. A family $\mathcal{U}$ of subsets of X is *point finite* (respectively *point countable*) iff each element of X belongs to only finitely many (respectively countably many) members of $\mathcal{U}$.
5. (X,T) is *metacompact* (respectively *meta-Lindelöf*) iff each open cover $\mathcal{U}$ of X has an open point finite (respectively point countable) refinement.
6. A set $C \subseteq \mathcal{P}(X)$ is *discrete* if $\forall x \in X$, there is a neighborhood U of x such that $U \cap A \neq \emptyset$ for at most one element $A \in C$.
7. A set $C \subseteq \mathcal{P}(X)$ is *σ-locally finite* (respectively *σ-point finite, σ-disjoint, σ-discrete*) if $C = \bigcup_{n\in\omega} C_n$ where each C_n is locally finite (respectively, point finite, pairwise disjoint, discrete).
8. (X,T) is *paradiscrete* (respectively *paradisjoint*) iff (X,T) is T_2 and every open cover $\mathcal{U}$ has a σ-discrete (respectively σ-disjoint) open refinement.
9. (X,T) is *subparacompact* iff (X,T) is T_2 and every open cover $\mathcal{U}$ has a σ-discrete closed refinement.
10. Assume that γ is a well ordered cardinal number, that is, γ is an initial ordinal. A set $C \subseteq \mathcal{P}(X)$ is *γ-locally finite* (respectively *γ-point finite, γ-disjoint, γ-discrete*) if $C = \bigcup_{\alpha\in\gamma} C_\alpha$ where each C_α is locally finite (respectively, point finite, pairwise disjoint, discrete).
11. X is a *T_0 space* if whenever x and y are distinct points in X, there is a neighborhood of one which does not contain the other.
12. X is a *T_1 space* if whenever x and y are distinct points in X, there is a neighborhood of each not containing the other.
13. X is a *T_2 (Hausdorff) space* if whenever x and y are distinct points in X, there are disjoint open neighborhoods U and V such that $x \in U$ and $y \in V$.
14. X is *regular* if whenever A is a closed set in X and $x \notin A$, there are disjoint open sets U and V such that $x \in U$ and $A \subseteq V$. X is a *T_3 space* if it is regular and T_1.
15. X is *normal* if whenever A and B are disjoint closed sets, there are disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$. X is a *T_4 space* if X is normal and T_1.

In section 2, we study metrization theorems that are provable in ZF^0 and, in section 3, we study various conditions on metric spaces. Most of these conditions are provable in ZFC, but we study the relationships between them in ZF^0. All proofs, unless specifically stated otherwise, are in ZF^0.

2. Metrization Theorems in ZF^0

The theory of metrization of topological spaces quite often occupies considerable space in general topology textbooks. The first metrization theorem was proved by Urysohn [ur] in 1925. This theorem is known as *Urysohn's Metrization Theorem* and it states :

A T_3 topological space with a countable base is metrizable.

Good and Tree ([gt], Corollary 4.8, p. 86) have shown that Urysohn's metrization theorem holds in ZF^0. In ZFC, it is an indispensable part of the theory of separable metric spaces, but it says nothing about non-separable metric spaces. In the early 1950's, Nagata [na], Smirnov [sm] and Bing [bi] generalized Urysohn's Metrization Theorem and proved the following *General Metrization Theorem*:

> *(General Metrization Theorem) A topological space is metrizable iff it is T_3 and has a σ-locally finite base.*

Actually, Bing proved something slightly different. He proved that "A space X is metrizable if and only if it is regular and has a σ-locally discrete base."

In [hkrs] it has been shown that the statement: *Every metric space has a σ-locally finite base* (form [232 B] in [hr]) implies Stone's Theorem: *MP : Metric spaces are paracompact.* Since the statement MP is not provable in ZF, (see [gtw], [hkrs] and [hr] for models where it fails), it follows that [232 B] is also not provable in ZF. Therefore, the proof of the $\rightarrow$ part of the General Metrization Theorem, in contrast with Urysohn's Metrization Theorem, requires some choice. It was shown in [cr], Theorem 7, that $\leftarrow$ does not require any choice. Thus, it is provable in ZF that:

Theorem 1. *If a regular space has a σ-locally finite base, then it is metrizable.*

It is known [hkrs] that for metric spaces the notions of σ-locally finite base, σ-discrete base, σ-disjoint base and σ-point finite base coincide. Therefore, in view of Theorem 1, one may think that it is possible to replace "σ-locally finite base" in Theorem 1 by "σ-discrete base", "σ-disjoint base" and "σ-point finite base" respectively. Since a discrete family of sets is clearly locally finite, it follows that we can replace "σ-locally finite base" by "σ-discrete base" in Theorem 1. Taking into account Theorem 1 and the following theorem from [kt1].

Theorem 2. *The following statements are equivalent:*
- *(i) Metric spaces are paracompact*
- *(ii) Every metric space has an σ-locally finite base.*

We get as a corollary:

Corollary 1. *The following are equivalent:*
- *(i) Metric spaces are paracompact.*
- *(ii) A topological space (X, T) is metrizable iff X is regular, T_1 and has a σ-locally finite base.*
- *(iii) A topological space (X, T) is metrizable iff X is regular, T_1 and has a σ-discrete base.*

Next we show by means of an example that "σ-locally finite base" cannot be replaced in Theorem 1 by: "σ-disjoint base" or "σ-point finite base", respectively.

Theorem 3. *(In ZF) There exists a (first countable) regular, T_1 space (X,T) with a σ-disjoint base (and hence a σ-point finite base) which is not metrizable.*

Proof. Let X be the closed interval $[0,1]$ with the topology T generated by the usual topology on $[0,1]$ together with the sets $\{a\}$ where a is an irrational number. We leave it to the reader to verify that (X,T) is a regular, first countable, T_1 space and that $\{\{a\} : a \text{ is irrational and } a \in [0,1]\} \cup \mathcal{B}$ is a σ-disjoint base for T where $\mathcal{B}$ is any countable base for the usual topology on $[0,1]$.

