PRODUCTS OF COMPACT SPACES AND THE AXIOM OF CHOICE II

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March 4, 2002

Abstract. This is a continuation of [dhhkr]. We study the Tychonoff Compactness Theorem for various definitions of compact and for various types of spaces, (first and second countable spaces, Hausdorff spaces, and subspaces of $\mathbb{R}^\kappa$). We also study well ordered Tychonoff products and the effect that the multiple choice axiom has on such products.

1. Introduction and Definitions

We started our study of the Tychonoff Compactness Theorem in [dhhkr], where we studied the Theorem using the definitions of compactness studied in [dhhrs]. In this paper we extend these results to various types of topological spaces, first countable, second countable, Hausdorff spaces and subsets of $\mathbb{R}^\kappa$. We also consider the role the multiple choice axiom plays when the products of topological spaces are well ordered.

It is well known (see [ty] or [k]) that the Tychonoff Compactness Theorem, "Products of compact spaces are compact" is equivalent to the Axiom of Choice (AC). In the last section we prove that this equivalence does not hold when "compact" in Tychonoff's Theorem is replaced by certain weaker forms of compactness. We construct a Fraenkel-Mostowski model in which one of the product theorems is true, but AC is false.

We start by defining some of the terms we shall be using. Let $S = (X, \mathcal{T})$ be a topological space.

1. $\mathcal{B} \subseteq \mathcal{T}$ is called a *base* for $\mathcal{T}$ if for each $x \in X$ and each neighborhood U of x there is a $V \in \mathcal{B}$ such that $x \in V \subseteq U$. (Thus, every element of $\mathcal{T}$ is a union of elements of $\mathcal{B}$.)
2. S is *first countable* if each point has a countable base.
3. S is *second countable* if $\mathcal{T}$ has a countable base.
4. S is *Lindelöf* if each open cover has a countable subcover.

1991 *Mathematics Subject Classification.* 03E25, 04A25, 54D30.

Key words and phrases. First Countable, Second Countable, Hausdorff, Products, Compactness, Choice, Multiple Choice.

5. S is said to be T_0 if whenever x and y are distinct points there is a neighborhood of one that does not contain the other.
6. S is said to be T_1 if whenever x and y are distinct points, each has a neighborhood not containing the other. (Points are closed.)
7. S is said to be T_2 (*Hausdorff*) if whenever x and y are distinct points, there is a neighborhood U of x and a neighborhood V of y such that $U \cap V = \emptyset$.

Clearly, every second countable space is first countable; every T_2 space is T_1; and every T_1 space is T_0.

Some of the weak forms of AC that we shall be using include the following:

8. AC(WO) is the axiom of choice for a well ordered family of sets.
9. $\mathrm{AC}_{\mathrm{WO}}$ is the axiom of choice for a family of well orderable sets.
10. AC(LO) is the axiom of choice for a linearly ordered family of sets.
11. $\mathrm{AC}(\aleph_0)$ is the axiom of choice for a countable number of sets.
12. $\mathrm{AC}_{\mathrm{fin}}(\aleph_0)$ is the axiom of choice for a countable number of finite sets.
13. MC is the the Multiple Choice Axiom: For every non-empty set X of non-empty sets, there is a function f such that for each $x \in X$, $f(x)$ is a non-empty finite subset of X.
14. BPI is the Boolean Prime Ideal Theorem: Every Boolean algebra has a prime ideal.

(The reader is referred to [hr] for more information about the statements given in items 8–14 above.)

It is known that each of AC(LO) and MC are equivalent to AC in ZF. However, they do not imply AC in ZF^0, ZF without the foundation axiom. See [le] and [tr]. Unless otherwise stated, all proofs are in ZF^0.

The types of compactness we will be studying are given in [dhhkr], but we shall repeat those definitions for convenience.

Definitions of Compact Spaces:.

1. A space is *compact A* if every open covering has a finite subcovering.
2. A space is *compact B* if every ultrafilter has an accumulation point.
3. A space is *compact C* if there is a subbase for the topology such that every open covering by elements of the subbase has a finite subcovering.
4. A space is *compact D* if every covering by elements of a nest of open sets has a finite subcovering.
5. A space is *compact E* if every sequence has a convergent subsequence.
6. A space is *compact F* if every infinite subset has a complete accumulation point.
7. A space is *compact G* if every infinite subset has an accumulation point.
8. A space is *compact H* if every countable open covering has a finite subcovering.

Product Topology: Let $\{X_i : i \in k\}$ be a set of topological spaces and let $X = \prod_{i\in k} X_i$ be their product. A base for the *product topology on* X is all sets of the form $\prod_{i\in k} U_i$ where U_i is open in X_i and all but a finite number of the $U_i = X_i$.

P(X,Y) is the statement: Products of compact X spaces are compact Y. $P_{\aleph_0}$(X,Y) is the statement: Countable products of compact X spaces are compact Y. Pr(X) is the statement: Projections in Tychonoff products of compact X spaces are closed.

One of the questions we will be concerned with is when the product of topological spaces of a certain type will have the same type. It is easy to show that a product of T_i spaces is T_i, for $i = 0, 1, 2$, but it is known, for example, that the product of two Lindelöf spaces need not be Lindelöf. (Sorgenfrey line: $\mathbb{R}$, where a base for the topology is the set of all half open intervals $[a, b)$.)

We shall study two types of product theorems. Let $P^1_t(X, Y)$ be the following statement: Every product of type t compact X spaces is compact Y, provided that the product itself is of type t. The same statement but without the requirement that the product is of type t will be called $P^2_t(X, Y)$. Clearly, $P^2_t(X, Y)$ implies $P^1_t(X, Y)$, for all X, Y. If t is some property such as Hausdorff that is preserved under arbitrary products, then P^1_t and P^2_t are identical and the superscript will be omitted.

The following results for general topological spaces were obtained in [dhhkr]:
(1) Each of the following statements is equivalent to AC:

P(A,A), P(A,C), P(A,D), P(A,F), P(A,G), P(A,H), P(B,A), P(B,C), P(B,D), P(B,F), P(B,G), P(B,H), P(C,A), P(C,C), P(C,D), P(C,F), P(C,G), P(C,H), P(D,A), P(D,C), P(D,D), P(D,F), P(D,G), P(D,H), and P(F,F).

$$AC \to P(F, A) \to P(F, C) \to P(F, B) \tag{2}$$

$$AC \to P(F, A) \to P(F, G) \to AC_{\mathrm{WO}} \tag{3}$$

$$AC \to P(F, A) \to P(F, D) \to P(F, H) \to AC_{\mathrm{WO}} \tag{4}$$

$$AC \to P(B, B) \to P(C, B) \to P(A, B) \leftarrow P(D, B) \leftarrow AC \tag{5}$$

It is easy to show that properties (1) through (5) also hold for $P^2_t(X, Y)$, where t is first or second countable. However, in the proof of Theorem 1 in [dhhkr] that each of P(A,G) and P(A,H) implies AC, the topology used is not even T_0. However, we show in section 4, that it is easy to extend the topology so that it is T_1. In addition, it is shown in [lr2] that P_{T_2}(A,A) $\leftrightarrow$ BPI and it is shown in [he] that P_{T_2}(B,B) is provable in ZF. We show in the introduction that if BPI holds, then compact D implies compact G. Therefore, P_{T_2}(A,A) + P_{T_2}(D,D) $\to$ P_{T_2}(D,G).

