

# PARACOMPACTNESS OF METRIC SPACES AND THE AXIOM OF MULTIPLE CHOICE

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ABSTRACT. The axiom of multiple choice implies that metric spaces are paracompact but the reverse implication cannot be proved in set theory without the axiom of choice.

## 1. Background, Definitions and Summary of Results.

Working in set theory without the axiom of choice we study the deductive strength of the assertion MP: *Metric spaces are paracompact*. (Definitions are given below.) MP was first proved in 1948 by A. H. Stone ([17]) using the axiom of choice (AC). A considerably shortened proof was given by Mary Ellen Rudin in [15]. In Rudin's proof the use of the axiom of choice is evident since the proof uses a well-ordering of an arbitrary open cover of a metric space. More recently, Good and Tree have shown that a metric space is paracompact provided it has a well-ordered dense subset. If we let (\*) and (\*\*) represent the following statements:

- (\*) Every open cover of every metric space can be well-ordered.
- (\*\*) Every metric space has a well-ordered dense subset.

then the results of Rudin and Good/Tree can be written respectively as  $(*) \Rightarrow \text{MP}$  and  $(**) \Rightarrow \text{MP}$ . Both are theorems in set theory without the axiom of choice. However (\*) and (\*\*) are both equivalent to the axiom of choice. (Let  $X$  be any set and  $d$  the metric defined on  $X$  by  $d(y, z) = 1$  if  $y \neq z$ . Since  $X$  is the only dense subset of  $X$ , (\*\*) gives a well-ordering of  $X$ . Similarly, applying (\*) to the open cover  $\{\{x\} : x \in X\}$  gives a well ordering of  $X$ .) Therefore neither the theorem of Rudin nor that of Good and Tree help us in placing MP in the deductive hierarchy of weak versions of the axiom of choice. (Although, as we will mention later, several of our results rely heavily on the proof given in [15].)

Some progress has been made in determining the deductive strength of MP. Recently Good, Tree and Watson [5] have constructed models of both Zermelo-Fraenkel set theory (ZF) and Zermelo-Fraenkel set theory with the axiom of foundation modified to permit the existence of atoms (ZF<sup>0</sup>) in which MP is false.

One of our purposes in this paper is to show that the axiom of multiple choice (MC) implies MP. This is one of our theorems that depends on [15]. We will also

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add to the independence results of [5] by showing that MP does not imply MC in  $\text{ZF}^0$ .

It is evident that MC plays a prominent role in our paper. MC was first studied by Lévy in [11]. Lévy showed that MC does not imply AC in  $\text{ZF}^0$ . It follows from results of Felgner and Jech [3] and H. Rubin [9] that MC and AC are equivalent in full Zermelo-Fraenkel set theory.

We will be using the following terminology from topology.

**Definition 1.** Let  $(X, T)$  be a topological space.

0. Let  $\mathcal{U}, \mathcal{V} \subseteq \mathcal{P}(X)$ . We say that  $\mathcal{U}$  *covers*  $X$  or that  $\mathcal{U}$  is a *covering* of  $X$  iff  $\bigcup \mathcal{U} = X$ . We say that  $\mathcal{V}$  is a *refinement* of  $\mathcal{U}$  iff  $\bigcup \mathcal{V} = X$  and every member of  $\mathcal{V}$  is included in a member of  $\mathcal{U}$ .
1. A family  $\mathcal{U}$  of subsets of  $X$  is *locally finite* (respectively *locally countable*) iff each point of  $X$  has a neighborhood meeting only a finite number (respectively countable number) of elements of  $\mathcal{U}$ .
2.  $(X, T)$  is *paracompact* (respectively *para-Lindelöf*) iff  $(X, T)$  is  $T_2$  and every open cover  $\mathcal{U}$  of  $X$  has an open locally finite (respectively locally countable) refinement.
3. A family  $\mathcal{U}$  of subsets of  $X$  is *point finite* (respectively *point countable*) iff each element of  $X$  belongs to only finitely many (respectively countably many) members of  $\mathcal{U}$ .
4.  $(X, T)$  is *metacompact* (respectively *meta-Lindelöf*) iff each open cover  $\mathcal{U}$  of  $X$  has an open point finite (respectively point countable) refinement.
5. A set  $C \subseteq \mathcal{P}(X)$  is *discrete* if  $\forall x \in X$ , there is a neighborhood  $U$  of  $x$  such that  $U \cap A \neq \emptyset$  for at most one element  $A \in C$ .
6. A set  $C \subseteq \mathcal{P}(X)$  is  *$\sigma$ -locally finite* (respectively  *$\sigma$ -point finite*,  *$\sigma$ -disjoint*,  *$\sigma$ -discrete*) if  $C = \bigcup_{n \in \omega} C_n$  where each  $C_n$  is locally finite (respectively, point finite, pairwise disjoint, discrete).
7.  $(X, T)$  is *paradiscrete* (respectively *paradisjoint*) iff  $(X, T)$  is  $T_2$  and every open cover  $\mathcal{U}$  has a  $\sigma$ -discrete (respectively  $\sigma$ -disjoint) open refinement.
8.  $(X, T)$  is *subparacompact* iff  $(X, T)$  is  $T_2$  and every open cover  $\mathcal{U}$  has a  $\sigma$ -discrete closed refinement.
9. Assume that  $\gamma$  is a well ordered cardinal number, that is,  $\gamma$  is an initial ordinal. A set  $C \subseteq \mathcal{P}(X)$  is  *$\gamma$ -locally finite* (respectively  *$\gamma$ -point finite*,  *$\gamma$ -disjoint*,  *$\gamma$ -discrete*) if  $C = \bigcup_{\alpha \in \gamma} C_\alpha$  where each  $C_\alpha$  is locally finite (respectively, point finite, pairwise disjoint, discrete).

We will consider the following statements all of which are known to be theorems of  $\text{ZF}^0 + \text{AC}$ .

**Definition 2.**

1. AC: The *Axiom of Choice*. For every family  $\mathcal{A} = \{A_i : i \in k\}$  of non-empty pairwise disjoint sets there exists a set  $C$  which consists of one and only one element from each element of  $\mathcal{A}$ .
2. MC: The *Multiple Choice Axiom*: For every family  $\mathcal{A} = \{A_i : i \in k\}$  of non-empty pairwise disjoint sets there exists a family  $\mathcal{F} = \{F_i : i \in k\}$  of finite non-empty sets such that for every  $i \in k$ ,  $F_i \subseteq A_i$ .
3.  $\omega$ -MC: For every family  $\mathcal{A} = \{A_i : i \in k\}$  of non-empty pairwise disjoint sets there exists a family  $\mathcal{F} = \{F_i : i \in k\}$  of countable non-empty sets such that for every  $i \in k$ ,  $F_i \subseteq A_i$ .

4. OP: The ordering principle: Every set can be linearly ordered.
5. MP: Metric spaces are paracompact.
6. MM: Metric spaces are metacompact.
7. MPL: Metric spaces are para-Lindelöf.
8. MML: Metric spaces are meta-Lindelöf.
9. MPDJ: Metric spaces are paradisjoint.
10. MPDC: Metric spaces are paradiscrete.
11. MSP: Metric spaces are subparacompact.
12. MCLFR: Every open cover  $\mathcal{U}$  of a metric space  $(X, d)$  has a closed locally finite refinement  $\mathcal{V}$ .

The following statements are also provable in  $\text{ZF}^0 + \text{AC}$ .

