NON-CONSTRUCTIVE PROPERTIES OF THE REAL NUMBERS

Paul Howard, Kyriakos Keremedis, Jean E. Rubin,
Adrienne Stanley, and Eleftherios Tachtsis

July 17, 2000

Abstract. We study the relationship between various properties of the real numbers and weak choice principles.

1. Introduction.

It is a well known result of ZFC (Zermelo-Fraenkel set theory with the axiom of choice AC) that $\aleph_1$ is a regular cardinal, Form 34 in [1], (i.e., $\aleph_1$ is not the limit of an increasing sequence $(a_n)_{n\in\omega}$ of ordinals in $\aleph_1$). On the other hand, this statement is not provable in ZF-AC. Indeed, Form 34 implies that *the real line* $\mathbb{R}$ *cannot be written as a countable union of countable subsets* (Form 38 in [1]. For a proof of this fact see [2], p.148) and the latter statement is not valid in the Feferman/Levy model $\mathcal{M}9$ in [1]. A recent work concerning the cofinality of $\aleph_1$ is due to C. Good and I. Tree [5]. In particular, the authors in [5] show that Form 34 is implied by each one of the following statements:

1. $\aleph_1$ with the order topology is not paracompact.
2. $\aleph_1$ with the order topology has the property that every infinite subset has a limit point in $\aleph_1$.
3. $\aleph_1$ with the order topology does not contain a countable discrete family of open subsets.
4. $\aleph_1$ with the order topology is not Lindelöf.

In addition, P. Howard and J. E. Rubin [1], prove (see Note 107 in [1]) that Form 34 implies each one of the latter statements. In this paper we study weak forms of the axiom of choice and their relationships to each other. We are mainly concerned with properties of the real numbers that require some form of choice in their proof.

Before we set out with results, let us state some of the well known weak choice principles we are going to use.

1. AC($\mathbb{R}$) (Form 79 in [1]) is the proposition:
   For every family $\mathcal{A} = \{A_i : i \in k\}$ *of non empty subsets of* $\mathbb{R}$ *there exists a set* $c = \{c_i : i \in k\}$ *such that for all* $i \in k$, $c_i \in A_i$.
   (AC($\mathbb{R}$) is equivalent to the statement that $\mathbb{R}$ can be well ordered.)
2. AC(WO, $\mathbb{R}$) is the proposition:
   For every family $\mathcal{A} = \{A_i : i \in \mu\}$, μ *an ordinal number, of non-empty subsets of* $\mathbb{R}$ *there exists a set* $c = \{c_i : i \in \mu\}$ *such that for all* $i \in \mu$, $c_i \in A_i$.

1991 *Mathematics Subject Classification.* 03E25, 04A25.
Key words and phrases. real numbers, axiom of choice, weak choice principles.

3. AC(WO, LO) is the statement:
Every well ordered family $\mathcal{A} = \{A_i : i \in k\}$, k an ordinal number, of sets such that $\bigcup \mathcal{A}$ is linearly ordered has a choice set.
4. CAC($\mathbb{R}$) (Form 94 in [1]) is AC($\mathbb{R}$) restricted to countable families.
5. $\text{CAC}_\omega(\mathbb{R})$ (Form 5 in [1]) is CAC($\mathbb{R}$) restricted to countable families of countable subsets of $\mathbb{R}$.
6. CUC($\mathbb{R}$) (Form 6 in [1]) is the proposition: *The union of a countable family of countable subsets of $\mathbb{R}$ is countable.*
7. CMC, the Countable Multiple Choice Axiom (Form 126 in [1]) is the proposition:
For every family $\mathcal{A} = \{A_i : i \in \omega\}$ of disjoint non-empty sets there exists a set $\mathcal{F} = \{F_i : i \in \omega\}$ of finite non-empty sets such that for all $i \in \omega$, $F_i \subseteq A_i$.
8. DMC, the Dependent Multiple Choice Axiom (Form 106 in [1]), is the statement:
If R is a binary relation on a non-empty set E such that $(\forall x \in E)(\exists y \in E)(x\ R\ y)$, then there exists a sequence $(F_n)_{n\in\omega}$ of non-empty finite subsets of E such that $(\forall n \in \omega)(\forall x \in F_n)(\exists y \in F_{n+1})(x\ R\ y)$.
9. DC$\mathbb{R}$, Dependent Choice for Relations on $\mathbb{R}$ (Form 211 in [1]), is the proposition:
If R is a binary relation on $\mathbb{R}$ such that $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x\ R\ y)$, then there exists a sequence $(x_n)_{n\in\omega}$ of real numbers such that $(\forall n \in \omega)(x_n\ R\ x_{n+1})$.

Other statements we use from [1] include the following: (Forms 34 and 38 are mentioned in the introduction.)