The proof that T is not metrizable will be by contradiction. Therefore, we assume that T is metrizable and that T is the topology determined by the metric d. Since each $a \in [0,1] \setminus \mathbb{Q}$ is open in T for each such a, there is some positive number ϵ such that the ball $D(a,\epsilon) = \{x \in X : d(a,x) < \epsilon\}$ contains only the number a. Therefore for each $a \in [0,1] \setminus \mathbb{Q}$, there is some $n \in \omega \setminus \{0\}$ such that $D(a,\frac{1}{n}) = \{a\}$. Hence, if we let $\mathcal{D}_n = \{a \in [0,1] \setminus \mathbb{Q} : D(a,\frac{1}{n}) = \{a\}\}$ then $[0,1] \setminus \mathbb{Q} = \bigcup_{n\in\omega\setminus\{0\}} \mathcal{D}_n$. By the Baire Category Theorem, $[0,1]$ is not the countable union of nowhere dense sets. (This can be proved in ZF, by contradiction. Assume that $[0,1] = \bigcup_{n\in\omega} N_n$ where N_n is nowhere dense for each $n \in \omega$. Since $\mathbb{Q}$ can be well ordered without AC, we can choose an open interval I_0 with rational endpoints whose closure is contained in the complement of N_0. Similarly, we can choose and open interval $I_1 \subseteq I_0$ with rational endpoints whose closure is contained in the complement of N_1. Continuing in this way, we obtain a nested sequence of closed sets $(\bar{I_n})_{n\in\omega}$ the intersection of which must be non-empty and contained in the complement of $\bigcup_{n\in\omega} N_n$.)

It follows that there is some $n_0 \in \omega \setminus \{0\}$ such that $\mathcal{D}_{n_0}$ is not nowhere dense (in the usual topology on $[0,1]$) and therefore, there is some rational number $q \in \mathbb{Q}$ such that q is a limit point of D_{n_0} (in the usual topology on $[0,1]$.) Since q is rational, every neighborhood of q in T is a neighborhood of q in the usual topology. Therefore, the T neighborhood $D(q,\frac{1}{n_0})$ contains a point a of $\mathcal{D}_{n_0}$. But then $D(a,\frac{1}{n_0})$ contains a number different from a contradicting the assumption that $a \in \mathcal{D}_{n_0}$. □

3. Properties of Metric Spaces in ZF^0

We list some properties of metric spaces from [hr]. (The numbers of the statements are those given in [hr].) For definitions and references see [hr].

[0 L] Caristi's Fixed Point Theorem: If (X,ρ) is a complete metric space and $\phi : X \to \mathbb{R}$ is bounded above and upper semi-continuous then in the Brøndsted ordering ($x \leq y$ iff $\rho(x,y) \leq \phi(y) - \phi(x)$) every $f : X \to X$ satisfying $\forall t \in X$, $t \leq f(t)$ has a fixed point.

[0 AK] Every separable metric space is second countable.

[1 I] In every metric space (X,d) there is an ϵ-lattice for some $\epsilon > 0$ which is not a strict upper bound of d.

[1 CS] Every W compact pseudometric space is A-U compact.

[1 CT] Every sequentially compact pseudometric space is A-U compact.

[**1 BK**] Every open cover $\mathcal{U}$ of a metric space (X, d) can be written as a well ordered union $\bigcup\{U_\alpha : \alpha \in \gamma\}$ where γ is an ordinal and each U_α is pairwise disjoint.

[**1 CE**] Every open cover $\mathcal{U}$ of a metric space (X, d) can be written as a well ordered union $\bigcup\{U_\alpha : \alpha \in \gamma\}$ where γ is an ordinal and each U_α is discrete.

[**8 F**] Every pseudometric space with a countable base is separable.

[**8 V**] Every pseudometric Lindelöf space is separable.

[**8 W**] Subspaces of separable pseudometric spaces are separable.

[**8 X**] Every sequentially bounded pseudometric space is totally bounded.

[**8 Y**] Every totally bounded pseudometric space is separable.

[**8 Z**] Every sequentially bounded pseudometric space is separable.

[**8 AC**] Every sequentially compact pseudometric space is totally bounded.

[**8 AD**] Every totally bounded, complete pseudometric space is compact.

[**8 AE**] Every sequentially compact pseudometric space is compact.

[**8 AF**] In a pseudometric space, every infinite subset has an accumulation point if and only if the space is complete and totally bounded.

[**8 AG**] Compact pseudometric spaces are separable.

[**8 AH**] In a metric space X, if x is an accumulation point of a subset $A \subseteq X$ then there is a sequence of elements of A which converges to x.

[**8 AI**] A function from one metric space to another is continuous if and only it is sequentially continuous.

[**8 AL**] For pseudometric spaces, every Cauchy filter converges if and only if every Cauchy sequence converges.

[**9 X**] In every sequentially compact pseudometric space, every infinite set has an accumulation point.

[**10 G**] For all metric spaces X and Y and continuous functions f from X onto Y such that $f^{-1}(y)$ is finite for all $y \in Y$, if X has a dense Dedekind finite subset, then so does Y.

31 $UT(\aleph_0, \aleph_0, \aleph_0)$: The countable union theorem: The union of a countable set of countable sets is countable.

34 $\aleph_1$ is regular.

[**43 F**] Baire Category Theorem for Sequentially Complete Metric Spaces: Every sequentially complete metric space is Baire.

[**43 J**] Baire Category Theorem for Cantor Complete Metric Spaces: Every Cantor complete metric space is Baire.

[**43 M**] If (X, ρ) is a complete metric space and $\phi : X \to \mathbb{R}$ is bounded above and upper semi-continuous then in the Brøndsted ordering, $(x \le y \text{ iff } \rho(x, y) \le \phi(y) - \phi(x))$, there is a maximal element.

[**43 AD**] Every Frechet complete (pseudo)metric space is Baire.

[**43 AH**] Ekeland's Variational Principle: If (E, d) is a non-empty complete metric space, $f : E \to \mathbb{R}$ is lower semi-continuous and bounded from below, and ϵ is a positive real number, then there exists $a \in E$ such that for all $x \in E$, $f(a) \le f(x) + \epsilon d(x, a)$.

[**64 D**] Metric C spaces are limited amorphous. (A space is C if every open covering has an amorphous refinement. A space is *limited amorphous* if each amorphous subset is relatively compact.)