**Definition 3.**

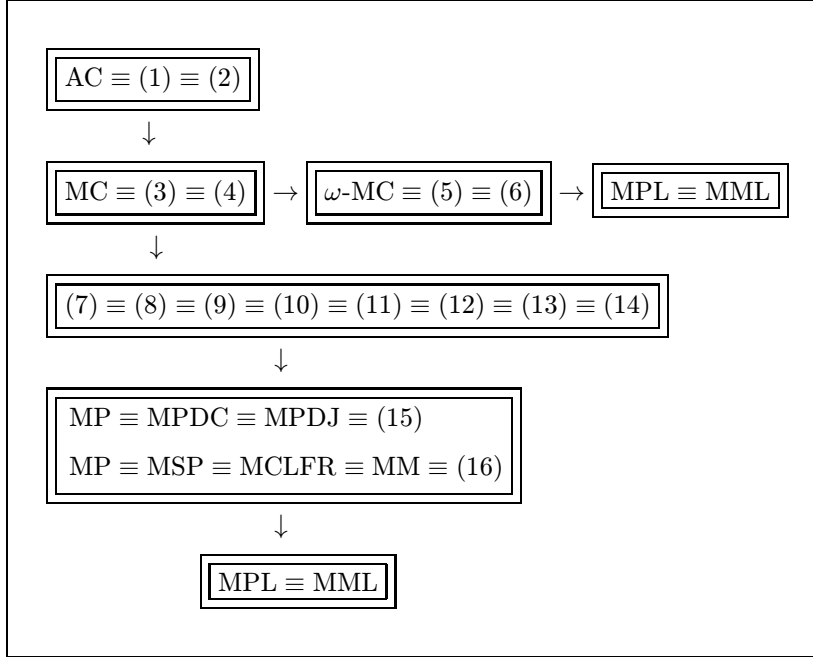
- (1) Every open cover  $\mathcal{U}$  of a metric space can be written as a well ordered union  $\bigcup\{U_\alpha : \alpha \in \gamma\}$  where  $\gamma$  is an ordinal and each  $U_\alpha$  is pairwise disjoint.
- (2) Every open cover  $\mathcal{U}$  of a metric space can be written as a well ordered union  $\bigcup\{U_\alpha : \alpha \in \gamma\}$  where  $\gamma$  is an ordinal and each  $U_\alpha$  is discrete.
- (3) Every open cover  $\mathcal{U}$  of a metric space can be written as a well ordered union  $\bigcup\{U_\alpha : \alpha \in \gamma\}$  where  $\gamma$  is an ordinal and each  $U_\alpha$  is locally finite.
- (4) Every open cover  $\mathcal{U}$  of a metric space can be written as a well ordered union  $\bigcup\{U_\alpha : \alpha \in \gamma\}$  where  $\gamma$  is an ordinal and each  $U_\alpha$  is point finite.
- (5) Every open cover  $\mathcal{U}$  of a metric space can be written as a well ordered union  $\bigcup\{U_\alpha : \alpha \in \gamma\}$  where  $\gamma$  is an ordinal and each  $U_\alpha$  is locally countable.
- (6) Every open cover  $\mathcal{U}$  of a metric space can be written as a well ordered union  $\bigcup\{U_\alpha : \alpha \in \gamma\}$  where  $\gamma$  is an ordinal and each  $U_\alpha$  is point countable.
- (7) Every metric space has a  $\sigma$ -discrete base.
- (8) Every metric space has a  $\gamma$ -discrete base, for some ordinal  $\gamma$ .
- (9) Every metric space has a  $\sigma$ -locally finite base.
- (10) Every metric space has a  $\gamma$ -locally finite base, for some ordinal  $\gamma$ .
- (11) Every metric space has a  $\sigma$ -disjoint base.
- (12) Every metric space has a  $\gamma$ -disjoint base, for some ordinal  $\gamma$ .
- (13) Every metric space has a  $\sigma$ -point finite base.
- (14) Every metric space has a  $\gamma$ -point finite base, for some ordinal  $\gamma$ .
- (15) Every open cover  $\mathcal{U}$  of a metric space  $(X, d)$  has a refinement  $\mathcal{V}$  which can be written as a well ordered union  $\bigcup\{V_\alpha : \alpha \in \gamma\}$  where each  $V_\alpha$  is locally finite.
- (16) Every open cover  $\mathcal{U}$  of a metric space  $(X, d)$  has a refinement  $\mathcal{V}$  which can be written as a well ordered union  $\bigcup\{V_\alpha : \alpha \in \gamma\}$  where each  $V_\alpha$  is point finite.

It is clear that every paracompact space is metacompact, therefore, MP implies MM. (It is also clear that  $\text{MC} \rightarrow (10)$ ,  $(9) \rightarrow (13) \rightarrow (14) \rightarrow \text{MM} \rightarrow \text{MML}$ , and  $\text{MP} \rightarrow \text{MPL} \rightarrow \text{MML}$ .) The concept of metacompactness has been studied by A. H. Stone [17] and Arens and Dugundji [1]. With regard to property (7), it is a theorem of Bing [2] that a topological space is metrizable if and only if it is  $T_1$ , regular and has a  $\sigma$ -discrete base. With regard to (9), Nagata [13] and Smirnov [16] have proven independently that a topological space is metrizable if and only if it is  $T_1$ , regular and has a  $\sigma$ -locally finite base. The proofs of Bing, Nagata, and Smirnov just mentioned take place in set theory with the axiom of choice.

Thus, as the referee suggested, this may lead to another area of study. What is the strength of the converse of some of our statements? For example, what is

the strength of the statement “Regular  $T_1$  spaces with  $\sigma$ -locally finite bases are metrizable”? Various authors have studied conditions for metrizability. (See, for example, [10] pp124ff, [4], [6], [12], [13], [14], and [16].) In some cases, the proofs of these theorems use some form of AC. It might be interesting to study whether the statements, themselves, imply some form of choice.

We close this section with a summary of the results beginning with the following diagram.



The independence results are

A. Statement (8) does not imply MC in  $ZF^0$  (Theorem 9). Therefore, from the diagram we can conclude that none of (7) through (14) imply MC.

B. MML is not provable in  $ZF^0$  (Theorem 10). Consequently, it follows that none of the statements listed in the diagram are provable in  $ZF^0$ .

C. (5) does not imply MM in  $ZF^0$  (Theorem 11). Thus, neither MPL nor MML implies MM.

The proof of B is similar to a proof in [7] where it is observed that MM fails in the permutation model of [5].

## 2. The implications.

In our first theorem in this section we prove the equivalences of AC, MC and  $\omega$ -MC which are shown in the diagram above.

### Theorem 1.

(A) *The following are equivalent: AC.*

- (1): Every open cover  $\mathcal{U}$  of a metric space  $(X, d)$  can be written as a well ordered union  $\bigcup\{U_\alpha : \alpha \in \gamma\}$  where  $\gamma$  is an ordinal and each  $U_\alpha$  is disjoint.
- (2): Every open cover  $\mathcal{U}$  of a metric space  $(X, d)$  can be written as a well ordered union  $\bigcup\{U_\alpha : \alpha \in \gamma\}$  where  $\gamma$  is an ordinal and each  $U_\alpha$  is discrete.
- (B) The following are equivalent: MC.
- (3) Every open cover  $\mathcal{U}$  of a metric space  $(X, d)$  can be written as a well ordered union  $\bigcup\{U_\alpha : \alpha \in \gamma\}$  where  $\gamma$  is an ordinal and each  $U_\alpha$  is locally finite.
- (4) Every open cover  $\mathcal{U}$  of a metric space  $(X, d)$  can be written as a well ordered union  $\bigcup\{U_\alpha : \alpha \in \gamma\}$  where  $\gamma$  is an ordinal and each  $U_\alpha$  is point finite.
- (C) The following are equivalent:  $\omega$ -MC.
- (5) Every open cover  $\mathcal{U}$  of a metric space  $(X, d)$  can be written as a well ordered union  $\bigcup\{U_\alpha : \alpha \in \gamma\}$  where  $\gamma$  is an ordinal and each  $U_\alpha$  is locally countable.
- (6) Every open cover  $\mathcal{U}$  of a metric space  $(X, d)$  can be written as a well ordered union  $\bigcup\{U_\alpha : \alpha \in \gamma\}$  where  $\gamma$  is an ordinal and each  $U_\alpha$  is point countable.

*Proof.* We shall prove part (B). The proofs of the others are similar.

MC  $\rightarrow$  (3): This follows from Levy's [11] characterization of MC, MC iff every set  $A$  can be written as a well ordered union of finite sets, and the fact that any finite collection of sets is locally finite.

(3)  $\rightarrow$  (4): This holds because locally finite implies point finite.