In [dhhrs], section 6, we give some relationships between the forms of compactness if BPI holds. Here, we note that we can add to those relationships by showing that compact D implies compact G. (BPI implies that every set can be linearly ordered. Suppose X is compact D and Y is an infinite subset of X that has no accumulation points. Then every subset of Y is closed. Let $N = \{U_y = X \setminus \{z \in Y : z < y\} : z \in y\}$, where $<$ is a linear ordering of Y which has no minimum element. Then N is a nest of open sets which covers Y, but has no finite subcover.) Thus, the arrow diagram which gives the relationship between the forms when BPI holds can be changed to:

$$\begin{tikzcd}
\boxed{A,\ B,\ C} \arrow[r] & D \arrow[r] \arrow[d] & H \\
F \arrow[r] & G &
\end{tikzcd}$$

2. First and Second Countable Spaces

In this section we shall study product theorems for first and second countable spaces. We shall use the notation $\mathrm{P}^{\aleph_0}_{\mathrm{fc}}(X, Y)$ to stand for the statement "Countable products of first countable compact X spaces are compact Y." Likewise, we have $\mathrm{P}^{\aleph_0}_{\mathrm{sc}}(X, Y)$ about second countable spaces. Note that first and second countability are preserved under countable products.

It follows from [dhhrs] that the following are the arrow diagrams for first countable spaces.

$$\begin{tikzcd}
E & H \arrow[l] & D \arrow[l] & A \arrow[l] \arrow[r] \arrow[d] & C \arrow[r] & B \\
 & & F \arrow[r] & G & &
\end{tikzcd}$$

Assuming AC,

$$\begin{tikzcd}
\boxed{A,\ B,\ C,\ D,\ F} \arrow[r] & \boxed{E,\ H} \arrow[r] & G
\end{tikzcd}$$

For second countable spaces we have the following relationships between the forms of compactness:

$$\begin{tikzcd}
F \arrow[r] & G & E \arrow[l] & \boxed{A, D, H} \arrow[l] \arrow[r] & C \arrow[r] & B
\end{tikzcd}$$

Consequently, we immediately have that the statements $\mathrm{P}^{1}_{\mathrm{sc}}(X, Y)$ for $X, Y =$ A, D, or H, are equivalent.

With ZFC:

$$\begin{tikzcd}
\boxed{A, B, C, D, E, F, H} \arrow[r] & G
\end{tikzcd}$$

(In addition in ZFC, if the space is second countable and T_1, then all the forms of compactness are equivalent. See [dhhrs].)

In a topological space X, let us refer to a point that has only one neighborhood (which must be X) as a *trivial point.* A *trivially compact* topological space is space that has a trivial point (so a space X is trivially compact if every open cover contains X). A space X is *indiscrete* if its topology is $\{\emptyset, X\}$. It is provable in ZFC that a product, $\prod_{i\in I} X_i$, of first countable spaces is first countable iff each X_i is first countable and all but countably many of the X_i's are indiscrete spaces, and that a product of T_2 spaces is second countable iff each factor is second countable and all but countably many factors are one-point spaces, [w]. In ZF, we show in the next lemma that uncountable products are not first countable unless all but κ of the factors are trivially compact, where κ is a countable union of finite sets.

Lemma 1. *Let $X_i, i \in I$ be topological spaces such that $\prod_{i\in I} X_i$ is first countable and non-empty.*

(a) X_i is first countable, for all $i \in I$.

(b) For each $s \in \prod_{i\in I} X_i$, the set $J_s := \{i \in I : \pi_i(s)$ is not a trivial point in $X_i\}$ is a countable union of finite sets.

Proof. Part (a) is clear.

For (b), let $s \in \prod_{i\in I} X_i$, and let $B_s = \{u_n : n \in \omega\}$ be a neighborhood base for s. For each $i \in J_s$, define $f(i)$ as the least $n \in \omega$ such that

$$u_n \subseteq \pi_i^{-1} v$$

for some neighborhood v of $\pi_i(s)$ in X_i such that $v \neq X_i$.

Given $n \in \omega$, since u_n is an open set in $\prod_{i\in I} X_i$, it contains a basic neighborhood of the form $\pi_{i_0}^{-1} v_{i_0} \cap \cdots \cap \pi_{i_k}^{-1} v_{i_k}$, where $v_{i_0}, \ldots, v_{i_k}$ are open sets in $X_{i_0}, \ldots, X_{i_k}$, respectively. In this case, $f^{-1}(n) \subseteq \{i_0, \ldots, i_k\}$. This means that f defines a partition of J_s into countably many finite pieces. □

Corollary 1. *$AC_{\mathrm{fin}}(\aleph_0)$ implies that if $\prod_{i\in I} X_i$ is first countable, then for each $s \in \prod_{i\in I} X_i$, the set J_s (as in Lemma 1(b)) is countable.*

Proof. $\mathrm{AC}_{\mathrm{fin}}(\aleph_0)$ implies that countable unions of finite sets are countable. □

The following can be checked case by case:

Lemma 2. *If S is a compact X space, and T is a trivially space, then $S \times T$ is compact X, where X is A, B, C, D, E, G, H.* □

Notice that the previous lemma does not apply when X is F. For example, suppose T is an infinite but Dedekind finite set given the indiscrete topology, and let $S = \{0, 1\}$ with the discrete topology. Then, $S \times T$ is an infinite set with no complete accumulation point, since a neighborhood of the form $\{i\} \times T$ does not have the same cardinality as $S \times T$.

Theorem 1. *Let X, Y be definitions of compactness, with Y different from F. If $AC_{\mathrm{fin}}(\aleph_0)$ holds, then $P^{\aleph_0}_{\mathrm{fc}}(X,Y) \to P^1_{\mathrm{fc}}(X,Y)$ and $P^{\aleph_0}_{\mathrm{sc}}(X,Y) \to P^1_{\mathrm{sc}}(X,Y)$*

Proof. We give the proof of $\mathrm{P}^{\aleph_0}_{\mathrm{fc}}(X,Y) \to \mathrm{P}^1_{\mathrm{fc}}(X,Y)$. The proof of $\mathrm{P}^{\aleph_0}_{\mathrm{sc}}(X,Y) \to \mathrm{P}^1_{\mathrm{sc}}(X,Y)$ is similar.