[**67 U**] Every open cover $\mathcal{U}$ of a metric space (X, d) can be written as a well ordered union $\bigcup\{U_\alpha : \alpha \in \gamma\}$ where γ is an ordinal and each U_α is locally finite.

[**67 V**] Every open cover $\mathcal{U}$ of a metric space (X, d) can be written as a well ordered union $\bigcup\{U_\alpha : \alpha \in \gamma\}$ where γ is an ordinal and each U_α is point finite.

[**76 A**] Every open cover $\mathcal{U}$ of a metric space can be written as a well ordered union $\bigcup\{U_\alpha : \alpha \in \gamma\}$ where γ is an ordinal and each U_α is locally countable.

[**76 B**] Every open cover $\mathcal{U}$ of a metric space can be written as a well ordered union $\bigcup\{U_\alpha : \alpha \in \gamma\}$ where γ is an ordinal and each U_α is point countable.

[**94 H**] Every subspace of a separable (pseudo)metric space is separable.

[**94 I**] Every separable metric space is Lindelöf.

[**94 J**] Every second countable metric space is Lindelöf.

[**94 K**] Every second countable (pseudo)metric space is separable.

[**126 M**] Weierstrass compact pseudometric spaces are countably compact.

[**126 N**] Weierstrass compact pseudometric spaces are compact.

[**126 O**] Every compact pseudometric space has a dense subset which can be written as a countable union of finite sets.

[**126 P**] Every compact pseudometric space has a dense subset which can be written as a well ordered union of finite sets.

173. MPL: Metric spaces are para-Lindelöf.

[**173 A**] MML: Metric spaces are meta-Lindelöf.

232. Every metric space (X, d) has a σ-point finite base.

[**232 A**] Every metric space has a γ-point finite base, for some ordinal γ.

[**232 B**] Every metric space has a σ-locally finite base.

[**232 C**] Every metric space has a γ-locally finite base, for some ordinal γ.

[**232 D**] Every metric space (X, d) has a σ-discrete base.

[**232 E**] Every metric space (X, d) has a γ-discrete base, for some ordinal γ.

[**232 F**] Every metric space (X, d) has a σ-disjoint base.

[**232 G**] Every metric space (X, d) has a γ-disjoint base for some ordinal γ.

[**232 H**] MP: Every metric space is paracompact. [hkrs].

[**232 I**] Every open cover $\mathcal{U}$ of a metric space (X, d) has a refinement $\mathcal{V}$ which covers X and which can be written as a well ordered union $\bigcup\{V_\alpha : \alpha \in \gamma\}$ where each V_α is locally finite. [hkrs]

[**232 J**] MM: Every metric space is metacompact. [hkrs]

[**232 K**] Every open cover $\mathcal{U}$ of a metric space (X, d) has a refinement $\mathcal{V}$ which covers X and which can be written as a well ordered union $\bigcup\{V_\alpha : \alpha \in \gamma\}$ where each V_α is point finite. [hkrs]

(Note that forms [232 H]-[232 K] were forms 383, [383 A]-[383 C] in [hr], but it is shown in [hkrs] that 232 implies 383 and it is shown in [hkt] that 383 implies 232.)

280. There is a complete separable metric space with a subset which does not have the Baire property.

340. Every Lindelöf metric space is separable.

341. Every Lindelöf metric space is second countable.

381. DUM: The disjoint union of metrizable spaces is metrizable.

382. DUMN: The disjoint union of metrizable spaces is normal.

First note that in the above list of statements, each statement that has the form **n** is equivalent (in ZF^0) to each statement that has the form [**n X**] for each **X**. The first two statements, [**0 L**] and [**0 AK**] are provable in ZF^0. The next five statements of the form [**1 X**] are equivalent to AC in ZF^0. The statements of the form [**8 X**], for some X are equivalent to the Countable Axiom of Choice, CAC: Every countable family of non-empty sets has a choice function. The statement [**9 X**] is equivalent to: Every Dedekind finite set is finite. (A set is *Dedekind finite* if it is not equivalent to a proper subset of itself.) The form [**10 G**] is equivalent CAC($< \aleph_0$): Every countable family of finite sets has a choice function. Each statement of the form [**43 X**] is equivalent to the Principle of Dependent Choice, DC: If S is a relation on a non-empty set A and $(\forall x \in A)(\exists y \in A)(xSy)$ then there is a sequence $a(0), a(1), a(2), \ldots$ of elements of A such that $(\forall n \in \omega)(a(n)\ S\ a(n+1))$.

Form **[64 D]** is equivalent to: There are no amorphous sets. (A set is *amorphous* if it is infinite, but not the disjoint union of two infinite sets.) Forms of the form **[67 X]** are equivalent to the Axiom of Multiple Choice, MC: For every set M of non-empty sets there is a function f such that $(\forall x \in M)(\emptyset \neq f(x) \subseteq x$ and $f(x)$ is finite). (MC implies AC in ZF, but not in ZF^0. See [le], [rr], and [je] Chapter 9.)

The two statements **[76 A]** and **[76 B]** are equivalent to a countable multiple choice axiom: For every family X of non-empty sets, there is a function f such that for each $x \in X$, f(x) is a non-empty countable subset of x. Forms of the form **[94 X]** are equivalent to CAC($\mathbb{R}$): Every countable family of non-empty subsets of $\mathbb{R}$ has a choice function. Each statement of the form **[126 X]** is equivalent to the multiple choice axiom for countable sets: For every countable set X of non-empty sets there is a function f such that for all $y \in X$, $f(y)$ is a non-empty finite subset of y. Form 340 implies 341, because separable metric spaces are second countable.

It is shown in [kt2] that 9 does not imply 341. The authors use a permutation model and construct a compact metric space that is not second countable. (In the permutation model. the set of atoms is a set of circles, center at the origin, radius $\frac{1}{n}$ for each positive integer n, permutations are rotations around the circles, and supports are finite.) The result is transferable to ZF using the results of Jech/Sochor ([js1, js2]) and Pincus ([pi1, pi2]). (Also see notes 18 and 103 in [hr].)