10. Form 34: $\aleph_1$ is regular
11. Form 35: The union of countably many meager subsets of $\mathbb{R}$ is meager. (A set is *meager* if it is the union of a countable family of nowhere dense sets.)
12. Form 36: If $A \subseteq \mathbb{R}^n$ and $A \bigcap B$ is countable for every bounded B then A is countable.
13. Form 38: $\mathbb{R}$ is not the union of a countable family of countable sets.
14. Form 51: Every linear ordering has a cofinal sub well ordering.
15. Form 170: $\aleph_1 \leq 2^{\aleph_0}$.
16. Form 130: $\mathcal{P}(\mathbb{R})$ is well orderable.
17. Form 203: Every partition of $\mathcal{P}(\omega)$ into non-empty subsets has a choice function.
18. Form 212: If R is a relation on $\mathbb{R}$ such that $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x\ R\ y)$, then there is a function $f : \mathbb{R} \to \mathbb{R}$ such that $(\forall x \in \mathbb{R})(x\ R\ f(x))$.
19. Form 368: The set of all denumerable subsets of $\mathbb{R}$ has cardinality $2^{\aleph_0}$.
20. Form 369: If $\mathbb{R}$ is partitioned into two sets, at least one of them has cardinality $2^{\aleph_0}$.

In the next section we shall derive relationships between these statements and other properties of the real numbers.

2. Results.

Lemma 1. *CAC($\mathbb{R}$) implies CUC($\mathbb{R}$).*

Proof. Let $\mathcal{A} = \{A_i : i \in \omega\}$ be a countable family of countable subsets of $\mathbb{R}$. For each $i \in \omega$, put $S_i = \{f \in (A_i)^\omega : f \text{ is a bijection}\}$. Clearly $S_i \subseteq \mathbb{R}^\omega$.

Since $|\mathbb{R}^\omega| = |\mathbb{R}|$, identify $\mathbb{R}^\omega$ with $\mathbb{R}$. By CAC($\mathbb{R}$), let $\{f_i : i \in \omega\}$ be a choice set for $\{S_i : i \in \omega\}$. On the basis of the f_i's we can easily construct a bijection $f : \omega \to \cup\mathcal{A}$. □

Theorem 1.
(i) CAC($\mathbb{R}$) implies form 35.

(ii) Form 35 implies form 38.

Proof. (i). Fix $M = \{M_n : n \in \omega\}$ a family of meager subsets of $\mathbb{R}$. We will show that $\cup M$ can be expressed as a countable union of nowhere dense sets of $\mathbb{R}$. It is a well known fact that the set of all open subsets of $\mathbb{R}$ has power 2^ω. Thus, the set of all open dense subsets of $\mathbb{R}$ and consequently the set $\mathcal{N}_\mathbb{R}$ of all closed nowhere dense subsets of $\mathbb{R}$ has size at most 2^ω. Therefore we can view $\mathcal{N}_\mathbb{R}$ as a subset of $\mathbb{R}$. Consider now the family $E = \{E_n : n \in \omega\}$, $E_n = \{f \in {\mathcal{N}_\mathbb{R}}^\omega : M_n \subseteq \bigcup_{k\in\omega} f(k)\}$. We assert that each E_n is a non-empty set. Indeed, let $\mathcal{A} = \{A_k : k \in \omega\}$ be a family of nowhere dense sets such that $M_n = \cup\mathcal{A}$. Then $M_n \subseteq \bigcup_{k\in\omega} \overline{A_k}$ and for each $k \in \omega$, $\overline{A_k} \in \mathcal{N}_\mathbb{R}$. Now define a function f on ω such that for each $k \in \omega$, $f(k) = \overline{A_k}$. Clearly $f \in E_n$. By CAC($\mathbb{R}$), let $\mathcal{E} = \{f_n : n \in \omega\}$ be a choice set for E. Then $\{f_n(k) : n, k \in \omega\}$ is a countable cover of $\cup M$ consisting of closed nowhere dense sets. Clearly $\cup M = \bigcup_{n,k\in\omega}(f_n(k) \cap (\cup M))$ and $f_n(k) \cap (\cup M)$ is a nowhere dense set. Thus, $\cup M$ is a meager set as required.

Part (ii) is straight forward because a countable set is meager. □

Theorem 2. *CAC($\mathbb{R}$) implies that $\aleph_1$ is a regular cardinal (form 34).*

Proof. Assume on the contrary that there exists an increasing sequence $(a_n)_{n\in\omega}$ of ordinals in $\aleph_1$ such that $\lim_{n\in\omega} a_n = \aleph_1$. Without loss of generality, we may assume that for each $n \geq 1$, $a_n \backslash a_{n-1}$ is an infinite set. Let $f : \mathbb{R} \to \aleph_1$ be a surjection (see [2], p.148). For each $n \geq 1$, set $S_n = \{g \in \mathbb{R}^\omega : f[Range(g)] = a_n \backslash a_{n-1}\}$.