Let X_i, for $i \in I$, be first countable compact X spaces such that $\prod_{i\in I} X_i$ is first countable. If $\prod_{i\in I} X_i$ is empty, then it is immediately compact Y; otherwise, fix $s \in \prod_{i\in I} X_i$. We have that

$$\prod_{i\in I} X_i \equiv \prod_{i\in J_s} X_i \times \prod_{i\in I\smallsetminus J_s} X_i,$$

where $J_s = \{i \in I : \pi_i(s)$ is not a trivial point in $X_i\}$ and is countable by Corollary 1. Now $\prod_{i\in I\smallsetminus J_s} X_i$ is trivially compact, since the projection of s into that space is a trivial point there. And since $\mathrm{P}^{\aleph_0}_{\mathrm{fc}}(X,Y)$ implies that $\prod_{i\in J_s} X_i$ is compact Y, by Lemma 2 we conclude that $\prod_{i\in I} X_i$ is compact Y. □

The fact that $\mathrm{AC}_{\mathrm{fin}}(\aleph_0)$ is needed for the results above is illustrated by the following theorem. However, some statements of the form $\mathrm{P}^{\aleph_0}_{\mathrm{fc}}(X,Y)$ might automatically imply $\mathrm{AC}_{\mathrm{fin}}(\aleph_0)$, and make it redundant for some of the instances of Theorem 1.

Theorem 2. *$P^1_{\mathrm{fc}}(X,Y)$ cannot be proved in ZF, where X is any of the definitions of compactness and Y is A, D, F, G, or H. Consequently, $P^2_{\mathrm{fc}}(X,Y)$ cannot be proved in ZF either, for the same values of X and Y.*

Proof. Consider the second Fraenkel model ($\mathcal{N}2$ in [hr]), in which there is a set A which has a countable partition into pairs with no choice function. In this model we have that 2^A is a product of two-element spaces which is first countable, but not compact A, D, F, G, or H (see [dhhrs]). □

Note that it does not follow immediately from the definitions that $\mathrm{P}^1_{\mathrm{fc}}(X,Y) \to \mathrm{P}^1_{\mathrm{sc}}(X,Y)$. The next theorem proves a result of this form.

Theorem 3. *$P^1_{\mathrm{fc}}(A,A) \to P^1_{\mathrm{sc}}(A,A)$ and $P^1_{\mathrm{sc}}(A,A) \leftrightarrow AC(\aleph_0)$.*

Proof. It is proved in [hkrs] that $\mathrm{P}^{\aleph_0}_{\mathrm{sc}}$(A,A) $\leftrightarrow$ AC($\aleph_0$), and thus both $\mathrm{P}^{\aleph_0}_{\mathrm{sc}}$(A,A) and $\mathrm{P}^{\aleph_0}_{\mathrm{fc}}$(A,A) imply $\mathrm{AC}_{\mathrm{fin}}(\aleph_0)$. It then follows from Theorem 1 that $\mathrm{P}^{\aleph_0}_{\mathrm{fc}}$(A,A) $\leftrightarrow$ $\mathrm{P}^1_{\mathrm{fc}}$(A,A). Likewise, it follows from Theorem 1 that $\mathrm{P}^{\aleph_0}_{\mathrm{sc}}$(A,A) $\leftrightarrow$ $\mathrm{P}^1_{\mathrm{sc}}$(A,A). The theorem follows easily since $\mathrm{P}^{\aleph_0}_{\mathrm{fc}}$(A,A) clearly implies $\mathrm{P}^{\aleph_0}_{\mathrm{sc}}$(A,A). □

We mention some Tychonoff-type product assertions that are provably false.

Theorem 4.
(a) $P^2_{\mathrm{sc}}(X,E)$ is false in ZF for all definitions X of compactness.
(b) $P^1_{\mathrm{sc}}(G,Y)$ is false in ZF for $Y \in \{A, B, C, D, E, F, H\}$.
(c) $P^1_{\mathrm{fc}}(X,Y)$ is false in ZFC for $X \in \{E,G,H\}$ and $Y \in \{A, B, C, D, F\}$.

Proof. The two-element space 2 with the discrete topology is compact X for $X =$ A, B, C, D, E, F, G, and H, while 2^{2^ω} is not compact E (see [dhhrs]). This proves (a). Parts (b) and

(c) are consequences of counterexamples that are not even products (again see [dhhrs]). For part (b) note that the topology $\mathcal{T} = \{(x, \infty) : x \in \mathbb{Z}\}$ on the integers witnesses in ZF that a second countable space can be compact G but not compact in any other sense. To prove (c) consider ω_1 with the order topology. Assuming ω_1 is regular, it is a first countable compact E, G, and H space that is not compact A, B, C, D, or F. □

From independence results found in [dhhrs] (Theorem 3(b) and Lemma 1(c)), it immediately follows that the following statements are not provable in ZF: $\mathrm{P}^2_{\mathrm{sc}}(B, X)$ for $X \in \{A, C, D, E, F, G, H\}$ and $\mathrm{P}^2_{\mathrm{sc}}(X, F)$ for $X \in \{A, B, C, D, E, G\}$.

Theorem 5. *$P^2_{\mathrm{sc}}(A, C)$ implies AC.*

Proof. We will refine the proof of P(A,A) → AC given in [rr].

Assume $\mathrm{P}^2_{\mathrm{sc}}$(A,C), and let $X = \{X_i : i \in I\}$ be a collection of non-empty sets; we will prove that X has a choice function. For each $i \in I$ define the space $\langle Y_i, T_i\rangle$ by $Y_i = X_i \cup \{\infty\}$, where ∞ stands for a fixed element not in $\bigcup X$, and $T_i = \{\emptyset, Y_i, \{\infty\}\}$. These spaces are compact A (actually trivially compact).

Let $P = \prod_{i\in I} Y_i$; clearly P is not empty. For every finite $E \subset I$, define $P_E = \prod_{i\in I\smallsetminus E} Y_i \times \prod_{i\in E}\{\infty\}$; every non-empty basic neighborhood in P is of the form P_E for some finite $E \subset I$. We have the following:

Claim 1: $s \in P$ is a trivial point in P if and only if s is a choice function for X. Indeed, a basic neighborhood P_E is different from P iff $E \neq \emptyset$, and P_E contains s iff $s(i) \notin X_i$ for $i \in E$.

Claim 2: Every finite open cover of P contains P as an element. To prove this we will show that if $S, S' \neq P$ are open sets, then $S \cup S' \neq P$; an easy inductive argument can then complete the proof.

