

# ELEMENTARY ABELIAN $p$ -GROUPS REVISITED

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ABSTRACT. For each prime  $p$ , a Fraenkel-Mostowski model is constructed in which there are two elementary Abelian  $p$ -groups with the same cardinality that are not isomorphic.

## Introduction.

In a paper published in 1977 Hickman [1] proves that for each prime  $p \geq 5$  there is a Fraenkel-Mostowski (FM) model in which the statement,

$S(p)$ : There exist two elementary Abelian  $p$ -groups with the same cardinality that are not isomorphic.

is true. Hickman says it is not known whether there exists such a model if  $p = 2$  or 3. (It is an easy exercise in ZFC to prove the negation of  $S(p)$  for every prime  $p$ .) In this note, for each prime  $p$ , we show that there is an FM model in which  $S(p)$  is true. Our result extends Hickman's to the primes 2 and 3, but since the construction is quite a bit different than Hickman's, we present the result here. It follows from the Jech-Sochor transfer results that the result is transferable to ZF, that is, for each prime  $p$ ,  $\text{Con}(\text{ZF} + S(p))$ .

An elementary Abelian  $p$ -group is an Abelian group in which all non-identity elements have order  $p$ . First note that an elementary Abelian  $p$ -group is just a vector space over the  $p$ -element field  $\{0, 1, \dots, p-1\}$ . Further, every vector space isomorphism is a group isomorphism and every group isomorphism is a vector space isomorphism.

## The Model.

Let  $\mathcal{N}$  be a model of  $\text{ZFU} + \text{AC}$  whose set of atoms  $A$  is written as a disjoint union  $A = B \cup C \cup R \cup S \cup T$  and the sets  $B, C, R, S$ , and  $T$  are indexed as follows:

$$B = \{b_{n,\lambda} : n \in \omega \wedge \lambda < \aleph_1\},$$

$$C = \{c_{n,\lambda} : n \in \omega \wedge \lambda < \aleph_1\},$$

$$R = \{r_{n,\lambda} : n \in \omega \wedge \lambda < \aleph_1\},$$

$$S = \{s_{n,\lambda} : n \in \omega \wedge \lambda < \aleph_1\},$$

$$T = \{t_{n,\lambda} : n \in \omega \wedge \lambda < \aleph_1\}.$$

For each  $\lambda < \aleph_1$ , we will let  $A_\lambda = B_\lambda \cup C_\lambda \cup R_\lambda \cup S_\lambda \cup T_\lambda$  where  $B_\lambda = \{b_{n,\lambda} : n \in \omega\}$ ,  $C_\lambda = \{c_{n,\lambda} : n \in \omega\}$  and so on. And let  $\phi_\lambda$  be the permutation of  $A$  which is the product of the cycles  $\phi_\lambda = \prod_{n \in \omega} (b_{n,\lambda}, c_{n,\lambda})(r_{n,\lambda}, s_{n,\lambda})$ . (So that  $\phi_\lambda(t_{n,\lambda}) = t_{n,\lambda}$  and  $\phi_\lambda$  is the identity on  $A - A_\lambda$ .) We let  $G$  be the group generated by the permutations  $\phi_\lambda$  for  $\lambda < \aleph_1$ ,  $\Gamma$  the filter of subgroups of  $G$  determined by finite supports and  $\mathcal{M}$  the model determined by  $\mathcal{N}$ ,  $G$  and  $\Gamma$ .

Note that for each  $\lambda < \aleph_1$ , if  $\psi \in G$  fixes any element of  $B_\lambda \cup C_\lambda \cup R_\lambda \cup S_\lambda$  then  $\psi$  fixes  $A_\lambda$  pointwise. Hence for each  $x \in \mathcal{M}$ , there is some finite set  $\{\lambda_1, \dots, \lambda_n\} \subset \aleph_1$  such that  $\text{fix}_G(\bigcup_{i=1}^n A_{\lambda_i}) =_{\text{def}} \{\phi \in G : \phi \text{ fixes } \bigcup_{i=1}^n A_{\lambda_i} \text{ pointwise}\}$  is in  $\Gamma$  and  $\forall \phi \in \text{fix}_G(\bigcup_{i=1}^n A_{\lambda_i}), \phi(x) = x$ . For the rest of the argument we will refer to  $\{\lambda_1, \dots, \lambda_n\}$  as a support of  $x$ .