(4)  $\rightarrow$  MC: Fix  $A = \{A_i : i \in k\}$  a family of pairwise disjoint non-empty sets. We may assume, without loss of generality, that each  $A_i$  is infinite. Let  $\{y_i : i \in k\}$  be distinct sets so that for each  $i \in k$ ,  $y_i \notin \bigcup A$ . The discrete topology on  $X = \bigcup_{i \in k} A_i \cup \{y_i\}$  is metrizable and  $\mathcal{U} = \{\{x, y\} : (\exists i \in k)(x \in A_i \wedge y = y_i)\}$  is an open cover of  $X$ . By the hypothesis, we have  $\mathcal{U} = \bigcup\{U_\alpha : \alpha \in \gamma\}$ , where  $\gamma$  is an ordinal and each  $U_\alpha$  is point finite. For each  $i \in k$ , let  $\alpha_i$  be the least  $\alpha$  such that  $y_i \in \bigcup U_\alpha$ .  $U_\alpha$  is point finite so  $y_i$  is in only finitely many members of  $U_\alpha$ . Therefore, if we define  $f(A_i) = (\bigcup\{z \in U_{\alpha_i} : y_i \in z\}) \setminus \{y_i\}$ , then  $f$  is the desired multiple choice function.  $\square$

In several of the theorems that follow we will need the following result which we refer to as Rudin's lemma. The argument given in [15] is a proof (in  $ZF^0$ ) of this lemma.

**M. E. Rudin's Lemma.** *For any metric space  $(X, d)$ , there is a function  $P$  such that for any ordinal  $\gamma$  and any function  $C : \alpha \mapsto C_\alpha$  from  $\gamma$  to the open subsets of  $X$  such that  $\mathcal{C} = \{C_\alpha : \alpha \in \gamma\}$  is a cover of  $X$ ,  $P(\mathcal{C})$  is an open, locally finite cover of  $X$  which refines  $\mathcal{C}$ .*

**Theorem 2.** (15) iff MP and (16) iff MM.

*Proof.* It is clear that MP implies (15) and MM implies (16). The proofs that (15) implies MP and (16) implies MM are almost identical. We will prove that (15) implies MP and indicate in parentheses the changes that would have to be made to get a proof that (16) implies MM. Note that in proving (15) implies MP we actually prove that for a given metric space  $(X, d)$  with metric topology  $T$ , there is a function  $F$  such that for every ordinal  $\gamma$  and every function  $V : \alpha \mapsto V_\alpha$  from  $\gamma$  into the set of locally finite subsets of  $T$  such that  $\mathcal{V} = \bigcup\{V_\alpha : \alpha < \gamma\}$ ,  $F(\mathcal{V})$  is an open locally finite refinement of  $\mathcal{V}$  which covers  $X$ .

Suppose  $(X, d)$  is a metric space and  $\mathcal{U}$  is an open cover of  $X$ . (15) implies that  $\mathcal{U}$  has an open refinement  $\mathcal{V} = \bigcup\{V_\alpha : \alpha \in \gamma\}$  where each  $V_\alpha$  is locally finite. ((16)

implies that  $\mathcal{U}$  has an open refinement  $\mathcal{V} = \bigcup\{V_\alpha : \alpha \in \gamma\}$  where each  $V_\alpha$  is point finite.) For each  $\alpha \in \gamma$ , let  $C_\alpha = \bigcup V_\alpha$ . Then  $\mathcal{C} = \{C_\alpha : \alpha \in \gamma\}$  is a well ordered covering of  $X$ . By Rudin's lemma there is an open, locally finite cover  $P(\mathcal{C}) = \mathcal{D}$  of  $X$  which refines  $\mathcal{C}$ . For each  $D \in \mathcal{D}$ , let  $\alpha(D)$  be the least ordinal  $\alpha$  such that  $D \subseteq C_\alpha$ . Let

$$\mathcal{W} = \{D \cap v : D \in \mathcal{D} \text{ and } v \in V_{\alpha(D)}\}.$$

( $\mathcal{W} = F(\mathcal{V})$  where  $F$  is the function mentioned in the opening paragraph of the proof and  $V : \alpha \mapsto V_\alpha$ .) It is clear that  $\mathcal{W}$  is an open cover of  $X$  which refines  $\mathcal{V}$  and therefore refines  $\mathcal{U}$ . To show that  $\mathcal{W}$  is locally finite (point finite), suppose  $x \in X$ . Since  $\mathcal{D}$  is locally finite there is a neighborhood  $G_x$  of  $x$  which meets only finitely many elements  $D_1, D_2, \dots, D_k$  of  $\mathcal{D}$ . This means that the only elements of  $\mathcal{W}$  meeting  $G_x$  are of the form  $D_i \cap v$  where  $1 \leq i \leq k$  and  $v \in V_{\alpha(D_i)}$ . Since each  $V_\alpha$  is locally finite we may choose neighborhoods  $G_i$  of  $x$ , where  $1 \leq i \leq k$ , such that  $G_i$  meets only finitely many elements of  $V_{\alpha(D_i)}$ . Then  $G_x \cap G_1 \cap \dots \cap G_k$  is a neighborhood of  $x$  meeting only finitely many elements of  $\mathcal{W}$ . (Since  $\mathcal{D}$  is locally finite only finitely many elements  $D_1, D_2, \dots, D_k$  of  $\mathcal{D}$  contain  $x$ . This means that the only elements of  $\mathcal{W}$  containing  $x$  are of the form  $D_i \cap v$  where  $1 \leq i \leq k$  and  $v \in V_{\alpha(D_i)}$ . Since each  $V_\alpha$  is point finite we may choose neighborhoods for  $1 \leq i \leq k$  only finitely many elements of  $V_{\alpha(D_i)}$  contain  $x$ . Therefore, only finitely many elements of  $\mathcal{W}$  contain  $x$ .)  $\square$

**Corollary.** (5) implies MPL and (6) implies MML.

To prove that statements (7) through (12) are equivalent we note first that the following implications are clear: (7) implies (8), (7) implies (9), (8) implies (10), and (9) implies (10). Below we show that (10) implies (9) (Theorem 3); (9) implies (11) and (10) implies (12) (Corollary 1 to Theorem 4); and (11) implies (7) and (12) implies (8) (Corollary 1 to Theorem 5). In what follows we will use  $N(\epsilon, x)$  to denote the ball of radius  $\epsilon$  and center at  $x$  in a metric space.

**Theorem 3.** (10) implies (9)

*Proof.* Let  $(X, d)$  be a metric space. Assuming (10), there is an ordinal  $\gamma$  such that  $(X, d)$  has a  $\gamma$ -locally finite base  $\mathcal{B} = \bigcup\{\mathcal{B}_\alpha : \alpha < \gamma\}$  where for each  $\alpha < \gamma$ ,  $\mathcal{B}_\alpha$  is locally finite. For each  $\alpha < \gamma$ , let  $\mathcal{C}_\alpha^0 = \mathcal{B}_\alpha$  and for each  $n \in \omega^+$  let  $\mathcal{C}_\alpha^n = \{B \in \mathcal{B}_\alpha : (\exists x \in X)(B \subseteq N(\frac{1}{n}, x))\}$ . Let  $\mathcal{C}^n = \bigcup\{\mathcal{C}_\alpha^n : \alpha < \gamma\}$ . Finally, define the function  $G^n$  with domain  $\gamma$  by  $G^n(\alpha) = \mathcal{C}_\alpha^n$ . Clearly  $\mathcal{C}^n$  is  $\gamma$ -locally finite. Further, since  $\mathcal{B}$  is a base for the metric topology on  $X$ ,  $\mathcal{C}^n$  covers  $X$ .

For each  $n \in \omega$ , let  $\mathcal{D}^n = F(\mathcal{C}^n)$  where  $F$  is the function described in the first paragraph of the proof of Theorem 2. Then for each  $n \in \omega$ ,  $\mathcal{D}^n$  is a locally finite refinement of  $\mathcal{C}^n$  which covers  $X$ . It is clear that  $\mathcal{D} = \bigcup\{\mathcal{D}^n : n \in \omega\}$  is  $\sigma$ -locally finite.