Let $S = \bigcup\{P_E : E \in \mathcal{E}\}$ and $S' = \bigcup\{P_E : E \in \mathcal{E}'\}$, where $\mathcal{E}, \mathcal{E}'$ are sets of finite subsets of I. Take $s, s' \in P$ such that $s \notin S$ and $s' \notin S'$. Then for each $E \in \mathcal{E}$ there exists $i \in E$ such that $s(i) \in X_i$, and for each $E \in \mathcal{E}'$ there exists $i \in E$ such that $s'(i) \in X_i$. It is clear that $t \in P$, defined by

$$t(i) = s(i), \text{ if } s(i) \in X_i$$

$$t(i) = s'(i) \text{ otherwise}$$

is neither in S nor in S'. This finishes the proof of Claim 2.

Finally, in order to reach a contradiction, assume that X has no choice function; by Claim 1, P has no trivial points. If $\mathcal{B}$ is a subbase for the product topology on P then every point has a neighborhood in $\mathcal{B}$ which is not P. This way, $\{u \in \mathcal{B} : u \neq P\}$ is an open cover of P with elements in $\mathcal{B}$. By $\mathrm{P}^2_{\mathrm{sc}}$(A,C), this cover has a finite subcover, but that is impossible by Claim 2. □

Thus, it follows that each of $\mathrm{P}^2_{\mathrm{sc}}(X, Y)$, for $X, Y =$ A or C, is equivalent to AC. Also, it follows from [he] Theorem 4.2, that $\mathrm{P}^2_{\mathrm{sc}}$(F,F) is equivalent to AC. In addition, Theorem 1 of [dhhkr] holds for second countable spaces: Each of $\mathrm{P}^2_{\mathrm{sc}}$(A,G) and $\mathrm{P}^2_{\mathrm{sc}}$(A,H) imply AC.

Consequently, (1) through (5) in the introduction also hold for $\mathrm{P}^2_{\mathrm{sc}}(X,Y)$. (In addition, since $\mathrm{P}^2_{\mathrm{fc}}(X,Y)$ implies $\mathrm{P}^2_{\mathrm{sc}}(X,Y)$, (1) through (5) hold for $\mathrm{P}^2_{\mathrm{fc}}(X,Y)$.)

Let U_ω be the statement: Every non-principal filter on ω can be extended to an ultrafilter. Recall that Pr(A) is the statement: Projections in Tychonoff products of compact A spaces are closed. We close this section with a result showing that some weak forms of AC imply Pr(A).

Theorem 6. *U_ω + AC($\aleph_0$) implies Pr(A) for first countable T_2 spaces.*

Proof. Let $\{(X_i,T_i) : i \in k\}$ be a family of first countable compact T_2 spaces and let (X,T) be their Tychonoff product. If $X = \emptyset$ then there is nothing to show. Let C be a closed set in X and $i \in k$. If $\pi_i(C)$ (the projection of C on its i^{th} coordinate), is not closed then $\pi_i(C)$ is an infinite subset of X_i and $X_i \setminus \pi_i(C)$ contains limit points from $\pi_i(C)$. Let x be such a limit point in $\pi_i(C)$ and, by AC($\aleph_0$), let $\{x_n : n \in \omega\}$ be a countably infinite subset of $\pi_i(C)$ converging to x. By AC($\aleph_0$) again, we choose for each $n \in \omega, y_n \in \pi_i^{-1}(x_n) \cap C$. Let $\mathcal{F}$ be a non-principal ultrafilter on $\{y_n : n \in \omega\}$ extending the filterbase

$$\{\pi_i^{-1}(V) \cap \{y_n : n \in \omega\} : V \text{ is a neighborhood of } x\}.$$

Using compactness and the fact that the spaces are T_2, we can readily verify that for every $j \in k$, $j \neq i$, there exists a unique $x_j \in X_j$ such that

$$\{\pi_j^{-1}(V) \cap \{y_n : n \in \omega\} : V \text{ is a neighborhood of } x_j\} \subset \mathcal{F}.$$

Clearly, the element $y \in X$, $\pi_i(y) = x$ and $\pi_j(y) = x_j$ for $j \neq i$ is in the closure of C, hence, in C. On the other hand, since $x \notin \pi_i(C)$, $y \notin C$. This contradiction finishes the proof of the theorem. □

3 HAUSDORFF SPACES

Properties (1)-(5) in the introduction do hold for T_1 spaces because Theorem 1 of [dhhkr] (Each of P(A,G) and P(A,H) implies AC) holds for T_1 spaces, (add the set of cofinite subsets to the topology on each Y_i). However, these properties do not necessarily hold for T_2 or Hausdorff spaces. We shall add a diagram to indicate the relationships that are known to hold for all spaces. First we have the relationships between the definitions of compact.

$$\begin{tikzcd}
H & D \arrow[l] & A \arrow[l] \arrow[r] \arrow[d] & C \arrow[r] & B \\
& F \arrow[r] & G & &
\end{tikzcd}$$

It is known that $\mathrm{P}_{\mathrm{T}_2}$(A,A) $\leftrightarrow$ BPI. (See Theorem 7 below.) Also, it is known (see, for example, [dhhrs]) that BPI implies that compact A, B, and C are equivalent. Thus,

$\mathrm{P}_{\mathrm{T}_2}$(A,A) implies $\mathrm{P}_{\mathrm{T}_2}$(C,C). In addition, using the preceding diagram and that if $X \to Y$, then $\mathrm{P}(Z,X) \to \mathrm{P}(Z,Y)$ and $\mathrm{P}(Y,Z) \to \mathrm{P}(X,Z)$, we obtain the following diagram (also note that products of T_2 spaces are always T_2):

$$\begin{tikzcd}
\mathrm{P}_{\mathrm{T}_2}(D,H) \arrow[d] & \mathrm{P}_{\mathrm{T}_2}(D,D) \arrow[l] \arrow[d] & & & \\
\mathrm{P}_{\mathrm{T}_2}(A,H) & \mathrm{P}_{\mathrm{T}_2}(A,D) \arrow[l] & \mathrm{P}_{\mathrm{T}_2}(A,A) \arrow[l] \arrow[r] \arrow[d] & \mathrm{P}_{\mathrm{T}_2}(C,C) \arrow[r] & \mathrm{P}_{\mathrm{T}_2}(A,C) \\
 & & \mathrm{P}_{\mathrm{T}_2}(A,G) & &
\end{tikzcd}$$

It has been shown in [he] that $\mathrm{P}_{\mathrm{T}_2}$(B,B) is provable in ZF. Since $\mathrm{P}_{\mathrm{T}_2}$(B,B) implies $\mathrm{P}_{\mathrm{T}_2}$(C,B) implies $\mathrm{P}_{\mathrm{T}_2}$(A,B), it follows that $\mathrm{P}_{\mathrm{T}_2}$(C,B) and $\mathrm{P}_{\mathrm{T}_2}$(A,B) are also provable in ZF. Also, $\mathrm{P}_{\mathrm{T}_2}(X,\mathrm{E})$ is always false (a counterexample is $2^{\mathbb{R}}$).