By CAC($\mathbb{R}$), we have that $S_n \neq \emptyset$ for every $n \geq 1$. Indeed, let $A = \{f^{-1}(x) : x \in a_n \backslash a_{n-1}\}$. Since A is a countable set of non-empty subsets of $\mathbb{R}$, CAC($\mathbb{R}$) implies that there is a choice set $c = \{r_x : x \in a_n \backslash a_{n-1}\}$ for A. As $|c| = \omega$, there exists a bijection $g : \omega \to c$. Clearly $g \in S_n$.

Identify $\mathbb{R}^\omega$ with $\mathbb{R}$ and let $\{g_n : n \geq 1\}$ be a choice set for the family $\{S_n : n \geq 1\}$ (which exists by CAC($\mathbb{R}$)). Clearly $G = \bigcup_{n\geq1} Range(g_n) = \bigcup_{n\geq1}\{g_n(i) : i \in \omega\}$ is a countable set. Let $G = \{r_n : n \in \omega\}$ be an enumeration of G. It is evident that for every $x \in \bigcup_{n\geq1}(a_n \backslash a_{n-1})$ there is an $n \in \omega$ such that $f(r_n) = x$. This fact immediately yields an injection $h : \bigcup_{n\geq1}(a_n \backslash a_{n-1}) \to G$ and consequently $\aleph_1$ is at most countable, a contradiction. □

In view of Theorem 2 and the discussion in the introduction we see that the statement "$\aleph_1$ is a regular cardinal" lies in strength between CAC($\mathbb{R}$) and Form 38.

Another implication whose status was unknown (see Table 1 in [1]) is, form 170 → CAC($\mathbb{R}$). By the fact that form 170 does not imply $\aleph_1$ is a regular cardinal (see [1]) and by Theorem 2, we deduce that

(*) Form 170 does not imply CAC($\mathbb{R}$).

Corollary 1. *The statement "$\aleph_1$ is a regular cardinal" holds in every Fraenkel-Mostowski model.*

Proof. . Since CAC($\mathbb{R}$) holds in all Fraenkel-Mostowski models ([1]) the conclusion follows from Theorem 2. □

Lemma 2.

(i) CAC($\mathbb{R}$) if and only if PCAC($\mathbb{R}$) (Every countable family of non-empty subsets of $\mathbb{R}$ has an infinite subfamily with a choice set).

(ii) CAC$_\omega$($\mathbb{R}$) if and only if PCAC$_\omega$($\mathbb{R}$) (PCAC($\mathbb{R}$) for countable families of countable subsets of $\mathbb{R}$).

Proof. (i). It suffices to show that PCAC($\mathbb{R}$) implies CAC($\mathbb{R}$) as the other direction is evident. Let $\mathcal{A} = \{A_n : n \in \omega\}$ be a countable family of non-empty sets and let $\mathcal{B} = \{\prod_{i\leq n} A_i : n \in \omega\}$. Apply the partial choice form to $\mathcal{B}$ and we obtain the corresponding choice form. The proof for (ii) is similar. □

Theorem 3.
(i) DC$\mathbb{R}$ implies CAC($\mathbb{R}$).
(ii) Form 203 implies DC$\mathbb{R}$.

Proof. (i). Fix $\mathcal{A} = \{A_i : i \in \omega\}$ a family of non-empty subsets of $\mathbb{R}$. Without loss of generality we may assume that the members of $\mathcal{A}$ are pairwise disjoint. Let $B_0 = \mathbb{R} \backslash (\bigcup \mathcal{A})$ and for each $n \geq 1$, let $B_n = A_{n-1}$. Then $\bigcup_{n\in\omega} B_n = \mathbb{R}$. Define a relation R on $\mathbb{R}$ by requiring $x\ R\ y$ if and only if $x \in B_n$, for some $n \in \omega$, then $y \in B_m$ for some $m > n$. It is straightforward to verify that R satisfies the hypotheses of DC$\mathbb{R}$. Thus, there exists a sequence $(x_n)_{n\in\omega}$ such that for each $n \in \omega$, $x_n\ R\ x_{n+1}$. Without loss of generality we may assume that $x_0 \notin B_0$. Then $\{x_n : n \in \omega\}$ is a choice set for the family $\{A_{i_n} : n \in \omega\}$, where i_n is the unique i such that $x_n \in A_i$. The conclusion now follows from Lemma 2.