We complete the proof of the theorem by showing that  $\mathcal{D}$  is a base for the metric topology on  $X$ . Assume  $x \in U$  and  $U$  is open in the metric topology. It suffices to find an element  $D$  of  $\mathcal{D}$  such that  $x \in D \subseteq U$ . Since  $U$  is open there is an  $n \in \omega$  such that  $N(\frac{1}{n}, x) \subseteq U$ .  $\mathcal{D}^{2n}$  is a refinement of  $\mathcal{C}^{2n}$  which covers  $X$  so for some  $D \in \mathcal{D}^{2n}$  and  $B \in \mathcal{C}^{2n}$ ,  $x \in D \subseteq B$ . Further, for some  $y \in X$ ,  $B \subseteq N(\frac{1}{2n}, y)$ . Since  $x \in N(\frac{1}{2n}, y)$ , we conclude that  $N(\frac{1}{2n}, y) \subseteq N(\frac{1}{n}, x)$ . Therefore,  $x \in D \subseteq N(\frac{1}{2n}, y) \subseteq U$ .  $\square$

**Theorem 4.** *If  $(X, d)$  is a metric space and  $\gamma$  is a well ordered cardinal  $\geq \omega$  such that  $(X, d)$  has a  $\gamma$ -locally finite base then  $(X, d)$  has a  $\gamma$ -disjoint base.*

*Proof.* Assume that  $(X, d)$  is a metric space and that  $\mathcal{B} = \bigcup_{\alpha \in \gamma} \mathcal{B}_\alpha$  (where each  $\mathcal{B}_\alpha$  is locally finite) is a  $\gamma$ -locally finite base for the metric topology. For each  $\alpha \in \gamma$  and  $n \in \omega$  we will define  $C_{\alpha, n}$  so that  $C_{\alpha, n}$  is a pairwise disjoint collection of open sets and  $\bigcup \{C_{\alpha, n} : \alpha \in \gamma \wedge n \in \omega\}$  is a base for the metric topology on  $X$ . Since for any well ordered infinite cardinal  $\gamma$ ,  $|\gamma \times \omega| = \gamma$ , this will give us a  $\gamma$ -disjoint base for the metric topology on  $X$ .

**Definition.**

1. For  $\alpha \in \gamma$ ,  $X_\alpha = \bigcup \mathcal{B}_\alpha$ .
2. For  $n \in \omega$ ,

$$X_{\alpha, n} = \{x \in X_\alpha : \text{there are exactly } n \text{ sets } U \in \mathcal{B}_\alpha \text{ such that } x \\ \text{is on the boundary of } U\}.$$

**Claim 1.**  $X_\alpha = \bigcup_{n \in \omega} X_{\alpha, n}$ .

**Proof.** (Note that this proof uses the assumption that  $\mathcal{B}_\alpha$  is *locally finite* and would not work if  $\mathcal{B}_\alpha$  were only *point finite*.) Assume  $x \in X_\alpha$ . Since  $\mathcal{B}_\alpha$  is locally finite there is a neighborhood  $N$  of  $x$  which meets only finitely many elements of  $\mathcal{B}_\alpha$ . Therefore  $x$  can be on the boundary of only finitely many elements of  $\mathcal{B}_\alpha$ . Hence  $x \in \bigcup_{n \in \omega} X_{\alpha, n}$ . This proves Claim 1.

We begin by defining  $C_{\alpha, 0}$  for arbitrary  $\alpha \in \gamma$ . The first step is to define an equivalence relation  $\sim$  on  $X_{\alpha, 0}$  by  $x \sim y \Leftrightarrow (\forall \mathcal{U} \in \mathcal{B}_\alpha)(x \in \mathcal{U} \leftrightarrow y \in \mathcal{U})$ . Given a finite subset  $S$  of  $\mathcal{B}_\alpha$ , the set  $R_S = \{x \in X_{\alpha, 0} : (\forall \mathcal{U} \in \mathcal{B}_\alpha)(x \in \mathcal{U} \leftrightarrow \mathcal{U} \in S)\}$  is either the empty set or one of the  $\sim$  equivalence classes. Further, every  $\sim$  equivalence class is an  $R_S$  for some non-empty, finite subset  $S$  of  $\mathcal{B}_\alpha$ .

**Claim 2.** The  $\sim$  equivalence classes are open.

**Proof.** Assume  $S$  is a finite, non-empty subset of  $\mathcal{B}_\alpha$ . By the remarks above it suffices to show that  $R_S$  is open. Assume  $x \in R_S$ . Choose a neighborhood  $N$  of  $x$  so that  $S' = \{\mathcal{U} \in \mathcal{B}_\alpha : \mathcal{U} \cap N \neq \emptyset\}$  is finite. This is possible since  $\mathcal{B}_\alpha$  is locally finite. Since  $x \in X_{\alpha, 0}$ ,  $x$  is on the boundary of no  $\mathcal{U} \in S' \setminus S$ . We may therefore choose a neighborhood  $N' \subset N$  of  $x$  so that  $N' \subseteq \mathcal{U}$  if  $\mathcal{U} \in S$  and  $N' \cap \mathcal{U} = \emptyset$  if  $\mathcal{U} \in S' \setminus S$ . Since  $N' \subseteq N$ ,  $N' \cap \mathcal{U} = \emptyset$  if  $\mathcal{U} \notin S'$ . We may therefore conclude that  $N' \subseteq R_S$  proving Claim 2.  $\square$

Let  $C_{\alpha, 0} = \{R_S : S \text{ is a finite subset of } \mathcal{B}_\alpha \text{ and } R_S \neq \emptyset\}$ . We have shown that  $C_{\alpha, 0}$  is a pairwise disjoint collection of open sets. We also claim

$$(*) \quad \text{If } x \in \mathcal{U} \in \mathcal{B}_\alpha \text{ and } x \in X_{\alpha, 0} \text{ then } (\exists \mathcal{V} \in C_{\alpha, 0})(x \in \mathcal{V} \subseteq \mathcal{U})$$

(We take  $\mathcal{V} = R_S$  where  $S = \{\mathcal{W} \in \mathcal{B}_\alpha : x \in \mathcal{W}\}$ .)

Now we construct  $C_{\alpha, n}$  for arbitrary  $n \in \omega$ . ( $C_{\alpha, 0}$  is a special case of this construction.) The first step is to define an equivalence relation  $\sim$  on  $X_{\alpha, n}$  by

$$x \sim y \Leftrightarrow [(\forall \mathcal{U} \in \mathcal{B}_\alpha)(x \in \mathcal{U} \leftrightarrow y \in \mathcal{U}) \wedge (\forall \mathcal{U} \in \mathcal{B}_\alpha)(x \in \text{Bndry}(\mathcal{U}) \leftrightarrow y \in \text{Bndry}(\mathcal{U}))].$$

If  $S, T \subseteq \mathcal{B}_\alpha$  with  $S$  finite and  $|T| = n$ , then the set

$$P_{S,T} =_{\text{def}} \{x \in X_{\alpha,n} : (\forall \mathcal{U} \in \mathcal{B}_\alpha)(x \in \mathcal{U} \leftrightarrow \mathcal{U} \in S) \wedge (\forall \mathcal{U} \in \mathcal{B}_\alpha)(x \in \text{Bndry}(\mathcal{U}) \leftrightarrow \mathcal{U} \in T)\}$$

is either empty or a  $\sim$  equivalence class. Further for every  $\sim$  equivalence class  $E$  there is a unique pair  $(S, T)$  such that  $S, T \subseteq \mathcal{B}_\alpha$ ,  $|T| = n$  and  $E = P_{S,T}$ .

Assume  $x \in X_{\alpha,n}$ . Let  $P_{S,T}$  be the  $\sim$  equivalence class of  $x$  (where  $S, T \subseteq \mathcal{B}_\alpha$  are finite and  $|T| = n$ ). Then there is a neighborhood  $N$  of  $x$  such that  $N \subseteq \bigcap S$  and  $N \cap \mathcal{U} = \emptyset$  if  $\mathcal{U} \in \mathcal{B}_\alpha \setminus (S \cup T)$ . The argument for the existence of  $N$  (which we omit) is similar to the proof of claim 2 and uses the fact that  $\mathcal{B}_\alpha$  is locally finite.

**Definition.** Recall that  $N(\epsilon, x)$  denotes the ball of radius  $\epsilon$  centered at  $x$ .

- (1)  $\epsilon'_x = \sup\{\epsilon : N(\epsilon, x) \subseteq \bigcap S \text{ and } (\forall \mathcal{U} \in \mathcal{B}_\alpha \setminus (S \cup T))(N(\epsilon, x) \cap \mathcal{U} = \emptyset)\}$  if the sup exists and  $\epsilon'_x = 1$  otherwise.
- (2)  $\epsilon_x = \frac{\epsilon'_x}{2}$

**Claim 3.** If  $x \in P_{S,T}$  where  $S, T \subseteq \mathcal{B}_\alpha$ ,  $S$  is finite, and  $|T| = n$  and if  $y \in \text{Bndry}(\mathcal{U})$  for some  $\mathcal{U} \in \mathcal{B}_\alpha \setminus T$ , then  $d(x, y) \geq \epsilon'_x$ .