Since 9 implies 10 and 340 implies 341, it also follows that 9 does not imply 340, and 10 does not imply 340 or 341.

Theorem 4. *341 implies 10.*

Proof. Let $A = \{A_i : i \in \omega\}$, where each A_i is finite and the A_i's are pairwise disjoint. (We shall show that 341 implies there is a choice function on A, thereby proving 10.) Let X be the one point compactification of $\bigcup A$ with the discrete topology. The space X is Lindelöf, so by 341, X is second countable. This implies that $\bigcup A$ is countable, which implies that there is a choice function on A. $\square$

Form 133 in [hr] is the statement: Every set that cannot be well ordered has an amorphous subset. (Form 133 is called inf 2 in [rr] where it is shown that inf 2 is equivalent to AC in ZF, but not in ZF^0. It is true in the models $\mathcal{N}1$, $\mathcal{N}24$, $\mathcal{N}24(n)$, and $\mathcal{N}26$ in [hr] in which AC is false.) We shall show that 133 implies 340, thereby showing that forms 340 and 341 are also true in these models.

Theorem 5. *Form 133 implies 340.*

Proof. Suppose X with the metric d is a Lindelöf metric space. We shall prove, assuming form 133, that X is separable. First we shall show:

Lemma 1. *If M is an amorphous subset of X, then the range of $d/M \times M$ $(= d[M \times M])$, is finite.*

Proof. For each $m \in M$, $d[m \times M]$ is finite because M is amorphous. Thus,

$$d[M \times M] = \bigcup_{m \in M} d[m \times M]$$

is a finite union of finite sets, so it is finite.

Lemma 2. *If X has an amorphous subset then X is not Lindelöf.*

Proof. Let M be an amorphous subset of X and let $S = \{d(a, b) : a, b \in M\}$. By Lemma 1, S is a finite set and it is clearly non-empty because M has at least two elements. Choose a positive ϵ which is less than $\frac{1}{2}$ of the minimum number in S. It follows from Lemma 1 that ϵ exists.

Let $B(\epsilon, m)$, where $m \in M$, be an open neighborhood about m with radius ϵ. Then

$$\{B(\epsilon, m) : m \in M\} \cup \{X \setminus \overline{M}\}$$

is an infinite open cover with no countable subcover, Thus, X is not Lindelöf.

It follows from Lemma 2 and 133 that X can be well ordered. We shall show that X is separable in the following lemma, thereby proving the Theorem.

Lemma 3. *Every well ordered, metric, Lindelöf space is separable.*

Proof. Let X be a well ordered, metric, Lindelöf space. We construct a covering of X as follows: Let n be any positive integer. Let x_1 be the first element of X and let $N_{1,n}$ be a neighborhood of x_1 of radius $1/n$. Let x_2 be the smallest element of X which is not in $N_{1,n}$ and let $N_{2,n}$ be a neighborhood of x_2 of radius $1/n$. Continuing this process, we construct an open covering $\mathcal{C}_n$ of X. Since $\mathcal{C}_n$ has no proper subcovering and X is Lindelöf, $\mathcal{C}_n$ must be countable. We can do this for any positive integer n, so it follows that we can construct a countable dense subset of X. Thus, X is separable.

This completes the proof of Theorem 5. □

In Table I of [6] it is indicated as unknown whether DUM (form 381) implies that $\aleph_1$ is a regular cardinal (form 34) in ZF. In fact this implication is provable in ZF; furthermore, we will consider the following three forms, each of which is an intermediate form strictly between 381 and 34:

418. DUM($\aleph_0$): The countable disjoint union of metrizable spaces is metrizable.

419. UT($\aleph_0$,cuf,cuf): The union of a countable set of cuf sets is cuf. (A set is *cuf* if it is a countable union of finite sets.)

[**419 A**] UT(cuf,cuf,cuf): The union of a cuf set of cuf sets is cuf.

420. UT($\aleph_0$,$\aleph_0$,cuf): The union of a countable set of countable sets is cuf.

We shall show that 419 and [419 A] are equivalent.

Theorem 6. *UT($\aleph_0$,cuf,cuf) if and only if UT(cuf,cuf,cuf)*

Proof. Clearly, UT(cuf,cuf,cuf) implies UT($\aleph_0$,cuf,cuf). For the other direction, assume UT($\aleph_0$,cuf,cuf) and let X be a cuf family of cuf sets, that is, $X = \bigcup\{F_j : j \in \omega\}$ where F_j is a finite set of cuf sets for each $j \in \omega$. Then, $\bigcup F_j$ is cuf, for each $j \in \omega$, and $\bigcup X = \bigcup\{\bigcup F_j : j \in \omega\}$, which, by hypothesis, is cuf. □

Theorem 7. *Let $\{\langle X_i, T_i\rangle : i \in \omega\}$ be a pairwise disjoint collection of metrizable topological spaces.*

(a) The following are equivalent:

(i) There is a choice function $i \mapsto d_i$, for $i \in \omega$, where d_i is a metric on X_i compatible with the topology T_i.

(ii) $X = \bigcup\{X_i : i \in \omega\}$, with the topology generated by $\bigcup\{T_i : i \in \omega\}$, is metrizable.

(b) In addition, if $\prod\{X_i : i \in \omega\}$ is not empty, then the following condition is equivalent to both (i) and (ii):

(iii) $Y = \prod\{X_i : i \in \omega\}$ with the product topology is metrizable.

Proof.

(a) (i) $\Longrightarrow$ (ii): We can assume without loss of generality that each d_i, for $i \in \omega$, is bounded by 1. (Replace d_i by d_i^*, where $d_i^*(x, y) = \min\{d_i(x, y), 1\}$.) Then we can define a metric d_X on X by

$$d_X(x, y) = \begin{cases} d_i(x, y), & \text{if } x, y \in X_i, \\ 1, & \text{if } x \in X_i \text{ and } y \in X_j \text{ with } i \neq j. \end{cases}$$

(ii) $\Longrightarrow$ (i): If d_X is a metric for X, we can define d_i as the restriction of d_X to X_i.

(b) Assume now that $Y = \prod\{X_i : i \in \omega\} \neq \emptyset$.