(ii). Let R be a relation on $\mathbb{R}$ such that for every $x \in \mathbb{R}$ there exists $y \in \mathbb{R}$ such that $(x\ R\ y)$. Since $|(0,1)| = |\mathbb{R}|$, we may consider R as a relation on $(0,1)$ such that $\mathrm{dom}(R) = (0,1)$. For each $n \in \omega$, set

$$A_n = \{f \in (0,1]^\omega : (\forall i \leq n)(f(i) \in (0,1) \wedge f(0)\ R\ f(1)\ R \ldots R\ f(n)) \wedge (\forall i > n)(f(i) = 1)\}.$$

For each $n \in \omega$ and $g \in A_n$, set $A_{(g,n)} = \{f \in A_{n+1} : (g|_{n+1} = f|_{n+1})\}$. Clearly the family $\mathcal{A} = \{A_{(g,n)} : n \in \omega, g \in A_n\} \cup \{(0,1]^\omega \backslash \bigcup \{A_{(g,n)} : n \in \omega, g \in A_n\}\}$ is a partition of $(0,1]^\omega$. Since we can identify $(0,1]^\omega$ with $\mathbb{R}$, let by Form 203, $\{f_{(g,n)} : n \in \omega, g \in A_n\}$ be a choice set for $\{A_{(g,n)} : n \in \omega, g \in A_n\}$. By induction we define a sequence $(x_n)_{n\in\omega}$ of real numbers and a sequence $(g_n)_{n\in\omega}$ of real valued functions such that $x_n\ R\ x_{n+1}$ for all $n \in \omega$. Let $g_0 \in A_0$ and define $g_{n+1} = f_{(g_n,n)}$. For each $n \in \omega$ define $x_n = g_n(n)$. Then the sequence $(x_n)_{n\in\omega}$ satisfies the conclusion of DC$\mathbb{R}$. □

In view of Theorem 3 and (*) above we see that Form 170 does not imply DC$\mathbb{R}$.

It is shown in [1] that the statement "$\aleph_1$ is a regular cardinal" implies none of the choice forms AC($\mathbb{R}$), CAC($\mathbb{R}$), CMC, DMC, DC$\mathbb{R}$ and form 203. It is unknown whether the implications 170 $\rightarrow$ CAC$_\omega$($\mathbb{R}$), 170 $\rightarrow$ 38 and 170 $\rightarrow$ CUC($\mathbb{R}$) are valid (see Table 1 in [1]). However, we show next that a stronger version of 170 implies both CAC$_\omega$($\mathbb{R}$) and 38 whereas 170 does not imply CUC($\mathbb{R}$).

Theorem 4.
(i) CAC$_\omega$($\mathbb{R}$) implies Form 38.
(ii) The statement "$\aleph_1$ can be embedded in every uncountable subset of $\mathbb{R}$" implies CAC$_\omega$($\mathbb{R}$).
(iii) 170 + CUC($\mathbb{R}$) implies 34.

Proof. (i). Suppose, by contradiction, that $\{A_n : n \in \omega\}$ is a countable collection of countable subsets of $\mathbb{R}$ that covers $\mathbb{R}$. For each $n \in \omega$, put $S_n = \{f \in (A_n)^\omega : f \text{ is a bijection}\}$. As $|\mathbb{R}^\omega| = |\mathbb{R}|$, let σ be a bijection from $\mathbb{R}^\omega$ onto $\mathbb{R}$. For each $n \in \omega$, let m_n be the least m such that $\sigma(S_n) \cap A_m \neq \emptyset$. By CAC$_\omega$($\mathbb{R}$), let $\{x_n : n \in \omega\}$ be a choice set for the family $\{\sigma(S_n) \cap A_{m_n} : n \in \omega\}$. Using $\{f_n = \sigma^{-1}(x_n) : n \in \omega\}$, we can construct a bijection $g : \omega \rightarrow \bigcup_{n\in\omega} A_n$. Thus $\{A_n : n \in \omega\}$ is not a cover of $\mathbb{R}$. So Form 38 holds.

(ii). Let $\mathcal{A} = \{A_i : i \in \omega\}$ be a family consisting of countable subsets of $\mathbb{R}$. If $\bigcup \mathcal{A}$ is countable then $\mathcal{A}$ has a choice set. Assume $\bigcup \mathcal{A}$ is an uncountable subset of $\mathbb{R}$. By our hypothesis we can view $\aleph_1$ as a subset of $\bigcup \mathcal{A}$. Since $\aleph_1$ is uncountable, there exists an infinite subfamily $\mathcal{A}^* = \{A_{i_j} : j \in \omega\}$ of $\mathcal{A}$ such that $\aleph_1 \cap A_{i_j} \neq \emptyset$ for every $j \in \omega$. Then $c = \{c_{i_j} : j \in \omega\}$, $c_{i_j} = \min(\aleph_1 \cap A_{i_j})$, is a choice set for $\mathcal{A}^*$. Since an infinite subfamily of $\mathcal{A}$ has a choice set, by Lemma 2, $\mathcal{A}$ has a choice set.