It is also shown in [he] that AC $\leftrightarrow$ $\mathrm{P}_{\mathrm{T}_2}$(F,F). Since F implies G, it is also true that:

$$\mathrm{P}_{\mathrm{T}_2}(F,F) \longrightarrow \mathrm{P}_{\mathrm{T}_2}(F,G)$$

We show in section 6 that $\mathrm{P}_{\mathrm{T}_2}$(F,G) does not imply AC.

The next collection of results shows that, in contrast to (1) in the introduction, none of the arrows in the sequence of implications

$$\mathrm{AC} \to \mathrm{P}_{\mathrm{T}_2}(\mathrm{A,A}) \to \mathrm{P}_{\mathrm{T}_2}(\mathrm{A,D}) \to \mathrm{P}_{\mathrm{T}_2}(\mathrm{A,H})$$

are reversible.

Theorem 7. *(Łoś, Ryll-Nardzewski* [lr2]*) $P_{\mathrm{T}_2}(A,A) \leftrightarrow$ BPI.*

Lemma 3. *Assuming AC(LO), a T_2 space X is compact D iff every well-ordered infinite subset of X has a complete accumulation point.*

Note that Lemma 3 is not very interesting in ZF, since ZF + AC(LO) implies AC. However, ZF^0 + AC(LO) does not imply AC (see [tr]).

Proof. Suppose X is not compact D, so there is a covering nest $\{U_i : i \in I\}$ of open sets that does not have X as a member. Under AC(LO), every linearly ordered set can be well-ordered, so choose a well-ordering $\leq$ of I (which is linearly ordered by the subset relation on the U_i's). Now we obtain a well-ordered nest by induction: For each ordinal α, let $W_\alpha = U_j$, where j is the $\leq$-least member of I such that U_j properly contains $\bigcup_{\beta<\alpha} W_\beta$, if such W_α exists. By throwing away all but some cofinal subset if necessary, we may assume that the W_α's are defined just for all α in some cardinal κ. Now using AC(LO) again, let $x_\alpha \in W_{\alpha+1} \setminus W_\alpha$ for each $\alpha \in \kappa$. Assuming X is Hausdorff, this family of x_α's will have no complete accumulation point.

Conversely, assume that (X, T) is a compact D T_2 space. We will show that every well-ordered subset $Y = \{y_j : j \in \aleph\}$ has a complete accumulation point. We consider two cases:

(i) $\aleph$ is a regular cardinal. For every $j \in \aleph$ put $U_j = \{O \in T : O \cap Y \subset \{y_i : i \in j\}\}$ and let $U = \{\bigcup U_j : j \in \aleph\}$. Assuming that Y has no c.a.p. (complete accumulation point) and taking into consideration that every point $x \in X$ has a neighborhood O_x meeting Y in a set of size $< \aleph$, we see that U is a well ordered open cover of X. Hence, for some $j \in \aleph$, $X = \bigcup U_j$ which is a contradiction.

(ii) $\aleph$ is a singular cardinal. Let $k = cf(\aleph)$ (the cofinality of k) and fix $\{k_n : n \in k\}$ a cofinal set of regular cardinals of $\aleph$. For every $n \in k$ let $B_n = \{x \in X : x \text{ is a c.a.p. of } \{y_i : i \in k_n\}\}$. By part (i) each B_n is non-empty. Let $G = \{g_n : n \in k\}$ be a choice set of $B = \{B_n : n \in k\}$. Since k is a regular cardinal it follows by part (i) again, that G has a c.a.p. g, and this g must be a c.a.p. of Y. Indeed, if V_g is a neighborhood of g then $|V_g \cap \{y_i : k_n \in i \in k_{n+1}\}| = k_{n+1}$ for all $n \in k$. Thus,

$$|V_g \cap \{y_i : i \in \aleph\}| = \sum |V_g \cap \{y_i : k_n \in i \in k_{n+1}\}| = \sum k_n = \aleph.$$

The equalities in the above display are justified because of AC(LO). □

As before, U_ω is the statement: Every non-principal filter on ω can be extended to an ultrafilter. Similarly, let U_{WO} be the statement: Every non-principal filter on a well-ordered set can be extended to an ultrafilter. Note that these principles are always true in permutation models, where AC holds in the subuniverse of pure sets.

Theorem 8.
(a) $U_\omega + AC(\aleph_0) \to P_{\mathrm{T}_2}(A, H)$
(b) $U_{\mathrm{WO}} + AC(LO) \to P_{\mathrm{T}_2}(A, D)$

Proof. (a) Let $\{(X_i, T_i) : i \in k\}$ be a family of T_2 compact A spaces and let X be their Tychonoff product. In view of AC($\aleph_0$) it suffices to show that every countably infinite subset $S = \{g_n : n \in \omega\}$ of X has a limit point. Choose, by U_ω, a non-trivial ultrafilter $\mathcal{F}$ on S. It is straightforward to verify that for each $j \in k$ there is a unique $b_j \in X_j$ such that $\{\pi_j^{-1}(V_{b_j}) \cap S : V_{b_j} \text{ is a neighborhood of } b_j\} \subset \mathcal{F}$. (The existence follows from compactness and the fact that $\mathcal{F}$ is an ultrafilter, and the uniqueness follows from T_2.) Now it is clear that the element $b \in X$ given by $b(i) = b_i$ is a limit point of S finishing the proof of the theorem.

(b) Choose a family of compact A Hausdorff spaces. By AC(LO) and Lemma 3, it suffices to show that every well-ordered infinite subset S of their product has a limit point. If we choose, by U_{WO}, an ultrafilter $\mathcal{F}$ on S such that each subset of S with cardinality smaller than that of S is not in $\mathcal{F}$, then the proof proceeds just as in the proof of part (a). □

Corollary. *P_{T_2}(A,D) does not imply P_{T_2}(A,A).*

Proof. The corollary follows from Theorem 7 and Theorem 8(b), since in ZF^0, U_ω + AC(LO) does not imply BPI.

(Construct a permutation model as follows: Let the set of atoms, A, be uncountable. The group of permutations, G, is the group of all permutations on A, and supports are countable. (See $\mathcal{N}12$ in [hr].) BPI is false because BPI implies every set of pairs has a choice function, but the set of pairs of atoms in this model has no choice function. Truss showed in [tr] that AC(LO) holds in this model.) □

Theorem 9. *P_{T_2}(A,H) does not imply P_{T_2}(A,D).*

Proof. We first give the description of a permutation model $\mathcal{N}$. The set of atoms $A = \bigcup\{A_i : i \in \omega_1\}$, where $A_i = \{a_{ix} : x \in C\}$ and C is the set of points on the unit circle centered at 0. The group of permutations $\mathcal{G}$ is the group of all permutations on A which rotate the A_i's by an angle $\theta_i \in \mathbb{R}$ and supports are countable.