(i) $\Longrightarrow$ (iii): As before, we can assume that each d_i, for $i \in \omega$, is bounded by 1. Then d_Y, defined by

$$d_Y(s, t) = \sum_{i \in \omega} 2^{-i} d_i(s(i), t(i)),$$

is a metric on Y compatible with the product topology.

(iii) $\Longrightarrow$ (i): Let d_Y be a metric on Y, and fix $s \in Y$. For each $i \in \omega$ and $x, y \in X_i$ we define d_i by

$$d_i(x, y) = d_Y(s_x, s_y),$$

where s_x is obtained from s by replacing $s(i)$ by x and s_y is obtained from s by replacing $s(i)$ by y. Then, d_i is a metric on X_i. □

The statement given in Theorem 7, part (ii) is form 418, DUM($\aleph_0$). Thus, it follows that the hypothesis of the following theorem can be replaced by DUM($\aleph_0$).

Theorem 8. *Assume that for every countable collection of metrizable spaces $\{Y_i : i \in \omega\}$ there is a choice function $i \to d_i$ such that d_i is a metric for Y_i, $i \in \omega$. Then every countable union of cuf sets is cuf*

Proof. Let $\{X_i : i \in \omega\}$ be a collection of cuf sets, which can be assumed to be pairwise disjoint, and take a set $Z = \{z_i : i \in \omega\}$ disjoint from $\bigcup\{X_i : i \in \omega\}$. We consider the collection of spaces $\{\langle Y_i, T_i\rangle : i \in \omega\}$, where for each $i \in \omega$ we have $Y_i = X_i \cup \{z_i\}$, and T_i is the topology generated by all the singletons $\{x\}$ with $x \in X_i$ together with all the sets $C \cup \{z_i\}$ with C a cofinite subset of X_i.

Each space Y_i is metrizable: choose a countable partition $\{F_j : j \in \omega\}$ of X_i in finite sets, and determine $d(x, y)$ by:
(i) $1/(j + 1)$, if $x = z_i$ and $y \in F_j$,
(ii) $1/(j + 1)$, if $x, y \in F_j$, and
(iii) $1/(j_1 + 1) - 1/(j_2 + 1)$, if $x \in F_{j_1}$, $y \in F_{j_2}$, and $j_1 < j_2$.

Therefore, by hypothesis, there is a choice function $i \mapsto d_i$ of metrics for the spaces Y_i, $i \in \omega$ (of course, these are not necessarily the same as the ones described above). Now, we have

$$\bigcup\{X_i : i \in \omega\} = \bigcup\{G_{i,j} : i, j \in \omega\},$$

with

$$G_{i,j} = \{x \in X_i : d_i(x, z_i) \geq 1/(j + 1)\}.$$

It is clear that each G_{ij} is finite (some of them can be empty). □

The principle "countable unions of countable sets are cuf" (UT($\aleph_0$,$\aleph_0$,cuf)), seems rather weak. However we have the following:

Theorem 9. *UT($\aleph_0$,$\aleph_0$,cuf) implies ω_1 is regular.*

Proof. Suppose that ω_1 is a countable union of countable sets. Then, by hypothesis, it is a countable union of finite sets. Now, a countable union of finite sets of ordinals can be easily proved to be countable, and this is contradiction. □

It is clear that form 31, UT($\aleph_0$,$\aleph_0$,$\aleph_0$), implies that cuf sets are countable, so 31 implies UT($\aleph_0$,cuf,cuf). In figure 1 below, we summarize the relationships between forms 31, 34, 381, 418, 419, and 420.

Next we look at the independence results for these forms.

Theorem 10. *For the following implications, (i) is not provable in ZF^0, and (ii) and (iii) are not provable in ZF.*
(i) $UT(\mathit{cuf}, \mathit{cuf}, \mathit{cuf}) \to 31$.
(ii) $34 \to UT(\aleph_0, \aleph_0, \mathit{cuf})$.
(iii) $31 \to DUM(\aleph_0)$.

Proof. For (i), consider the permutation model $\mathcal{N}2$ in [hr]. (In $\mathcal{N}2$, the set of atoms $A = \{a_i : i \in \omega\}$ and $B = \{\{a_{2i}, a_{2i+1}\} : i \in \omega\}$. The group of permutations consists of all permutations which leave B point-wise fixed and the supports are finite.) Form 31 fails in $\mathcal{N}2$, but in this model every set is a well-orderable union of finite sets. Since every cuf set in $\mathcal{N}2$ is countable in the model $\mathcal{N}$ of ZFA+AC from where $\mathcal{N}2$ was constructed, we have that a countable union of cuf sets in $\mathcal{N}2$ is countable in $\mathcal{N}$, and therefore, is an at most countable union of finite sets in $\mathcal{N}2$.

To prove (ii) we construct a permutation model in which the set A of atoms is a countable union of countable sets, but is not cuf. This permutation model can be easily embedded into a model of ZF using the techniques of Jech/Sochor, [js1] and [js2], in such a way that there is a set $\tilde{A}$ with the stated property but still $\aleph_1$ is regular.

To construct the permutation model, let $P = \{A_i : i \in \omega\}$ be a partition of A into countable sets; we use the group $\mathcal{G}$ of all permutations of A that fix P pointwise, and take supports of the form $\bigcup_{i\in n} A_i$ for each $i \in \omega$.

Clearly A is a countable union of countable sets in the resulting permutation model $\mathcal{N}$. To see that A is not cuf, let $R \in \mathcal{N}$ be a countable partition of A. It is easy to see that if $\bigcup_{i\in n} A_i$ is a support for a countable enumeration of R and $r \in R$ such that $r \cap A_{n+1} \neq 0$, then $A_{n+1} \subseteq r$. Thus R is not a partition into finite sets.

For (iii) we construct a permutation model in which the set A of atoms is a union of countably many metric spaces for which there is no choice function for the metrics, and in which Form 31 holds. This result is again transferable to ZF.