(iii). Let $(\alpha_n)_{n\in\omega}$ be an increasing sequence of ordinals in $\aleph_1$. Then $\bigcup_{n\in\omega}\alpha_n \subset \aleph_1$. By 170 we may view $\aleph_1$ and thus $\bigcup_{n\in\omega}\alpha_n$ as subsets of $\mathbb{R}$. Since each α_n is a countable set, we deduce by CUC($\mathbb{R}$) that $\bigcup_{n\in\omega}\alpha_n$ is a countable set and thus $\bigcup_{n\in\omega}\alpha_n \neq \aleph_1$. So $\aleph_1$ is regular. □

It is known that 170 does not imply 34. In the Cohen model $\mathcal{M}36$ (Figura's model) in [1], 170 is true whereas 34 is false. By Theorem 4, we have that CUC($\mathbb{R}$) fails in $\mathcal{M}36$ also.

It is still an open problem of whether 170 implies 38, but we show below that 170 does not imply form 36.

Theorem 5. *Form 36 iff CUC(*$\mathbb{R}$*) (form 6).*

Proof. By mimicking the proof of 36→38 in note 110 in [1] we get that 36→CUC($\mathbb{R}$). For the converse, fix $A \subseteq \mathbb{R}^n$ such that $A\cap B$ is countable for every bounded B. Since $|\mathbb{R}^n| = |\mathbb{R}|$, A can be considered as a subset of $\mathbb{R}$. Let $\mathcal{B}$ be the set of all open intervals with rational endpoints. By our hypothesis and the fact that $\mathcal{B}$ is countable we have that $\mathcal{A} = \{A\cap B : B \in \mathcal{B}\}$ is a countable family of countable sets. Thus, CUC($\mathbb{R}$) implies that $A = \cup\mathcal{A}$ is countable. □

It is shown in [1] that 170 does not imply 6, therefore, the following corollary follows immediately from Theorem 5.

Corollary 2. *Form 170 does not imply form 36.*

Lemma 3. *The conjunction of the statements "*$\aleph_1$ *is a regular cardinal" and "*$\aleph_1$ *can be embedded in every uncountable subset of* $\mathbb{R}$*" implies CUC(*$\mathbb{R}$*).*

Proof. Let $\mathcal{A} = \{A_i : i \in \omega\}$ be a family of countable subsets of $\mathbb{R}$. Since $\aleph_1$ is a regular cardinal, it cannot be written as a countable union of countable sets. Thus $\aleph_1$ cannot be embedded in $\bigcup\mathcal{A}$. Thus $\bigcup\mathcal{A}$ is countable. □

Theorem 6. *Form 51 implies AC(WO,LO).*

Proof. Let $\mathcal{A} = \{A_i : i \in \mu\}$ be a family where μ is an ordinal and $\bigcup\mathcal{A}$ is linearly ordered. For each $i \in \mu$ define $B_i = \prod_{j\le i} A_j$. Let $\le_i$ be a lexicographic order on B_i. Let $B = \bigcup_{i\in\mu} B_i$. For each $x, y \in B$ we say $x \preceq y$ if and only if one of the following conditions holds.

(1) $x = y$
(2) $x \in B_i$, $y \in B_j$ and $i < j$.
(3) $x, y \in B_i$ and $x \le_i y$.

Since $(B, \preceq)$ is a linearly ordered set there is a well-ordered subset C of B such that C is cofinal in B. Since C is cofinal in B there exists a strictly increasing sequence $(i_j)_{j<\kappa}$ of ordinals cofinal in μ such that $C \cap B_{i_j} \neq \emptyset$ for every $j < \kappa$. For each $j < \kappa$, let $c_j = \min(C \cap B_{i_j})$. Then $\{c_j : j < \kappa\}$ is a choice set for $\{B_{i_j} : j < \kappa\}$. For each $i \in \mu$ define $a_i = c_j(i)$, where j is least such that $i_j \ge i$. The $\{a_i : i \in \mu\}$ is a choice set for $\mathcal{A}$. □

Corollary 3. *AC(WO,LO) implies DC*$\mathbb{R}$*.*

Proof. It is known that the axiom of choice for well ordered families of non-empty sets implies the principle of dependent choices (see [2], Theorem 8.2). The proof of this fact adapts to show that AC(WO,LO) implies DC$\mathbb{R}$. (The only part of the proof of Theorem 8.2 we need to justify is the place where $\mathrm{AC}_{|\alpha|}$ is used. We claim that the sets S_ξ are linearly ordered. Indeed, letting $b = \sup(A)$, $A = \{u \in On : X \simeq u$ for some $X \subseteq \mathbb{R}\}$ we see that every well ordered subset X of $\mathbb{R}$ is order isomorphic with some ordinal of b. Thus, every element of S_ξ can be viewed as an element of $\mathbb{R}^b$. As $\mathbb{R}^b$ with the lexicographic ordering is linearly ordered, it follows that S_ξ is linearly ordered.