The following 2 claims are straightforward:

Claim 1. The family $\mathcal{A} = \{A_i : i \in \omega_1\}$ does not have a multiple choice function in $\mathcal{N}$.

Claim 2. AC($\aleph_0$) is true in $\mathcal{N}$.

Since U_ω and AC($\aleph_0$) are true in $\mathcal{N}$, it follows by Theorem 8(a) that $\mathrm{P}_{\mathrm{T}_2}$(A,H) also holds in $\mathcal{N}$. Furthermore, $\mathrm{P}_{\mathrm{T}_2}$(A,D) fails in $\mathcal{N}$. Indeed, for every $i \in \omega_1$ let $X_i = A_i \cup \{*_i\}$ carry the disjoint topology T_i induced from the (compact A) metric topology of A_i (A_i is identified with the unit circle) and the discrete topology on $\{*_i\}$. Clearly each X_i is a compact A T_2 space. Let X be the Tychonoff product of the family $\{(X_i, T_i) : i \in \omega_1\}$. Let $U_i = \cup\{\pi_j^{-1}(\{*_i\}) : j \in i\}$. Since AC($\aleph_0$) holds in $\mathcal{N}$, it follows that no U_i covers X. Since $\mathcal{A}$ has no choice set, we see that $U = \{U_i : i \in \omega_1\}$ is a nested open cover of X which has no finite subcover. □

This completes the proof that none of the implications

$$\mathrm{AC} \to \mathrm{P}_{\mathrm{T}_2}(\mathrm{A,A}) \to \mathrm{P}_{\mathrm{T}_2}(\mathrm{A,D}) \to \mathrm{P}_{\mathrm{T}_2}(\mathrm{A,H})$$

are reversible.

Let U be the statement: Every infinite set has a non-trivial ultrafilter. It is known that U is strictly weaker than BPI, so the next result shows that the implication $\mathrm{P}_{\mathrm{T}_2}$(A,A) → $\mathrm{P}_{\mathrm{T}_2}$(A,G) is not reversible in ZF.

Theorem 10. *U implies P_{T_2}(A,G).*

Proof. Let $\{(X_i, T_i) : i \in k\}$ be a family of compact A, T_2 spaces and let X be their Tychonoff product. Let S be an infinite subset of X and let $\mathcal{F}$ be a non-trivial ultrafilter on X. Continuing as in the proof Theorem 8(a) we can finish the proof of this theorem. □

4 WELL ORDERED PRODUCTS AND THE MULTIPLE CHOICE AXIOM

Our first theorem in this section gives a property of well ordered products of compact A spaces. We shall use the notation P^{WO}(X,Y) to mean that a well ordered product of compact X spaces is compact Y.

Theorem 11. *P^{WO}(A,D) if and only if P^{WO}(A,A).*

Proof. It suffices to show ($\rightarrow$) as the other implication is clear. Fix $\{(X_n, T_n) : n \in \aleph\}$ a well ordered family of compact A spaces and let (X, T) be their Tychonoff product. Without loss of generality we may assume that for every $v \in \aleph$, $Y_v = \prod_{n \in v} X_n$ is compact A. Let π_v denote the projection of X to X_v. Let $\mathcal{G}$ be a family of closed subsets of X having the finite intersection property. Let $\mathcal{Q} = \{\pi_v^{-1}(Q_v) : v \in \aleph\}, Q_v = \bigcap\{\overline{\pi_v(G)} : G \in \mathcal{G}\}$. It can be readily verified that $\mathcal{Q}$ is a family of closed sets of X having the finite intersection property. Since for every $v \in \aleph$, Y_v is compact A, it follows that $\mathcal{W} = \bigcap\{\pi_v^{-1}(Q_v) : v \in \aleph\}$ is a nest of non-empty closed subsets of X. Since X is compact D, it follows that $\bigcap \mathcal{W} \neq \emptyset$. Clearly, any point $x \in \bigcap \mathcal{W}$ is in $\bigcap G$ and, consequently, X is compact A as required, finishing the proof of the theorem. □

Our next result gives a property of the multiple choice axiom, MC.

Theorem 12. *MC implies that compact D spaces are compact A.*

Proof. Let (X, T) be a compact D space and $\mathcal{U} = \{U_i : i \in k\}$ be an open cover of X. Using Levy's Lemma (form [67 B] in [hr]: Each set can be covered by a well ordered family of finite sets) we express k as $\bigcup\{k_j : j \in \aleph\}$, where each k_j is a finite subset of k and $\aleph$ is a well ordered cardinal. For every $j \in \aleph$ put $O_j = \bigcup\{U_i : i \in k_j\}$. Let m be the least cardinal number for which there exists a subfamily F of $\{O_j : j \in \aleph\}$ covering X. We show that m is finite. Assume on the contrary that m is infinite. For every $j \in m$ put $Q_j = \bigcup\{O_i : i \in j\}$. Since X is compact D, it follows that some Q_j covers X. Thus, $\{O_i : i \in j\}$ is a subfamily of $\{O_j : j \in \aleph\}$ of cardinality $< m$ covering X. This is a contradiction. Hence, $\mathcal{U}$ has a finite subcover and X is compact A as required. □

Theorem 13. *MC + P^{WO}(A,A) if and only if AC.*

Proof. It suffices to show that MC + P^{WO}(A,A) implies P(A,A), as P(A,A) is equivalent to AC and the implication in the other direction is clear.

Let $\{X_i : i \in k\}$ be a family of compact A spaces and let $X = \prod_{i \in k} X_i$ be the Tychonoff product. Use Levy's Lemma again (see the proof of Theorem 12) to express k as a well ordered union of finite disjoint subsets of k. That is, $k = \bigcup\{k_v : v \in \aleph\}$. For every $v \in \aleph$, let $Q_v = \prod_{j \in k_v} Y_j$. Clearly Q_v is compact A. The product $Q = \prod_{v \in \aleph} Q_v$ is a compact A space by P^{WO}(A,A), and Q is homeomorphic to X.

It now follows from Theorems 12 and 13 that

Corollary. *MC + P^{WO}(D,D) if and only if AC.*

It follows from Theorem 13 and its Corollary that neither P^{WO}(A,A) nor P^{WO}(D,D) is provable in ZF^0.

5 PRODUCTS OF SUBSETS OF $\mathbb{R}^\kappa$ AND CHOICE FOR CLOSED SETS

In this section we study products of subsets of $\mathbb{R}^\kappa$; the results are different for well orderable products and general products.