To construct the permutation model, let $\{\, R_{n,i} : n \in \omega,\ i \in \omega \,\}$ be a partition of A into continuum sized sets, and fix bijections $f_{n,i} : \mathbb{R} \to R_{n,i}$. Let $A_n = \bigcup\{\, R_{n,i} : i \in \omega\}$; in the permutation model the A_n's will be metric spaces whose union is not metrizable. Let D_n be the metric on each A_n such that each $f_{n,i}$ is an isometry $\mathbb{R} \to R_{n,i}$ and such that the distance between elements in different $R_{n,i}$'s is always 1.

Let G^+ be the group of permutations π of A such that:

(a) π is the identity except on at most finitely many $R_{n,i}$'s (i.e., for some $i_0 \in \omega$ and some $n_0 \in \omega$, π fixes $A \setminus \bigcup\{\, R_{n,i} : i \leq i_0,\ n \leq n_0 \,\}$ pointwise).

(b) $(\forall i \in \omega)\,(\exists j \in \omega)\,\pi R_{n,i} = R_{n,j}$ (thus $\pi A_n = A_n$ for all $n \in \omega$).

(c) If $\pi R_{n,i} = R_{n,j}$, then $f_{n,j}^{-1} \circ \pi \circ f_{n,i}$ is an affine transformation of $\mathbb{R}$.

Let $\mathcal{D} = \{\, \pi D_n : n \in \omega, \pi \in G^+ \,\}$. The group G we use to define the permutation model is the group generated by elements $g \in G^+$ such that g is an isometry of some $d \in D$. Let $\mathcal{W}$ be the set of relations w such that w is a well-ordering of some $R_{n,i}$. We take the set of supports to be the set of finite subsets of $\mathcal{D} \cup \mathcal{W}$, and call the resulting permutation model $\mathcal{N}$.

To see that DUM($\aleph_0$) is false in $\mathcal{N}$, let T_n be the topology on A_n induced by the metric D_n. Each T_n is preserved by every permutation in G, so the disjoint union topology T on $A = \bigcup\{\, A_n : n \in \omega \,\}$ exists in $\mathcal{N}$. Suppose $d \in \mathcal{N}$ is a metric on A compatible with T, and let S be a support for d. Pick $N \in \omega$ such that A_N has no metric in S and such that no subset of A_N has an ordering in S.

For any metric e on A_N in $\mathcal{D}$, we can find $\pi \in G$ such that $\pi A_{N,0} = A_{N,1}$, $\pi e = e$, and π fixes $A \setminus A_n$ pointwise. Fix such an e and π. For many choices of e it must be the case that $\pi d \neq d$. If we happened to choose such that $\pi d = d$, then replace e by the metric that is just like e except that distances on A_N are doubled, and change π accordingly. Now we have $\pi d \neq d$. Yet $\pi S = S$, and this is a contradiction. Thus A has no metric compatible with T, and this shows that DUM($\aleph_0$) is false.

Note that if $x \in \mathcal{N}$, then x has a support E such that no two elements in $E \cap \mathcal{W}$ have the same domain and no two elements in $E \cap \mathcal{D}$ have the same domain (If d and e in $\mathcal{D}$ have the same domain then e is supported by d together with finitely many elements of $\mathcal{W}$, since d and e differ on only finitely many $R_{n,i}$'s). Say E is a "standard support" if no two of its elements have the same domain. For a set s, let $\operatorname{fix}(s)$ denote the pointwise stabilizer of s in G.

Claim 1. Suppose E is a support for x with $w \in E \cap \mathcal{W}$. Suppose $\exists \pi \in G$ such that (1) $\pi x = x$, (2) $\operatorname{dom}(\pi w)$ is not the domain of any $v \in E \cap \mathcal{W}$, and (3) $\pi \in \operatorname{fix}(E \setminus \{w\})$. Then $E \setminus \{w\}$ is a support for x.

Proof of Claim 1. Let $\rho \in \operatorname{fix}(E \setminus \{w\})$; we will show that $\rho x = x$.

Let $m_\pi = \operatorname{dom} \pi w$ and $m_\rho = \operatorname{dom} \rho w$. Define σ by the conditions $\sigma \restriction m_\pi = \rho\pi^{-1} \restriction m_\pi$ and $\sigma \restriction m_\rho = \pi\rho^{-1} \restriction m_\rho$ and $\sigma \restriction A \setminus (m_\pi \cup m_\rho) = \mathrm{Id}$. Both π and ρ are in $\operatorname{fix}(E \setminus \{w\})$, so it follows that also $\sigma \in \operatorname{fix}(E \setminus \{w\})$ (to see that σ fixes a metric in E, notice that σ fixes the metric on each component and that the components are all the same distance apart). Furthermore, $\sigma w = w$, since $(\operatorname{dom} w) \cap (m_\pi \cup m_\rho) = 0$. Thus $\sigma \in \operatorname{fix}(E)$.

Now we have $\rho \restriction E = \sigma\pi \restriction E$, and E supports x, so $\rho x = \sigma\pi x$. And σ and π both fix x, so $\rho x = x$, as desired.

Now the main part of proving that Form 31 holds in $\mathcal{N}$ is establishing

Claim 2. Let S be a standard support for X, and suppose there is an element $y \in X$ such that y is not supported by S. Then X is not well-orderable in $\mathcal{N}$.

Proof of Claim 2. Let S_y be a standard support for y containing S such that no proper subset of S_y containing S is a support for y.

Case 1: $\exists w \in \mathcal{W} \cap S_y \setminus S$.

Let $m = \operatorname{dom} w$, and let n be the ordinal such that $m \subset A_n$. Let

$$B := \{\, R_{n,i} : i \in \omega \text{ and } R_{n,i} \text{ is not ordered by any } v \in \mathcal{W} \cap S_y \,\}.$$

Let H be the set of $h \in G$ such that $hm \in B$ and $h \in \operatorname{fix}(S_y \cup A \setminus (m \cup hm))$.

Since $H \subset \operatorname{fix}(S)$, $hy \in X$ for all $h \in H$. Consider the set $\{\,\{\, hy : h \in H \text{ and } hm = b\,\} : b \in B\,\}$. We will show that this is a partition (of a subset of X) with the same cardinality as that of B. Since B is clearly amorphous, it then follows that X cannot be well-ordered.