**Proof.** Assume the hypotheses. It suffices to show that  $y \notin N(\epsilon'_x, x)$ . But if  $y \in N(\epsilon'_x, x)$  then for every  $\mathcal{U} \in S$ ,  $y \notin \text{Bndry}(\mathcal{U})$ . Therefore we may conclude that  $\mathcal{U} \notin S$ . Also, as a consequence of  $y \in N(\epsilon'_x, x)$  and  $y \in \text{Bndry}(\mathcal{U})$ , we may conclude that  $N(\epsilon'_x, x) \cap \mathcal{U} \neq \emptyset$ . But for all  $V \in \mathcal{B}_\alpha \setminus (S \cup T)$ ,  $N(\epsilon'_x, x) \cap \mathcal{U} = \emptyset$ . This means that  $\mathcal{U} \in (S \cup T)$ . Since  $\mathcal{U}$  can be in neither  $S$  nor  $T$  we have a contradiction.  $\square$

For each pair  $(S, T)$  of finite subsets of  $\mathcal{B}_\alpha$  such that  $|T| = n$  and  $P_{S,T} \neq \emptyset$  we define

$$R_{S,T} = \bigcup \{N(\epsilon_x, x) : x \in P_{S,T}\}$$

and we let

$$C_{\alpha,n} = \{R_{S,T} : S \text{ and } T \subseteq \mathcal{B}_\alpha \text{ are finite, } |T| = n \text{ and } P_{S,T} \neq \emptyset\}.$$

$C_{\alpha,n}$  is clearly a collection of open sets. In addition

**Claim 4.** If  $S, S', T, T'$  are finite subsets of  $\mathcal{B}_\alpha$  such that  $|T| = |T'| = n$  and if  $S \neq S'$  or  $T \neq T'$  then  $R_{S,T} \cap R_{S',T'} = \emptyset$ .

**Proof.** Assume first that  $T \neq T'$ . Since  $|T| = |T'| = n$  there are sets  $\mathcal{U}_1$  and  $\mathcal{U}_2$  such that  $\mathcal{U}_1 \in T \setminus T'$  and  $\mathcal{U}_2 \in T' \setminus T$ . By Claim 3 if  $x \in P_{S,T}$  and  $y \in P_{S',T'}$  then  $d(x, y) \geq \epsilon'_x$  ( $y$  is in  $\text{Bndry}(\mathcal{U}_2)$  and  $\mathcal{U}_2 \notin T$ ) and  $d(x, y) \geq \epsilon'_y$ . This implies that  $N(\epsilon_x, x) \cap N(\epsilon_y, y) = \emptyset$ . (Since  $\epsilon_x = \frac{\epsilon'_x}{2}$  and similarly for  $\epsilon_y$ ). By the definition of  $R_{S,T}$  we conclude that  $R_{S,T} \cap R_{S',T'} = \emptyset$ .

Now assume that  $T = T'$  and  $S \neq S'$ . Assume without loss of generality that  $\mathcal{V} \in S \setminus S'$ . Then  $\mathcal{V} \notin T$  and therefore,  $\mathcal{V} \notin T'$ . Hence,  $\mathcal{V} \notin S' \cup T'$ . If  $x \in P_{S,T}$  and  $y \in P_{S',T'}$  then  $N(\epsilon_x, x) \subseteq \mathcal{V}$  and  $N(\epsilon_y, y) \cap \mathcal{V} = \emptyset$ . Therefore  $N(\epsilon_x, x) \cap N(\epsilon_y, y) = \emptyset$  and, as in the previous case,  $R_{S,T} \cap R_{S',T'} = \emptyset$ .  $\square$

By Claim 4 the collection  $C_{\alpha,n}$  is pairwise disjoint. All that remains is to show that

$$\mathcal{C} = \bigcup \{C_{\alpha,n} : \alpha \in \gamma \wedge n \in \omega\}$$

is a base for the metric topology. Assume  $x \in X$  and  $\epsilon > 0$ . It suffices to show that there is a set  $W \in \mathcal{C}$  such that  $x \in W \subseteq N(\epsilon, x)$ . Since  $\mathcal{B}$  is a base for the topology there is an  $\alpha \in \gamma$  and a  $\mathcal{U} \in \mathcal{B}_\alpha$  such that  $x \in \mathcal{U} \subseteq N(\epsilon, x)$ . It follows that  $x \in X_\alpha$  and, by Claim 1, there is an  $n \in \omega$  such that  $x \in X_{\alpha,n}$ . Therefore,  $x \in P_{S,T}$  for some finite sets  $S, T \subseteq \mathcal{B}_\alpha$  where  $|T| = n$  and  $\mathcal{U} \in S$ . If we let  $\mathcal{V} = R_{S,T}$  then the fact that  $\mathcal{V} \subseteq \mathcal{U}$  follows from the observation that for every  $\mathcal{U} \in S$ ,  $R_{S,T} \subseteq \mathcal{U}$ . We conclude that  $x \in \mathcal{V} \subseteq N(\epsilon, x)$ .  $\square$

**Corollary 1.** (9) implies (11) and (10) implies (12).

**Corollary 2.** MP implies MPDJ.

**Theorem 5.** If  $(X, d)$  is a metric space and  $\gamma$  is a well ordered cardinal  $\geq \omega$  such that  $(X, d)$  has a  $\gamma$ -disjoint base then  $(X, d)$  has a  $\gamma$ -discrete base.

*Proof.* Assume that  $(X, d)$  is a metric space with a  $\gamma$ -disjoint base  $\mathcal{C} = \{\mathcal{C}_\alpha : \alpha \in \gamma\}$  where, for  $\alpha \in \gamma$ ,  $\mathcal{C}_\alpha$  is a pairwise disjoint collection of open sets. For each  $U \in \mathcal{C}$ , and each  $n \in \omega$ , let  $U_n = \{x \in U : (\forall y \in X \setminus U)(d(x, y) > \frac{1}{n+1})\}$ . Then  $U_n$  is open. Further for each  $U \in \mathcal{C}$ ,  $U = \bigcup_{n \in \omega} U_n$ . For each  $\alpha \in \gamma$  and each  $n \in \omega$ , let  $\mathcal{D}_{\alpha,n} = \{U_n : U \in \mathcal{C}_\alpha\}$ . Since  $\mathcal{C}$  is a base for the topology on  $X$ , so is the set  $\mathcal{D} = \bigcup\{\mathcal{D}_{\alpha,n} : \alpha \in \gamma \text{ and } n \in \omega\}$ . Further, since  $|\gamma \times \omega| = \gamma$ ,  $\mathcal{D}$  has cardinality  $\gamma$ . Therefore the proof will be complete once we have shown that each set  $\mathcal{D}_{\alpha,n}$  is discrete.

Fix  $\alpha \in \gamma$ ,  $n \in \omega$  and  $x \in X$ . We need to find an open set  $V$  containing  $x$  such that  $V$  meets at most one element of  $\mathcal{D}_{\alpha,n}$ . We consider two cases:

**Case 1.**  $x \notin \bigcup \mathcal{C}_\alpha$ . Let  $V = N(\frac{1}{n+1}, x)$ . Choose a  $W \in \mathcal{D}_{\alpha,n}$ . Then  $W = U_n$  for some  $U \in \mathcal{C}_\alpha$  and  $x$  is in  $X \setminus U$ . Hence, for any  $y \in W = U_n$ ,  $d(x, y) > \frac{1}{n+1}$ . Therefore,  $V \cap W = \emptyset$ . This completes the proof in Case 1.

**Case 2.**  $x \in \bigcup \mathcal{C}_\alpha$ . Let  $V$  be the unique element of  $\mathcal{C}_\alpha$  such that  $x \in V$ . Using the fact that  $(\forall U \in \mathcal{C}_\alpha)(U_n \subseteq U)$ , we conclude that the open set  $V$  meets only one element, namely  $V_n$ , of  $\mathcal{D}_{\alpha,n}$ . (Unless  $V_n$  is empty in which case  $V$  meets no elements of  $\mathcal{D}_{\alpha,n}$ .)