Theorem 14. *Let* $\{X_i : i \in I\}$ *be a collection of compact C topological spaces such that* $\{c \subset X_i : i \in I, c \neq \emptyset,\ c \text{ is closed}\}$ *has a choice function. Assume also that there is a sequence* $\{S_i : i \in I\}$ *such that for each* $i \in I$, S_i *is a subbase for* X_i *that witnesses the C-compactness of* X_i. *Then* $\prod_{i\in I} X_i$ *is compact C.*

Proof. Consider the collection $T = \{U_{i,s} : i \in I, s \in S_i\}$ of open sets in $\prod_{i\in I} X_i$, where $U_{i,s} = \prod_{j\in I} N_j$ with $N_i = s$ and $N_j = X_j$ for $j \neq i$. It is easy to see that the set of finite intersections of elements of T is a base for the Tychonoff topology of $\prod_{i\in I} X_i$, and therefore T is a subbase for that topology.

Suppose now that $C \subset T$ is a cover for $\prod_{i\in I} X_i$. Then there exists $i_0 \in I$ such that $\bigcup\{s : U_{i_0,s} \in C\} = X_{i_0}$; otherwise, we have that for every $i \in I$, $D_i = X_i \setminus \bigcup\{s : U_{i,s} \in C\}$ is a non-empty closed subset of X, and by hypothesis there is an element in $\prod_{i\in I} D_i$, which cannot be covered by any element of C. Since $\{s : U_{i_0,s} \in C\}$ is a cover of X_i by elements of S_i, it has a finite subcover F. Clearly, $\{U_{i_0,s} : s \in F\}$ is a finite subcover of C. $\square$

The existence of the witnessing family of subbases is satisfied, for example, in the case of a power of a fixed space.

For well orderable products, we have:

Theorem 15. *Let* $\{X_\alpha : \alpha < \lambda\}$ *be a collection of compact A topological spaces such that* $\{c \subset X_\alpha : \alpha < \lambda, c \neq \emptyset,\ c \text{ is closed}\}$ *has a choice function. Then* $\prod_{\alpha<\lambda} X_\alpha$ *is compact A.*

Proof. Let C be a choice function for $\{c \subset X_\alpha : \alpha < \lambda, c \neq \emptyset,\ c \text{ is closed}\}$. To see that $X = \prod_{\alpha<\lambda} X_\alpha$ is compact A, let F be a family of closed subsets of X with the finite intersection property (fA .i.p.); we will show that $\bigcap F \neq \emptyset$.

By simultaneous recursion on α we define a family $\{\mathcal{F}_\alpha : \alpha < \lambda\}$ of filters on X, and a sequence $\langle x_\alpha \in X_\alpha : \alpha < \lambda\rangle$, satisfying:

(i) $\mathcal{F}_0$ is the filter generated by F.

(ii) $\mathcal{F}_\alpha \subset \mathcal{F}_\beta$ for all $\alpha < \beta < \lambda$.

(iii) For all $\beta < \lambda$ and all $\alpha < \beta$, $\pi_\alpha^{-1}(u) \in \mathcal{F}_\beta$, for every neighborhood u of x_α.

Once this construction is completed, the filter $\mathcal{G}$ generated by $\bigcup\{\mathcal{F}_\alpha : \alpha < \lambda\}$ will converge to the point $x = \langle x_\alpha \in X_\alpha : \alpha < \lambda\rangle \in X$. Then every neighborhood v of x intersects every element of F; since each element of F is closed, $x \in \bigcap F$.

$\mathcal{F}_0$ is given by (i). Assuming that the filter $\mathcal{F}_\delta$ is defined, we take $x_\delta = C(\bigcap\{\overline{\pi_\delta(y)} : y \in \mathcal{F}_\delta\})$ (we have that $\bigcap\{\overline{\pi_\delta(y)} : y \in \mathcal{F}_\delta\} \neq \emptyset$ because each X_α is compact A). We also define $\mathcal{F}_{\delta+1}$ as the filter generated by the collection $\mathcal{F}_\delta \cup \{\pi_\delta^{-1}(v) : v \text{ is a neighborhood of } x\}$. If γ is a limit ordinal and $\mathcal{F}_\delta$ is defined for all $\delta < \gamma$, we define $\mathcal{F}_\gamma$ as the filter generated by $\bigcup\{\mathcal{F}_\delta : \delta < \gamma\}$. This finishes the recursion; clearly conditions (ii) and (iii) are satisfied. $\square$

Corollary. *(a) Any product $\prod_{i\in I} X_i$ of closed compact C sets of reals, with a witnessing family of subbases $\{S_i : i \in I\}$, is compact C.*

(b) Well ordered products of compact A sets of reals are compact A.

(c) More generally, any product of compact C linearly ordered spaces with the order topology (and a family of witnessing subbases) which are conditionally complete (any subset with upper bounds has a least upper bound) is compact C. If the product is well orderable and the spaces are compact A (equivalently, if they have a maximum and a minimum element), then the product is compact A.

Proof. Part (c) implies both parts (a) and (b). For (c), if the product is empty, then it is compact A. Assume then that the product is not empty; because of the previous theorems, we only need to prove that the collection of all closed subsets from all the factor spaces has a choice function.

Fix one element x in the product; we will give a rule for choosing an element from a given closed subset c of the i-th factor space using only the element x_i. If x_i is an upper bound for c, we pick the least upper bound of c. If not, we pick the greatest lower bound of $(x_i, \infty) \cap c$. In either case the element we picked is in c, since c is closed. □

Notice that arbitrary products of compact A sets of reals are not necessarily compact A: consider the space 2^A in the model $\mathcal{N}2$ in [hr], where A is the set of atoms.

Theorem 16. *If κ is a well-orderable cardinal then the collection*

$$\{c \subset \mathbb{R}^\kappa : c \neq \emptyset, c \textit{ is compact } A\}$$

has a choice function.

Proof. Let c be a fixed compact A subset of $\mathbb{R}^\kappa$. We will describe a rule for choosing an element of c.

For each $\alpha < \kappa$, let c_α be the projection by π_α of c on $\mathbb{R}$. Since projections are continuous, c_α is compact A. Therefore, $c_\alpha \subset \mathbb{R}$ is bounded. Let $r_\alpha = \inf(c_\alpha)$ and $s_\alpha = \sup(c_\alpha)$. Then $Y = \prod_{\alpha<\kappa}[r_\alpha, s_\alpha]$ contains c, and Y is compact A by the previous corollary. Also, for each $\alpha < \kappa$, let $\mathcal{B}_\alpha$ be the restriction to $[r_\alpha, s_\alpha]$ of the base $\mathcal{B}$ for the topology of $\mathbb{R}$ which contains all open intervals with rational endpoints. Clearly, $\mathcal{B}_\alpha$ is a countable base for $[r_\alpha, s_\alpha]$.