We now define elementary Abelian  $p$ -groups  $G_1$  and  $G_2$  (with  $p = 2$ ) both of which are in  $\mathcal{M}$  with empty support:  $G_1$  is the vector space over the two element field  $\{0, 1\}$  with basis  $B \cup C$  and  $G_2$  is the vector space over the two element field with basis  $R \cup S \cup T$ . (More formally, we could define  $G_1$  to be the set of all functions  $\mathbf{v} : B \cup C \rightarrow \{0, 1\}$  such that  $\mathbf{v}^{-1}(1)$  is finite, together with the operation  $+$  of coordinatewise addition mod 2, similarly for  $G_2$ .) If  $\mathbf{v} \in G_2$  and  $\mathbf{v} = \mathbf{v}_1 + \dots + \mathbf{v}_n$  where  $\mathbf{v}_1, \dots, \mathbf{v}_n \in R \cup S \cup T$ , we will say that  $\mathbf{v}_1, \dots, \mathbf{v}_n$  occur in  $\mathbf{v}$ .

Since  $\forall \gamma, \lambda < \aleph_1$  and  $\forall n \in \omega, \phi_\gamma(t_{n,\lambda}) = t_{n,\lambda}$  (and since  $t_{n,\lambda} + t_{n,\lambda} = 0$ ) we have:

LEMMA 1. For all  $\mathbf{v} \in G_2$ , for all  $\lambda, \gamma < \aleph_1$  and for all  $n \in \omega, t_{n,\lambda}$  does not occur in  $\mathbf{v} + \phi_\gamma(\mathbf{v})$ .

THEOREM 2. In  $\mathcal{M}$ ,  $|G_1| = |G_2|$ .

PROOF. It is clear that the function  $H_1^* : B \cup C \rightarrow R \cup S$  defined by  $H_1^*(b_{n,\lambda}) = r_{n,\lambda}$  and  $H_1^*(c_{n,\lambda}) = s_{n,\lambda}$  is in  $\mathcal{M}$  and has empty support. We note that  $H_1^*$  is a one to one function from a basis for  $G_1$  onto an independent subset of  $G_2$  and therefore can be extended uniquely to a one to one homomorphism  $H_1$  from  $G_1$  into  $G_2$ . Similarly, if we define  $H_2^* : R \cup S \cup T \rightarrow G_1$  by  $H_2^*(r_{n,\lambda}) = b_{2n,\lambda}, H_2^*(s_{n,\lambda}) = c_{2n,\lambda}$  and  $H_2^*(t_{n,\lambda}) = b_{2n+1,\lambda} + c_{2n+1,\lambda}$  then

- (1)  $H_2^*$  has empty support. (Note that  $\phi_\lambda(t_{n,\lambda}) = t_{n,\lambda}$  and  $\phi_\lambda(b_{2n+1,\lambda} + c_{2n+1,\lambda}) = c_{2n+1,\lambda} + b_{2n+1,\lambda} = b_{2n+1,\lambda} + c_{2n+1,\lambda}$ .)
- (2)  $H_2^*$  maps a basis for  $G_2$  one to one onto an independent subset of  $G_1$ .

Therefore,  $H_2^*$  has empty support and can be extended uniquely to a one to one homomorphism  $H_2$  from  $G_2$  into  $G_1$ . Since  $|G_1| \leq |G_2|$  and  $|G_2| \leq |G_1|$  in  $\mathcal{M}$ , we conclude that  $|G_1| = |G_2|$  in  $\mathcal{M}$ .

THEOREM 3. In  $\mathcal{M}$ ,  $G_1$  and  $G_2$  are not isomorphic.

PROOF. We argue by contradiction that there is no isomorphism in  $\mathcal{M}$  from  $G_1$  onto  $G_2$ . Suppose that  $H$  is such an isomorphism with support  $E = \{\lambda_1, \dots, \lambda_n\}$ . The set  $X = \bigcup_{j=1}^n (B_{\lambda_j} \cup C_{\lambda_j})$  is countable and hence there exists  $n_0 \in \omega$  and  $\lambda_0 < \aleph_1$  such that  $t_{n_0,\lambda_0}$  occurs in no element of  $H''X (= \{H(x) : x \in X\})$ .

Since  $H$  is an isomorphism,  $H''(B \cup C)$  is a basis for  $G_2$  and therefore,

$$(3) \quad t_{n_0,\lambda_0} = \mathbf{v}_1 + \dots + \mathbf{v}_k + \mathbf{u}_1 + \dots + \mathbf{u}_r$$

where  $\mathbf{v}_1, \dots, \mathbf{v}_k \in H''((B \cup C) - X), \mathbf{u}_1, \dots, \mathbf{u}_r \in H''X$  and the elements  $\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{u}_1, \dots, \mathbf{u}_r$  are pairwise distinct. Furthermore, except for the order in which the elements are written, this is the only way of writing  $t_{n_0,\lambda_0}$  as a sum of pairwise distinct elements from  $H''(B \cup C)$ . By our choice of  $t_{n_0,\lambda_0}$ ,  $t_{n_0,\lambda_0}$  does not occur in  $\mathbf{u}_1 + \dots + \mathbf{u}_r$  (and hence  $k > 0$ ). We will arrive at a contradiction (and hence complete the proof) by showing that  $t_{n_0,\lambda_0}$  does not occur in  $\mathbf{v}_1 + \dots + \mathbf{v}_k$ .