This completes the proof in Case 2 and the proof of Theorem 5.  $\square$

**Corollary 1.** (11) implies (7) and (12) implies (8).

**Corollary 2.** MPDJ implies MPDC.

**Theorem 6.** MP, MPDJ, and MPDC are equivalent.

*Proof.* From Theorem 4, Corollary 2, we have that MP implies MPDJ, and from Theorem 5, Corollary 2, MPDJ implies MPDC. It remains to prove that MPDC implies MP. Let  $\mathcal{U}$  be an open cover of the metric space  $(X, d)$  and let  $\mathcal{V} = \bigcup\{V_n : n \in \omega\}$  be a  $\sigma$ -discrete open refinement of  $\mathcal{U}$ . By Theorem 2, MP iff (15). Since a  $\sigma$ -discrete set is  $\sigma$ -locally finite, it follows that MPDC implies MP.  $\square$

In our next theorem, we show that MP is also equivalent to statements about closed refinements of an open covering of a metric space.

**Theorem 7.** MP, MSP, and MCLFR, are equivalent.

*Proof.* We shall show first that MP and MSP are equivalent.

(MP  $\rightarrow$  MSP): Fix  $(X, d)$ , a metric space, and let  $\mathcal{U}$  be an open cover of  $X$ . We can assume  $\mathcal{U}$  is a locally finite family. Since MP  $\rightarrow$  MD there is an open  $\sigma$ -disjoint

refinement,  $\mathcal{V} = \bigcup_{m \in \omega} \mathcal{V}_m$ , of  $\mathcal{U}$  where each  $\mathcal{V}_m$  is a disjoint family of open sets. For each  $n \in \mathbb{Z}^+$  and  $V \in \mathcal{V}$  define

$$C(V, n) = \{x \in V : N(1/n, x) \subseteq V\}.$$

We now define the family

$$\mathcal{W} = \bigcup_{n \in \mathbb{Z}^+} \bigcup_{m \in \omega} \mathcal{W}_{m,n}$$

where

$$\mathcal{W}_{m,n} = \{C(V, n) : V \in \mathcal{V}_m\}.$$

We will show that each  $\mathcal{W}_{m,n}$  is a closed discrete family and that  $\mathcal{W}$  is a cover of  $X$  which refines  $\mathcal{V}$ . The fact that  $\mathcal{W}$  refines  $\mathcal{V}$  is trivial.

To show that for all  $n \in \mathbb{Z}^+$  and  $V \in \mathcal{V}$ ,  $C(V, n)$  is closed we assume the contrary. Let  $n \in \mathbb{Z}^+$  and  $V \in \mathcal{V}$  so that there exists  $x \in \overline{C(V, n)} \setminus C(V, n)$ . Since  $N(1/n, x) \cap C(V, n) \neq \emptyset$ , let  $y \in C(V, n)$  so that  $d(x, y) < 1/n$ . But then  $x \in N(1/n, y) \subseteq V$ . Therefore,  $x \in V \setminus C(V, n)$  and  $N(1/n, x) \setminus V \neq \emptyset$ . Let  $r = \min\{k \in \mathbb{Z}^+ : N(1/k, x) \subseteq V\}$ . Clearly  $r > n$ . Consequently, there exists  $y \in (X \setminus V) \cap N(1/(2n) + r/2, x)$ . Since  $x \in \overline{C(V, n)}$  there exists  $z \in C(V, n) \cap N(1/2n - r/2, x)$ . But then

$$d(y, z) < 1/2n - r/2 + 1/2n + r/2 = 1/n.$$

Thus,  $y \in N(1/n, z) \subseteq V$ . This is a clear contradiction so we have for every  $n \in \mathbb{Z}^+$  and for every  $V \in \mathcal{V}$ ,  $C(V, n)$  is closed.

To verify that  $\mathcal{W}$  is a cover of  $X$ , let  $V \in \mathcal{V}$ . We only need to show  $V \subseteq \bigcup\{C(V, n) : n \in \mathbb{Z}^+\}$ . Let  $x \in V$ . Since  $V$  is open there exists  $n \in \mathbb{Z}^+$  so that  $N(1/n, x) \subseteq V$ . But then  $x \in C(V, n)$ . So  $\mathcal{W}$  is a closed cover of  $X$ .

To finish we only need to show that each  $\mathcal{W}_{m,n}$  is a discrete family. Let  $x \in X$ . We have already seen that  $N(1/n, x) \cap C(V, n) \neq \emptyset$  then  $x \in V$ . Since  $\mathcal{V}_m$  is a disjoint family  $N(1/n, x)$  meets  $C(V, n)$  for at most one  $V \in \mathcal{V}_m$ . Thus,  $\mathcal{W}_{m,n}$  is a discrete family and so  $(X, d)$  is subparacompact.

We shall show next that  $\text{MSP} \rightarrow \text{MPDJ}$  and then it follows from Theorem 6 that  $\text{MSP} \rightarrow \text{MP}$ .

( $\text{MSP} \rightarrow \text{MPDJ}$ ): Let  $\mathcal{U}$  be an open cover of a metric space  $(X, d)$ . For each  $x \in X$  define

$$n_x = \min\{n \in \mathbb{Z}^+ : \{U \in \mathcal{U} : N(1/n, x) \subseteq U\} \neq \emptyset\}.$$

For each  $m \in \mathbb{Z}^+$  define  $\mathcal{U}'_m = \{N(1/2m, x) : x \in X \wedge n_x = m\}$ . Clearly  $\mathcal{U}' = \bigcup_{m \in \mathbb{Z}^+} \mathcal{U}'_m$  is an open cover of  $X$ . Let  $\mathcal{V} = \bigcup_{n \in \omega} \mathcal{V}_n$  be a  $\sigma$ -discrete closed refinement of  $\mathcal{U}'$ . For each  $m \in \mathbb{Z}^+$  define

$$\mathcal{V}_{m,n} = \{V \in \mathcal{V}_n : \{U \in \mathcal{U}'_m : V \subseteq U\} \neq \emptyset\}.$$

Since  $\mathcal{V}_{m,n} \subseteq \mathcal{V}_n$ ,  $\mathcal{V}_{m,n}$  is a discrete family of closed sets. Fix  $m \in \mathbb{Z}^+$ ,  $n \in \omega$ . For each  $V \in \mathcal{V}_{m,n}$  and  $x \in V$  define

$$k_x = \min\{k \in \mathbb{Z}^+ : k \geq m \wedge N(1/k, x) \cap \bigcup \mathcal{V}_{m,n} \subseteq V\}.$$

For each  $V \in \mathcal{V}_{m,n}$ , define  $W(V) = \bigcup \{N(1/2k_x, x) : x \in V\}$ .

We will show that for every  $n \in \omega$  and  $m \in \mathbb{Z}^+$  that the family  $\mathcal{W}_{m,n} = \{W(V) : V \in \mathcal{V}_{m,n}\}$  is a disjoint family of open sets which refines  $\mathcal{U}$ . Suppose not. Fix  $n \in \omega$  and  $m \in \mathbb{Z}^+$ . Let  $V, V' \in \mathcal{V}_{m,n}$  so that  $W(V) \cap W(V') \neq \emptyset$ . So there exist  $y \in V$  and  $y' \in V'$  so that  $N(1/2k_y, y) \cap N(1/2k_{y'}, y') \neq \emptyset$ . Without loss of generality assume  $k_y \leq k_{y'}$ . So

$$d(y, y') < 1/2k_y + 1/2k_{y'} \leq 1/k_y.$$

But then  $y' \in N(1/k_y, y) \cap V' = \emptyset$ . So we conclude that  $\mathcal{W}_{m,n}$  is a disjoint family of open sets.

To see that  $\mathcal{W}_{m,n}$  refines  $\mathcal{U}$  fix  $V \in \mathcal{V}_{m,n}$ . By the definition of  $\mathcal{V}_{m,n}$  there exists  $U' \in \mathcal{U}'_m$  so that  $V \subseteq U'$ . Let  $x \in X$  so that  $V \subseteq N(1/2m, x)$  and  $m = n_x$ . Let  $y \in V$ . We only need to show  $N(1/2k_y, y) \subseteq N(1/m, x)$ . Let  $z \in N(1/2k_y, y)$ . So  $d(y, z) < 1/2k_y$ . Since  $y \in V$ ,  $d(x, y) < 1/2m$ . Since  $k_y \geq m$ ,  $d(x, z) < 1/m$ . Thus,  $N(1/2k_y, y) \subseteq N(1/m, x)$ . So  $\mathcal{W}_{m,n}$  refines  $\mathcal{U}$ .