Thus, the family $G = \{S_\xi : \xi \in \alpha\}$ is a well ordered family of linearly ordered sets. Therefore, by the last remark it has a choice set.) □

Other implications whose status was unknown (see Table 1 in [1]) are, Form 51→Form 170, Form 203→Form 170, Form 203→Form 368, Form 203→Form 369, and 212→ 369. In Lemma 4 (i) below we show that AC(WO,$\mathbb{R}$) implies Form 170 so we conclude by Theorem 6, that Form 51→Form 170. In Lemma 4 (ii)-(v), we prove the latter four implications and in (vi) we state an old result of Tarski [6].

Lemma 4.
(i) AC(WO,$\mathbb{R}$) implies $\aleph_1 \le 2^{\aleph_0}$ (form 170).
(ii) Form 203 implies $\aleph_1 \le 2^{\aleph_0}$.
(iii) Form 203 implies Form 368.
(iv) Form 203 implies Form 369.
(v) Form 212 implies Form 369.
(vi) Form 368 implies Form 170.

Proof. (i) As in Theorem 2, there is a function f from $\mathbb{R}$ onto $\aleph_1$. Consequently, AC(WO,$\mathbb{R}$) implies that $\{f^{-1}(i) : i \in \aleph_1\}$ has a choice function. Therefore, $\aleph_1 \le 2^{\aleph_0}$.

(ii) Each well ordering of ω is a subset of $\omega \times \omega$ and $|\omega \times \omega| = |\omega|$. Consequently, each well ordering of ω corresponds to a real number. Therefore, $W = \{R : R$ well orders $\omega\}$ is in one-to-one correspondence with a subset of $\mathbb{R}$. Define an equivalence relation $\equiv$ on W so that if R_1, R_2 are in W, then $R_1 \equiv R_2$ if and only if R_1 and R_2 have the same order type. Let $W_\equiv$ be the set of equivalences classes of W. It follows from Form 203 that $W_\equiv$ has a choice set and, since each countable ordinal is included in the set of order types, it follows that $\aleph_1 \le 2^{\aleph_0}$.

(iii) Let $\mathcal{D}$ be the set of all denumerable subsets of $\mathbb{R}$. For each $X \in \mathcal{D}$, let $\mathcal{D}_X = \{f \in X^\omega : f$ is a bijection $\}$. For all $X, Y \in \mathcal{D}$, if $X \ne Y$, then $\mathcal{D}_X \cap \mathcal{D}_Y = \emptyset$. Since $|\mathbb{R}^\omega| = |\mathbb{R}|$, let $F : \mathbb{R}^\omega \to \mathbb{R}$ be a bijection. Let $\mathcal{A} = \{F[\mathcal{D}_X] : X \in \mathcal{D}\}$ and let $\mathcal{B} = \mathcal{A} \cup \{\mathbb{R} \setminus \bigcup \mathcal{A}\}$. Thus, $\mathcal{B}$ is a partition of $\mathbb{R}$ so 203 implies that $\mathcal{A}$ has a choice set $C = \{r_X : X \in \mathcal{D}\}$. It is clear that $|\mathbb{R}| \le |\mathcal{D}| = |C| \le |\mathbb{R}|$. Consequently, $\mathcal{D}$ has cardinality $2^{\aleph_0}$.

(iv) Suppose 369 is false. Since $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$, we may assume that $\mathbb{R} \times \mathbb{R}$ can be partioned into two sets, A and B, each of which has cardinality different from $2^{\aleph_0}$. Let $S = \{(\mathbb{R} \times \{x\}) \cap A : x \in \mathbb{R}\}$, then $(S, (\mathbb{R} \times \mathbb{R}) \setminus S)$ is a partition of $\mathbb{R} \times \mathbb{R}$. If $(\mathbb{R} \times \{x\}) \cap A \ne \emptyset$ for all $x \in \mathbb{R}$, it follows from 203 that $2^{\aleph_0} \le A$ which contradicts our assumption on A. Thus, there is an $x \in \mathbb{R}$ such that $(\mathbb{R} \times \{x\}) \cap A = \emptyset$. Then, $\mathbb{R} \times \{x\} \subseteq B$ and this contradicts our assumption on B.