Fix $b_0 \neq b_1$ both in B and for $i \in 2$ fix $h_i \in H$ such that $h_i m = b_i$. To establish the claim it just remains to show that $h_0 y \neq h_1 y$. Suppose by way of contradiction that $h_0 y = h_1 y$, whence $h_1^{-1} h_0 y = y$. Note that $h_1^{-1} h_0(\operatorname{dom} w) = h_1^{-1} b_0 = b_0$, and that $(h_1^{-1} h_0) \in \operatorname{fix}(S_y \setminus \{w\})$. By Claim 1, $S_y \setminus \{w\}$ is a support for y. But this violates the minimality condition by which S_y was chosen, so we have a contradiction.

Case 2: $\mathcal{W} \cap S_y \setminus S = 0$.

In this case, $\exists d \in \mathcal{D} \cap S_y \setminus S$ such that d is a metric on A_n for some n, and such that $\pi y \neq y$ for some $\pi \in F := \operatorname{fix}((S_y \setminus \{d\}) \cup (A \setminus A_n))$.

Furthermore, we may assume that there is a metric $c \in \mathcal{D}$ on A_n such that $\pi c = c$ (recall that G by definition is generated by isometries of elements of $\mathcal{D}$).

Let $S' = S \cup \{c\}$. Clearly S' is a standard support for X. However, $\pi \in \mathrm{fix}(S')$, so S' is not a support for y. Let S'_y be a standard support for y containing S' such that no proper subset of S'_y containing S is a support for y. Since S'_y is a standard support, the only metric on A_n in S'_y is c. If there were no well-orderings in $S'_y \setminus S'$, we would have $\pi \in \mathrm{fix}(S'_y)$; this is impossible since $\pi y \neq y$. Thus we may apply Case 1 with S' in place of S and S'_y in place of S_y. This completes the proof of Claim 2.

Now using Claim 2 we can prove that a countable union of countable sets is countable in $\mathcal{N}$. To this end, let $X = \{\, x_n : n \in \omega \,\}$ be a countable set of countable sets in $\mathcal{N}$, and let E be a support for a well-ordering of X, so E is also a support for each $x_n \in X$. Since each x_n is well-orderable in $\mathcal{N}$, it follows from Claim 2 that E is also a support for each element of x_n. Since E is a support for each element of $\bigcup X$, it is a support for any well-ordering of $\bigcup X$, which must then be countable. □

It is unknown whether UT($\aleph_0$,$\aleph_0$,cuf) implies UT(cuf, cuf,cuf). We plan to study union theorems in more detail in another paper. We give some more independence results in the next theorem.

Theorem 11. *The following implications are not provable in ZF:*
(i) $8 \to 173$.
(ii) $8 \to 232$.
(iii) $8 \to 381, 382$.
(iv) $43 \to 173, 232, 381, 382$.

Proof.
(i) First we shall construct a permutation model in which form 8 is true and 173 is false. The set of atoms $A = \cup\{A_n : n \in \aleph_1\}$, where $A_n = \{a_{nx} : x \in B\}$ and B is the set of points on the unit circle centered at 0. The group of permutations $\mathcal{G}$ is the group of all permutations on A which rotate the A_n's by an angle $\theta_n \in \mathbb{R}$ and supports are countable. Let $\mathcal{N}$ be the permutation model. Then DC, and consequently CAC, holds in $\mathcal{N}$. Define a function $d : A \times A \to \mathbb{R}$ by requiring:

$$d(x,y) = d(y,x) = \left\{ \begin{array}{c} 1 \text{ if } x \in A_n, y \in A_m \text{ and } n \neq m \\ \rho(x,y) \text{ if } x, y \in A_n \end{array} \right\}.$$

where ρ is the bounded Euclidean metric. Then (A,d) is a non para-Lindelöf metric space. The proof follows ideas from Theorem 10 in [hkrs]. Consider the open cover $\mathcal{U} = \{D(x,\varepsilon) : x \in A_n, n \in \mathbb{N}, \varepsilon > 0\}$. Suppose that $\mathcal{U}$ has a locally countable open refinement $\mathcal{V} \in \mathcal{N}$. Let E be a support for $\mathcal{V}$ and let $n \in \aleph_1$ be such that $A_n \cap E = \emptyset$. Choose an element $x \in A_n$. Then $x \in V \subseteq D(y,\varepsilon) \subseteq A_n$ for some $V \in \mathcal{V}$, $y \in A_n$ and $\varepsilon > 0$. Now $D(y,\varepsilon) = \widehat{ab}$ (= the arc determined by $a, b \in A_n$). Let $c, d \in A_n$ such that $x \in int(\widehat{cd}) \subseteq V$, where $int(\widehat{cd})$ is the interior of the arc $\widehat{cd}$. For each central angle θ such that $0 < \theta < \widehat{xd}$ consider the permutation π_θ which is the identity outside A_n and rotates A_n by θ. Then $V = \pi_\theta(V) \in \pi_\theta(\mathcal{V}) = \mathcal{V}$ and the set $\{\pi_\theta(V) : 0 < \theta < \widehat{xd}\}$ is an uncountable subset of $\mathcal{V}$ each element of which contains x.

It follows from the transfer theorems of Jech/Sechor ([js1, js2]) and Pincus ([pi1, pi2]) that this result also holds in ZF. (Also see [hr] notes 18 and 103.)

(ii) This follows from (i) because 232 implies 173.

(iii) This follows from [ta], Theorem 2.4 and Remark 1. (For every regular cardinal λ, $\mathrm{DC}_\kappa, \kappa < \lambda$, does not imply either of 381 (DUM) or 382 (DUMN) in ZF).

(iv) This follows from the preceding results because 43 (DC) is true in the model given in part (i) and 43 is transferable.

Corollary 1. *Theorem 11 holds if 8 is replaced by any one of 9, 10, 31, 34, 64, 94, 126, 340, 341 or 418.*

Proof. This Corollary holds because 8 implies each of the ten forms given above. □

Corollary 2. *Theorem 11 holds if 43 is replaced by any one of 8, 9, 10, 31, 34, 64, 94, 126, 340, 341 or 418.*

Proof. Form 43 implies each of the eleven forms given above. □

An implication diagram which also shows independence results is given in the next diagram.