To complete the proof we only need to see that

$$\bigcup_{m \in \mathbb{Z}^+} \bigcup_{n \in \omega} \mathcal{W}_{m,n}$$

covers  $X$ . But since for every  $V \in \mathcal{V}$ ,  $V \subseteq W(V)$  and  $\mathcal{V}$  covers  $X$ , this is an easy consequence.

To finish the proof of Theorem 7, we shall show that  $\text{MP} \leftrightarrow \text{MCLFR}$ .

(MP  $\rightarrow$  MCLFR): Fix  $\mathcal{U}$  an open cover of the metric space  $(X, d)$ . For every  $U \in \mathcal{U}$  and  $x \in U$  let  $B(1/n_{Ux}, x)$  be the closed ball centered at  $x$  where  $n_{Ux}$  is the first  $n$  satisfying  $B(1/n, x) \subseteq U$ . Let  $\mathcal{W}'$  be a locally finite open refinement of  $\mathcal{V} = \{N(1/n_{Ux}, x) : U \in \mathcal{U}, x \in U\}$ . Put  $\mathcal{W} = \{\bar{w} : w \in \mathcal{W}'\}$ . Clearly  $\mathcal{W}$  is a locally finite closed cover of  $X$ . Furthermore, as each member  $w$  of  $\mathcal{W}$  is included in a member  $V$  of  $\mathcal{V}$  which is closed and is contained in a member  $U$  of  $\mathcal{U}$ , it follows that  $\bar{w} \subseteq U$  and  $\mathcal{W}$  is a refinement of  $\mathcal{U}$  as required.

(MCLFR  $\rightarrow$  MP): Fix  $\mathcal{U}$  an open cover of the metric space  $(X, d)$  and let  $\mathcal{V}$  be a closed locally finite refinement of  $\mathcal{D} = \{N(1/2n, x) : x \in X, n \in \omega \text{ and } N(2/n, x) \text{ is included in some member of } \mathcal{U}\}$ .

For every  $x \in X$  let  $N(1/n_x, x)$  meet finitely many members of  $\mathcal{V}$ , where  $n_x$  is the first  $n$  such that  $N(1/n, x)$  meets finitely many members of  $\mathcal{V}$ . Let  $\mathcal{Q}$  be a locally finite closed refinement of  $\mathcal{D} = \{N(1/n_x, x) : x \in X\}$ . For each  $V \in \mathcal{V}$  put

$$V^* = X \setminus \bigcup \{Q \in \mathcal{Q} : Q \cap V = \emptyset\}.$$

The set  $V^*$  is open because the union of a locally finite collection of closed sets is closed. Therefore,  $\mathcal{V}^* = \{V^* : V \in \mathcal{V}\}$  is clearly an open cover of  $X$ .

$\mathcal{V}^*$  is locally finite. To see this, fix  $x \in X$  and let  $G_x$  be a neighborhood of  $x$  meeting finitely many members of  $\mathcal{Q}$ , say  $Q_1, Q_2, \dots, Q_n$ . Now each  $Q_j$ ,  $j \leq n$ , being included in a member of  $\mathcal{D}$  meets only a finite number of members from  $\mathcal{V}$ . Hence, each  $Q_j$  can only meet a finite number of members from  $\mathcal{V}$  and consequently,  $G_x$  ( $\subseteq Q_1 \cup Q_2 \cup \dots \cup Q_n$ ) meets finitely many elements from  $\mathcal{V}^*$ .

Fix  $H \in \mathcal{V}^*$  and let

$$A_H = \{V \in \mathcal{V} : V^* = H\}.$$

(The function  $V \rightarrow V^*$  is not necessarily one to one.)

**Claim.** For every  $H \in \mathcal{V}^*$ ,  $A_H$  is a finite set.

**Proof of the Claim.** Fix  $M \in A_H$  and  $x \in M$ . Then there is a  $Q \in \mathcal{Q}$  such that  $x \in Q$ . For every  $V \in A_H$ , it must be the case that  $Q \cap V \neq \emptyset$ . (Otherwise,  $Q \subseteq X \setminus V^*$  and, therefore,  $x \notin V^*$ . But this contradicts the fact that  $V^* = M^*$ .) Therefore, if  $A^H$  were infinite,  $Q$  would meet infinitely many elements of  $\mathcal{V}$ . This is not possible because  $Q \subseteq D$ , for some  $D \in \mathcal{D}$  and therefore meets only finitely many elements of  $\mathcal{V}$ . Thus,  $A_H$  is finite, finishing the proof of the claim.

For every  $V \in A_H$  and positive  $n \in \omega$  we let

$$L_n(V) = \{x \in X : V \subseteq N(1/2n, x) \text{ and } N(2/n, x) \text{ is included in some member of } \mathcal{U}\}.$$

Let

$$n_V = \min\{n \in \omega : L_n(V) \neq \emptyset\}.$$

and

$$U_V = \cup\{N(1/2n_V, x) : x \in L_{n_V}(V)\}.$$

$V \subseteq U_V$  (for every  $x \in L_{n_V}(V)$ ,  $V \subseteq N(1/2n_V, x)$  and consequently  $V \subseteq U_V$ ) and  $U_V$  is included in some member of  $\mathcal{U}$ . (Fix  $x \in L_{n_V}(V)$  then  $N(1/2n_V, x) \subseteq N(2/n_V, x) \subseteq G$  for some  $G \in \mathcal{U}$ . Note that for every  $y \in L_{n_V}(V)$ ,  $N(1/2n_V, y) \subseteq N(2/n_V, x)$ . Indeed, for every  $z \in N(1/2n_V, y)$  and  $t \in V$  we have

$$d(z, x) \leq d(z, y) + d(y, t) + d(t, x) \leq 3/2n_V < 2/n_V.$$

Hence,  $z \in N(2/n_V, x)$  and  $N(1/2n_V, y) \subseteq N(2/n_V, x)$ . It now follows that  $U_V \subseteq G \in \mathcal{U}$ . For every  $H \in \mathcal{V}^*$  put

$$O_H = \{H \cap U_V : V \in A_H\}.$$

It is straightforward to verify that the collection  $\mathcal{V}^\# = \cup\{O_H : H \in \mathcal{V}^*\}$  is an open locally finite refinement of  $\mathcal{U}$  as required.  $\square$

It follows from the next theorem that (14) implies (10), MM implies MP, and MML implies MPL.

**Theorem 8.** *Let  $\mathcal{U}$  be a  $\gamma$ -point finite open cover of a metric space  $(X, d)$  for some ordinal  $\gamma$ . Then there exists  $\mathcal{V}$ , a  $\gamma$ -locally finite refinement of  $\mathcal{U}$ , so that for every  $x \in X$  and  $U \in \mathcal{U}$  there exists  $V \in \mathcal{V}$  such that  $x \in V \subset U$ .*

*proof.* Let  $\mathcal{U} = \bigcup_{\alpha \in \gamma} \mathcal{U}_\alpha$  be an open cover of a metric space  $(X, d)$  so that  $\gamma$  is an ordinal and for every  $\alpha \in \gamma$ ,  $\mathcal{U}_\alpha$  is a point finite family. For each  $U \in \mathcal{U}$  and  $n \in \mathbb{Z}^+$  define

$$V(U, n) = \bigcup\{N(1/2n, x) : N(1/n, x) \subset U\}.$$

Clearly, for every  $U \in \mathcal{U}_\alpha$  and  $n \in \mathbb{Z}^+$  we have  $V(U, n) \subset U$ . For  $\alpha \in \gamma$ ,  $n \in \mathbb{Z}^+$  let

$$\mathcal{V}_{\alpha, n} = \{V(U, n) : U \in \mathcal{U}_\alpha\}.$$

**Claim.** For every  $\alpha \in \gamma$ ,  $x \in X$ ,  $n \in \mathbb{Z}^+$

$$\{V \in \mathcal{V}_{\alpha, n} : N(1/2n, x) \cap V \neq \emptyset\} \subset \{V(U, n) : x \in U \in \mathcal{U}_\alpha\}.$$

**Proof.** Fix  $\alpha \in \gamma, x \in X, n \in \mathbb{Z}^+$ . Let  $V \in \mathcal{V}_{\alpha, n}$  so that

$$N(1/2n, x) \cap V \neq \emptyset.$$

We only need to show that if  $U \in \mathcal{U}_\alpha$  and  $V(U, n) = V$ , then  $x \in U$ . Let  $U \in \mathcal{U}_\alpha$  so that  $V(U, n) = V$ . For each  $y \in N(1/2n, x) \cap V$ , let

$$Z(U, y) = \{z \in U : y \in N(1/2n, z) \subset N(1/n, z) \subset U\}.$$

If  $y \in N(1/2n, x) \cap V$  and  $z \in Z(U, y)$ , then  $d(x, z) < 1/n$ , because  $N(1/n, z) \subset U$  and  $x \in U$ .