(v) Using essentially the same proof as that in part (iv), it is easy to see that 212 may be used instead of 203 to get the same contradiction. □

For our last result, we prove some additional independence results. All the models mentioned are in [1]. G. H. Moore (*Zermelo's Axiom of Choice*, Springer-Verlag, New York (1982) p 324) asked whether 38 implies 36 ((1.4.10) on p 36 of Moore's book). We answer that question in the negative. In Theorem 5 we prove that 36 is equivalent to 6 and in Lemma 5 part (v) we prove that 34 is true in $\mathcal{M}6$. However, 34 implies 38, so 38 is true in $\mathcal{M}6$ and 6 is false. (See [1] p 152.)

Lemma 5.
(i) Form 170 is true in $\mathcal{M}1$.
(ii) Form 369 is false in $\mathcal{M}1$.
(iii) AC(WO,LO) is false in $\mathcal{N}53$.
(iv) Form 369 is true in $\mathcal{M}12(\aleph)$.

(v) Forms 34 and 170 are true in $\mathcal{M}6$.

Proof. (i). Let $\mathcal{M}1$ be the basic Cohen model Since GCH holds in the ground model, the reals in $\mathcal{M}$ have cardinality $\aleph_1$. Since cardinalities are preserved in the construction of $\mathcal{M}1$, in $\mathcal{M}_1$, $\aleph_1 \leq |\mathbb{R}|$. Thus, Form 170 is true in $\mathcal{M}1$.

(ii). G. P. Monro (Fund. Math. **80** (1973) pp 101-104) has shown that every cardinal that cannot be well ordered is decomposable (i.e. is the sum of two smaller positive cardinals) (Form 277 in [1]) is true in $\mathcal{M}1$. Since $\mathbb{R}$ cannot be well ordered in $\mathcal{M}1$, it follows that Form 369 is false.

(iii). In $\mathcal{N}53$, the set $Q = \{Q_n : n \in \omega\}$ is well ordered, $\bigcup Q$ can be linearly ordered, but, because supports are finite, Q has no choice function. Therefore, AC(WO,LO) is false.

(iv) In $\mathcal{M}12(\aleph)$, every uncountable set of reals has a perfect subset. Since it can be shown in ZF^0, (see [0 AB] in [1]) that every perfect subset of $\mathbb{R}$ has cardinality $2^{\aleph_0}$, it follows that Form 369 is true.

(v) GCH holds in the ground model for $\mathcal{M}6$. Consequently, forms 34 and 170 hold in the ground model. Sageev has shown that in $\mathcal{M}6$, any set of mutually incompatible conditions are countable. It follows from this that alephs are preserved. (See Sageev Ann. Math. Logic **8** (1975) p 130ff.) Therefore, 34 and 170 hold in $\mathcal{M}6$. □

3. Summary.

The following implications are clear: 130 $\to$ AC($\mathbb{R}$); AC($\mathbb{R}$) $\to$ 369; AC($\mathbb{R}$) $\to$ 212; AC($\mathbb{R}$) $\to$ AC(WO,$\mathbb{R}$); 212 $\to$ DC$\mathbb{R}$; AC($\mathbb{R}$) $\to$ 203; DMC $\to$ DC$\mathbb{R}$; DMC $\to$ CMC; CMC $\to$ CAC($\mathbb{R}$); AC(WO,LO) $\to$ AC(WO,$\mathbb{R}$); AC(WO,$\mathbb{R}$) $\to$ CAC($\mathbb{R}$); and 6 $\to$ CAC$_\omega$($\mathbb{R}$). The references for the remaining implications are as follows: CAC($\mathbb{R}$) $\to$ 35 (Theorem 1(i)); 203 $\to$ DC$\mathbb{R}$ (Theorem 3(ii)); 51 $\to$ AC(WO,LO) (Theorem 6); AC(WO,LO) $\to$ DC$\mathbb{R}$ (Corollary 3); DC$\mathbb{R}$ $\to$ CAC($\mathbb{R}$) (Theorem 3(i)); CAC($\mathbb{R}$) $\to$ 34 (Theorem 2); AC(WO,$\mathbb{R}$) $\to$ 170 (Lemma 4(i)); 203 $\to$ 170 (Lemma 4(ii); 203 $\to$ 368 (Lemma 4(iii)); 203 $\to$ 369 (Lemma 4 (iv)); 212 $\to$ 369 (Lemma 4 (v)); 35 $\to$ 38 (Theorem 1(ii)); 34 $\to$ 38 ([2] p 148); CAC$_\omega$($\mathbb{R}$) $\to$ 38 (Theorem 4(i)); CAC($\mathbb{R}$) $\to$ 6 (Lemma 1); and 6 $\equiv$ 36 (Theorem 5).