LEMMA 4. Assume  $1 \leq j \leq k$ , then there is a  $\lambda$  not in  $E$  and an  $i$  with  $1 \leq i \leq k$  and  $i \neq j$  such that  $t_{n_0,\lambda_0} \neq \mathbf{v}_i + \dots + \mathbf{v}_k$ .

PROOF. Assume that  $\mathbf{v}_j = H(b_{n,\lambda})$  where  $\lambda \notin E$ . (The proof is similar if  $\mathbf{v}_j = H(c_{n,\lambda})$  where  $\lambda \notin E$ .) Since  $\phi_\lambda$  fixes  $X$  pointwise,  $\phi_\lambda(H) = H$ . Further  $\phi_\lambda(b_{n,\lambda}) = c_{n,\lambda}$  so

$$(4) \quad \phi_\lambda(\mathbf{v}_j) = \phi_\lambda(H(b_{n,\lambda})) = H(\phi_\lambda(b_{n,\lambda})) = H(c_{n,\lambda}) \neq H(b_{n,\lambda}) = \mathbf{v}_j$$

since  $H$  is one to one. Applying  $\phi_\lambda$  to both sides of the equality (3) gives

$$(5) \quad t_{n_0,\lambda_0} = \phi_\lambda(\mathbf{v}_1) + \cdots + \phi_\lambda(\mathbf{v}_k) + \mathbf{u}_1 + \cdots + \mathbf{u}_r.$$

( $\phi_\lambda$  fixes  $t_{n_0,\lambda_0}$  and fixes  $X$  pointwise. Therefore,  $\phi_\lambda$  fixes  $H''X$  pointwise.)

Since  $\phi_\lambda(B \cup C) = B \cup C$ , we have  $\phi_\lambda(H''(B \cup C)) = H''(B \cup C)$ . Therefore (5) expresses  $t_{n_0,\lambda_0}$  as a sum of vectors from the basis  $H''(B \cup C)$ . Hence the right hand sides of (3) and (5) contain the same terms and so  $\phi_\lambda(\mathbf{v}_j) = \mathbf{v}_i$  for some  $i$  such that  $1 \leq i \leq k$ . By (4),  $\phi_\lambda(\mathbf{v}_j) \neq \mathbf{v}_j$ , hence  $i \neq j$ . This completes the proof of Lemma 4.

It follows that  $\mathbf{v}_1 + \cdots + \mathbf{v}_k$  can be written as a sum of vectors of the form  $\mathbf{v}_j + \phi_\lambda(\mathbf{v}_j)$ . Since (by Lemma 1)  $t_{n_0,\lambda_0}$  does not occur in  $\mathbf{v}_j + \phi_\lambda(\mathbf{v}_j)$  it follows that  $t_{n_0,\lambda_0}$  does not occur in  $\mathbf{v}_1 + \cdots + \mathbf{v}_k$ , which completes the proof of Theorem 3.

It is easy to see how the result can be generalized to any prime  $p$ . We shall indicate the construction for  $p = 3$ . Let  $A = B \cup C \cup D \cup R \cup S \cup T \cup W$ . The group  $G$  is generated, by  $\phi_\lambda = \prod_{n \in \omega} (a_{n,\lambda}, b_{n,\lambda}, c_{n,\lambda})(r_{n,\lambda}, s_{n,\lambda}, t_{n,\lambda})$ ,  $\lambda < \aleph_1$ , where  $\phi_\lambda(w_{n,\lambda}) = w_{n,\lambda}$  for all  $n \in \omega$ . Supports are again finite. The elementary Abelian  $p$ -groups, are defined so that  $G_1$  is the vector space over the three element field  $\{0, 1, 2\}$  with basis  $B \cup C \cup D$  and  $G_2$  is the vector space over the same three element field with basis  $R \cup S \cup T \cup W$ . The proofs of the lemmas are similar to that given above using the fact that for all  $\mathbf{v} \in G_2$ ,  $w_{n,\lambda}$  does not occur in  $\mathbf{v} + \phi_\lambda(\mathbf{v}) + \phi_\lambda^2(\mathbf{v})$ . The proofs of Theorems 2 and 3 are also similar to the proofs given above.

## REFERENCES

1. J. S. Hickman, *A remark on elementary Abelian groups*, Bull. Austral. Math. Soc. **16** (1977), 213-217.

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