Since each  $\mathcal{U}_\alpha$  is point finite, the above claim shows that each  $\mathcal{V}_{\alpha, n}$  is locally finite.

Fix  $x \in X, \alpha \in \gamma$  and  $U \in \mathcal{U}_\alpha$ . Let  $x \in U$ , then there exists  $n \in \mathbb{Z}^+$  so that  $N(1/n, x) \subset U$ . Let  $n_0$  be the smallest such  $n$ . Then  $x \in V(U, n) \subset U$ .  $\square$

**Corollary 1.** (14) implies (10).

**Corollary 2.** MM implies MP.

In the proof we could easily substitute point countable for point finite and thereby prove that MML implies MPL.

**Corollary 3.** MML implies MPL.

This completes the proof of all the implications given in the figure in section 1.

### 3. The Independence Results.

**Theorem 9.** (8) does not imply MC in  $ZF^0$ .

*Proof.* Let  $N$  be the basic Fraenkel model (determined by the group  $G$  of all permutations of the atoms  $A$  and using finite supports.) For each finite subset  $E$  of  $A$  and  $t \in N$ , let  $\text{Ob}_E(t)$  be the  $\text{fix}_G(E)$  orbit of  $t$ . That is,  $\text{Ob}_E(t) = \{\phi(t) : \phi \in G \wedge \phi \text{ fixes } E \text{ pointwise}\}$ . Let  $(X, d)$  be a metric space in  $N$  with support  $E$ . For each  $x \in X$ , let  $P_x = \{\text{Ob}_E(\langle y, z \rangle) : y, z \in \text{Ob}_E(x)\}$ . If  $\langle y_1, z_1 \rangle$  and  $\langle y_2, z_2 \rangle$  are in the same element of  $P_x$  then there is a  $\phi$  which fixes  $E$  pointwise such that  $\phi(\langle y_1, z_1 \rangle) = \langle y_2, z_2 \rangle$ . Therefore, since  $\phi$  fixes the metric  $d$ , we obtain

$$(*) \quad (\forall Y \in P_x)(\forall \langle y_1, z_1 \rangle, \langle y_2, z_2 \rangle \in Y)(d(y_1, z_1) = d(y_2, z_2)).$$

We will show that for each  $x \in X$ , there are only finitely many elements in  $P_x$ : Let  $F \cup E$  be a support of  $x$  where  $F$  is chosen so that  $F \cap E = \emptyset$ . Let  $K$  be a subset of  $A$  of cardinality  $2|F|$ . Let  $R_x = \{\langle \phi(x), \psi(x) \rangle : \phi, \psi \in \text{fix}_G(E) \text{ and } \phi(F) \cup \psi(F) \subseteq K\}$ . The set  $R_x$  is finite so it suffices to show that all  $\eta, \beta \in \text{fix}_G(E)$ , the pair  $\langle \eta(x), \beta(x) \rangle$  is in  $\text{Ob}_E(\langle y, z \rangle)$  for some  $\langle y, z \rangle$  in  $R_x$ . By the cardinality condition on  $K$ ,  $|\eta(F) \cup \beta(F)| \leq |K|$ . Therefore there is a  $\gamma \in \text{fix}_G(E)$  such that  $\gamma(\eta(F) \cup \beta(F)) \subseteq K$ . This means that the pair  $\langle \gamma(\eta(x)), \gamma(\beta(x)) \rangle$  is in  $R_x$ . We can conclude that  $\langle \eta(x), \beta(x) \rangle \in \text{Ob}_E(\langle y, z \rangle)$  where  $\langle y, z \rangle = \langle \gamma(\eta(x)), \gamma(\beta(x)) \rangle \in R_x$ .

Using (\*) and the fact that  $P_x$  is finite we conclude that  $\epsilon_x = \inf\{d(y, z) : y, z \in \text{Ob}_E(x) \wedge y \neq z\}$  is greater than zero. By definition for all  $y, z \in \text{Ob}_E(x)$ ,

$d(y, z) \geq \epsilon_x$ . Therefore for each positive integer  $n$  and  $x \in X$ , the set  $\mathcal{B}_{n,x} = \{N(\frac{\epsilon_x}{3^n}, y) : y \in \text{Ob}_E(x)\}$  is a discrete open cover of  $\text{Ob}_E(x)$ . We note

(a) For every positive integer  $n$  and  $x \in X$ , the set  $\mathcal{B}_{n,x}$  has support  $E$ . Hence,

$$\{\mathcal{B}_{n,x} : x \in X \wedge n > 0\}$$

is well ordered in  $N$ .

(b)  $\bigcup \{\mathcal{B}_{n,x} : x \in X \wedge n > 0\}$  is a base for the topology  $(X, d)$ .

Therefore  $(X, d)$  has a  $\aleph$ -discrete base in  $N$  for some well ordered cardinal  $\aleph$ .

This shows that (8) is true in  $N$ . Since MC is known to be false in  $N$  ([8]) the proof is complete.  $\square$

**Theorem 10.** *MML is not provable in  $ZF^0$ .*

*Proof.* We use a slight variation of the model given in [5]. The set of atoms  $A = \bigcup \{Q_n : n \in \omega\}$ , where  $Q_n = \{a_{n,q} : q \in \mathbb{R}\}$ . We let  $<$  be the lexicographic ordering on  $A$ . The group of permutations  $\mathcal{G}$ , is the group of all permutations on  $A$  which are a translation on  $Q_n$ , that is, if  $\phi \in \mathcal{G}$ , then  $\phi|_{Q_n}(a_{n,q}) = a_{n,q+r_n}$  for some  $r_n \in \mathbb{R}$ . Supports are finite. Let  $d$  be the metric on  $A$  given by:

$$d(a_{n,q}, a_{m,p}) = 1 \text{ if } n \neq m$$

and

$$d(a_{n,q}, a_{m,p}) = |q - p| / (1 + |q - p|) \text{ if } n = m.$$

It is shown in [5] that  $(A, d)$  is a metric space. Using the method used in [7], we shall show that  $(A, d)$  is not meta-Lindelöf. The set of intervals  $\mathcal{U} = \{(a, b) : (\exists n \in \omega)(a, b \in Q_n \wedge a < b)\}$  is an open cover without a point countable refinement in the model. For suppose  $\mathcal{V}$  is a point countable refinement of  $\mathcal{U}$  with support  $E$  and suppose  $Q_n \cap E = \emptyset$ . Then for some  $V \in \mathcal{V}$ ,  $V \subseteq (a, b) \subseteq Q_n$ . Choose  $x \in Q_n$  and an interval  $(c, d) \subseteq (a, b)$  such that  $x \in (c, d) \subset V$ . For each  $r \in \mathbb{R}$  define  $\phi_r$  to be the permutation which is the identity outside of  $Q_n$  and for  $a_{n,q} \in Q_n$ ,  $\phi_r(a_{n,q}) = a_{n,q+r}$ . Then the set  $\{\phi_r(V) : 0 < r < d - x\}$  is an uncountable subset of  $\mathcal{V}$ , each element of which contains  $x$ .  $\square$

**Theorem 11.** *(5) does not imply MM in  $ZF^0$ .*

*Proof.* We use the model  $M$  given in [5]. (It is the same as the model described in Theorem 10, except that the group  $\mathcal{G}$  is the group of permutations on  $A$  that are rational translations on each  $Q_n$ .) It is shown in [7] that MM is false in this model. (The argument is similar to the one given in our Theorem 10.) Let  $\mathcal{G}$  be the group of permutations of the atoms from which  $M$  is defined. For every  $x \in M$ , the  $\mathcal{G}$  orbit  $\{\phi(x) : \phi \in \mathcal{G}\}$  of  $x$  is countable in  $M$ . It is not hard to see that (5) holds in any Fraenkel-Mostowski model with this property.

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