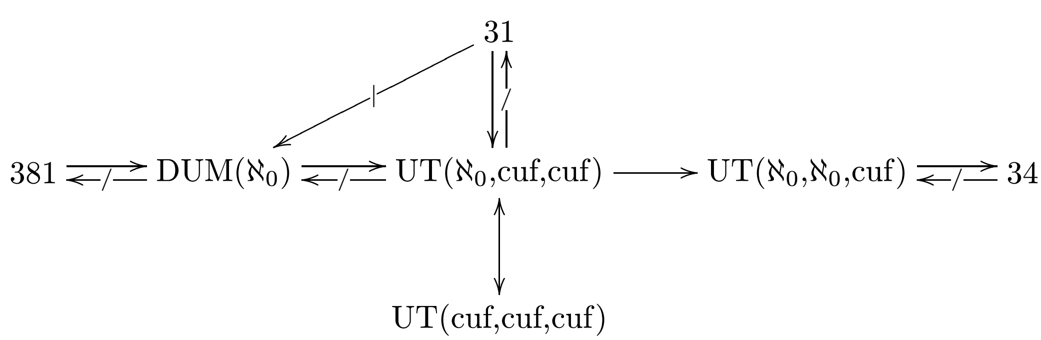
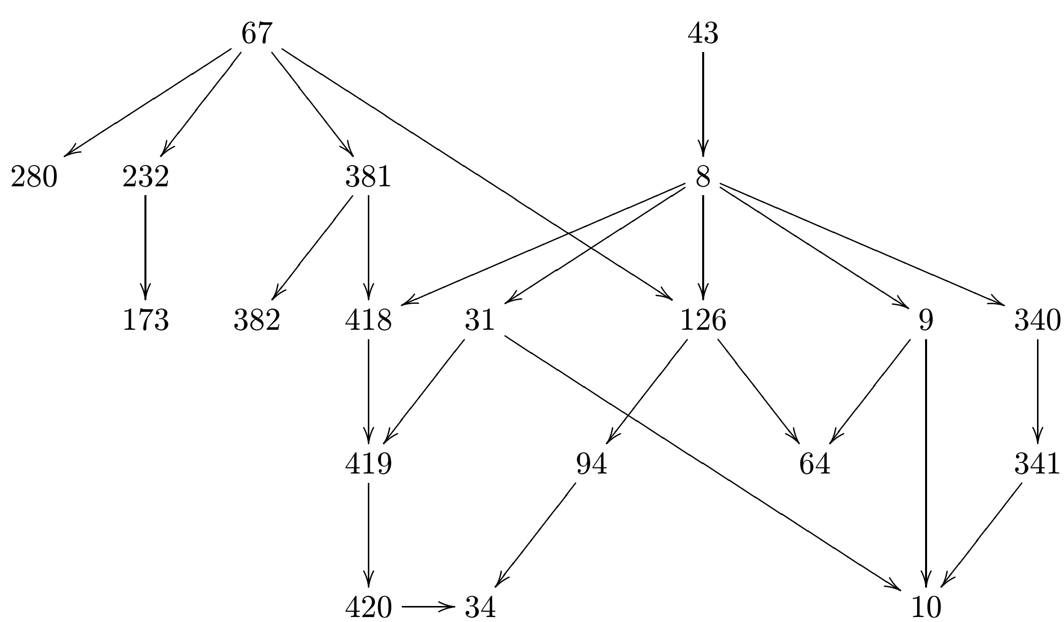


Figure 1

It is shown in [hr] that 381 is true and 31 is false in the following models: $\mathcal{N}2$, $\mathcal{N}2(n)$, $\mathcal{N}2^*(3)$, $\mathcal{N}6$, and $\mathcal{N}50(E)$. Also, additional results that are shown in the matrix below and are not proven here, can be found in [hr].

In the matrix below, if there is a "0" at position (m, n), it means that it is unknown whether statement m implies statement n, a "1" means that statement m implies statement n; a "3" means that there is a Cohen model in which statement m is true and statement n is false, and a "5" means that there is a Fraenkel-Mostowski (permutation) model in which statement m is true and statement n is false. The implications and independence results that are known about the statements given in this paper are summarized in the matrix and arrow diagram below.

	8	9	10	31	34	43	64	67	94	126	173	232	280	340	341	381	382	418	419	420
8	1	1	1	1	1	3	1	3	1	1	3	3	3	1	1	3	3	1	1	1
9	3	1	1	3	0	3	1	3	3	3	3	3	3	3	3	3	3	0	0	0
10	3	3	1	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
31	3	3	1	1	1	3	3	3	3	3	3	3	3	0	0	3	3	3	1	1
34	3	3	3	3	1	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
43	1	1	1	1	1	1	1	3	1	1	3	3	3	1	1	3	3	1	1	1
64	3	3	3	3	0	3	1	3	3	3	3	3	3	3	3	3	3	0	0	0
67	5	5	5	5	1	5	1	1	1	1	1	1	1	5	5	1	1	1	1	1
94	3	3	3	3	1	3	3	3	1	3	3	3	3	3	3	3	3	0	0	0
126	5	5	5	5	1	3	1	3	1	1	3	3	3	5	5	3	3	0	0	0
173	3	5	5	3	0	3	5	3	0	3	1	0	0	5	5	0	0	0	0	0
232	5	5	5	5	0	5	5	5	0	5	1	1	0	5	5	0	0	0	0	0
280	3	3	3	3	0	3	3	3	3	3	3	3	1	3	3	3	3	0	0	0
340	5	5	1	0	0	3	5	3	0	5	3	3	3	1	1	3	3	0	0	0
341	5	5	1	0	0	3	5	3	0	5	3	3	3	0	1	3	3	0	0	0
381	5	5	5	5	1	5	0	5	0	5	0	0	0	5	5	1	1	1	1	1
382	5	5	5	5	0	5	0	5	0	5	0	0	0	5	5	0	1	0	0	0
418	5	5	5	5	1	3	0	3	0	5	3	3	3	5	5	3	3	1	1	1
419	3	3	5	5	1	3	3	3	3	3	3	3	3	5	5	3	3	3	1	1
420	3	3	5	5	1	3	3	3	3	3	3	3	3	5	5	3	3	3	0	1



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