In addition we have the following independence results: (All the models given can be found in [1].) 6, 34, 35, 170 $\not\to$ 369, CAC($\mathbb{R}$), AC(WO,$\mathbb{R}$), (Lemma 5 (i), (ii), model $\mathcal{M}1$); AC(WO,$\mathbb{R}$), AC(WO,LO) $\not\to$ 79, 130, 203 (model $\mathcal{M}2$); 369 $\not\to$ AC($\mathbb{R}$), AC(WO,$\mathbb{R}$), 130, 170, 203, and 368 (model $\mathcal{M}5(\aleph)$); 130, DC$\mathbb{R}$, AC(WO,$\mathbb{R}$) $\not\to$ AC(WO,LO), (Lemma 5 (iii), $\mathcal{N}53$); 34, 170 $\not\to$ 5 (Lemma 5 (v), model $\mathcal{M}6$); 369 $\not\to$ CAC($\mathbb{R}$), DC$\mathbb{R}$, 34, 212 (Lemma 5 (iv), model $\mathcal{M}12(\aleph)$); DC$\mathbb{R}$ $\not\to$ DMC and CAC($\mathbb{R}$) $\not\to$ CMC (model $\mathcal{N}1$); 6, 38 $\not\to$ 34 (model $\mathcal{M}12(\aleph)$); 170 $\not\to$ 6, 34, AC(WO,$\mathbb{R}$) (Corollary 2, Theorem 5, model $\mathcal{M}36$); CAC($\mathbb{R}$) $\not\to$ 170, AC(WO,$\mathbb{R}$); DC$\mathbb{R}$ $\not\to$ AC(WO,LO), 203 (model $\mathcal{M}38$).

In the table below, the row and column headings represent the numbers of the forms in this paper. For the list of forms, see the Introduction. The numerals in the matrix have the following meaning: 0 means that it is unknown whether the row form implies the column form, 1 means that that the row form implies the column form; 3 means that the row form does not imply the column form in ZF; and 5 means that the row form does not imply the column form in ZF^0.

	5	6	34	35	38	51	79	92	94	106	126	130	170	203	211	212	337	368	369
5	1	0	3	0	1	3	3	3	3	3	3	3	3	3	3	3	3	3	3
6	1	1	3	0	1	3	3	3	3	3	3	3	3	3	3	3	3	3	3
34	3	3	1	0	1	3	3	3	3	3	3	3	3	3	3	3	3	3	3
35	0	0	0	1	1	3	3	3	3	3	3	3	3	3	3	3	3	3	3
38	3	3	3	0	1	3	3	3	3	3	3	3	3	3	3	3	3	3	3
51	1	1	1	1	1	1	0	1	1	5	5	0	1	0	1	0	1	0	0
79	1	1	1	1	1	3	1	1	1	3	3	0	1	1	1	1	3	1	1
92	1	1	1	1	1	3	3	1	1	3	3	3	1	3	0	0	3	0	0
94	1	1	1	1	1	3	3	3	1	3	3	3	3	3	0	0	3	3	0
106	1	1	1	1	1	3	3	3	1	1	1	3	3	3	1	0	3	3	0
126	1	1	1	1	1	3	3	3	1	3	1	3	3	3	0	0	3	3	0
130	1	1	1	1	1	3	1	1	1	3	3	1	1	1	1	1	3	1	1
170	3	3	3	0	0	3	3	3	3	3	3	3	1	3	3	3	3	0	3
203	1	1	1	1	1	3	0	0	1	3	3	0	1	1	1	0	3	1	1
211	1	1	1	1	1	3	3	3	1	3	3	3	3	3	1	0	3	3	0
212	1	1	1	1	1	3	0	0	1	3	3	0	0	0	1	1	3	0	1
337	1	1	1	1	1	0	3	1	1	5	5	3	1	3	1	0	1	0	0
368	0	0	0	0	0	3	0	0	0	3	3	0	1	0	0	0	3	1	0
369	0	0	3	0	0	3	3	3	3	3	3	3	3	3	3	3	3	3	1

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Paul Howard
Department of Mathematics, Eastern Michigan University, Ypsilanti, MI 48197 USA
email: phoward@emunix.emich.edu

Kyriakos Keremedis
Department of Mathematics, University of the Aegean, Karlovasi, Samos 83200 GREECE
email: kker@aegean.gr.

Jean E. Rubin
Department of Mathematics, Purdue University, West Lafayette, IN 47907 USA

email: `jer@math.purdue.edu`.

Adrienne Stanley
Department of Mathematics, University of Northern Iowa, Cedar Falls, IA 50614 USA
email: `stanley@cns.uni.edu`.

Eleftherios Tachtsis
Department of Mathematics, University of the Aegean, Karlovasi, Samos 83200 GREECE
email: `itah@aegean.gr`.