Let $\mathcal{G}$ be a subfamily of $\{c\} \cup \{\pi_\alpha^{-1}(b) : \alpha < \kappa, b \in \mathcal{B}_\alpha\}$ which contains c and is maximal with respect to the f.i.p.; such a family can be easily constructed by transfinite induction, since $\{c\} \cup \{\pi_\alpha^{-1}(b) : \alpha < \kappa, b \in \mathcal{B}_\alpha\}$ has cardinality κ.

Let $\mathcal{F}$ be the filter generated by $\mathcal{G}$, and for each $\alpha < \kappa$, define

$$A_\alpha = \{x \in [r_\alpha, s_\alpha] : \pi_\alpha^{-1}(b) \in \mathcal{F}, \text{ for every neighborhood } b \in \mathcal{B}_\alpha \text{ of } x\}.$$

We have that $A_\alpha \neq \emptyset$. Since $[r_\alpha, s_\alpha]$ is compact A, the filter $\mathcal{F}_\alpha$ generated by the sets $b \in \mathcal{B}_\alpha$ such that $\pi_\alpha^{-1}(b) \in \mathcal{F}$ has an accumulation point x. If $b \in \mathcal{B}_\alpha$ is a neighborhood of x, then it has a non-empty intersection with every member of $\mathcal{F}_\alpha$; consequently $\pi_\alpha^{-1}(b)$ meets every member of $\mathcal{F}$ and therefore $\pi_\alpha^{-1}(b) \in \mathcal{F}$ (by maximality).

Furthermore, A_α is a singleton. Otherwise, if $b_1, b_2 \in \mathcal{B}_\alpha$ are disjoint neighborhoods of two points in A_α, we would have $\pi_\alpha^{-1}(b_1), \pi_\alpha^{-1}(b_1) \in \mathcal{F}$, which is not possible because these sets are disjoint. Now, for each $\alpha < \kappa$, let x_α be the unique element of A_α, and let $x^c = \langle x_\alpha : \alpha < \kappa \rangle \in Y$. Then every neighborhood of x^c in Y contains a neighborhood which is a finite intersection of sets of the form $\pi_\alpha^{-1}(b_1)$, which are in $\mathcal{F}$. Therefore, every neighborhood of x^c intersects c; since c is closed, then $x^c \in c$. □

Notice that in the proof we used no arbitrary parameters. Therefore we can uniformly choose elements from the compact A subsets of many different spaces $\mathbb{R}^{\kappa_i}$ at the same time.

Corollary. *Arbitrary products of compact A spaces which are a subspace of* $\mathbb{R}^\kappa$ *for some well-orderable cardinal* κ*, are compact C. Also, if the index set is well orderable, then the product is compact A.*

Proof. Let $\{X_i : i \in I\}$ be a family of compact A spaces, such that for each $i \in I$, X_i is a subspace of $\mathbb{R}^{\kappa_i}$. Then for each $i \in I$, the collection of closed subsets of X_i is contained in the collection of compact A subsets of $\mathbb{R}^{\kappa_i}$. Therefore, by the previous theorem, we have a choice function for the closed subsets of all the factor spaces. Applying Theorems 14 and 15 we obtain the desired conclusions. □

We turn now to the study of compact F sets in this context.

Theorem 17. *Assume that* X *has a well-ordered base and a choice function for its closed subsets. If* X *is compact F, then it is compact A.*

Proof. Let $\mathcal{B}$ be a well ordered base for X, and let F be a choice function for the closed subsets of X, and suppose that X is not compact A.

From all the well-ordered open covers from $\mathcal{B}$ without a finite subcover, choose $C = \{u_\alpha : \alpha < \lambda\}$ with minimum λ, where λ is a cardinal. Let $\kappa = \mathrm{cof}(\lambda)$ and let $f : \kappa \to \lambda$ be an increasing cofinal sequence. For each $\alpha < \kappa$ let

$$x_\alpha = F(X \smallsetminus \bigcup_{\beta < f(\alpha)} u_\beta).$$

(Each set $X \smallsetminus \bigcup_{\beta < f(\alpha)} u_\beta$ is closed, and it is non-empty by the minimality of κ.) Then, since κ is a regular cardinal, $Y = \{x_\alpha : \alpha < \kappa\}$ has cardinality κ, and for every $\gamma < \kappa$, $|\{x_\alpha : \alpha < \gamma\}| < \kappa$.

Now, the set Y has no complete accumulation points because for every $x \in X$ there exists a neighborhood u_α of x, and we have that $|u_\alpha \cap Y| < \kappa$. Therefore, X is not compact F. □

Corollary. *For closed sets of reals, compact F implies compact A.*

Proof. From our previous results, closed sets of reals clearly satisfy the hypotheses of Theorem 17. □

Corollary. *Well-ordered products of closed compact F sets of reals are compact A.* □

However, it is not true in general that even well-ordered products of compact F sets of reals are compact F: consider the product space 2^ω in the model $\mathcal{M}1$ in [hr]; $2 = \{0, 1\}$ is clearly compact F, but 2^ω is not, because it is Hausdorff and contains an infinite Dedekind finite set.

The results in this section suggest the following questions: Are compact C (or F) sets of reals necessarily closed in $\mathbb{R}$?

Since choice for closed sets played an important role in the previous results, the study of the following principles becomes natural:

Definition. Let CSC(T_i), $i = 1, 2$ stand for the statements: If (X, τ) is a compact A T_i space then the family $\mathcal{G}$ of all non-empty closed subsets of X has a choice function.

Theorem 18. *CSC(T_1) is equivalent to AC.*

Proof. We only need to prove that CSC(T_1) implies AC. Fix $\mathcal{A} = \{A_i \ : i \in I\}$ a disjoint family of non-empty sets and let Z be the one point compactification of the disjoint union Y of the family of spaces $\{(A_i, \tau_i) : i \in I\}$, where τ_i is the cofinite topology on A_i. By CSC(T_1), let f be a choice function on the family of all non-empty closed subsets of Z. Clearly, the restriction h of f to the family $\mathcal{A}$ is a choice function, finishing the proof of the theorem. □

In the last part of this section we use the following statements from [hr]:

FORM 154. Tychonoff's Compactness Theorem for Countably Many T_2 Spaces: The product of countably many compact A T_2 spaces is compact A.

FORM 343. A product of non-empty, compact A T_2 spaces is non-empty.

Theorem 19. *(Keremedis–Tachtsis* [kt]*) CSC(T_2) $\leftrightarrow$ form 343.*

Theorem 20. *CSC(T_2) $\leftrightarrow$ $P^*_{T_2}(A, C)$, where $P^*_{T_2}(A,C)$ is the principle: If $\{(X_i, \tau_i) : i \in I\}$ is a family of non-empty compact A T_2 spaces then the Tychonoff product $X = \prod_{i\in I} X_i$ is compact C with respect to the standard subbase $\mathcal{H} = \{\pi_i^{-1}(O) : O \in \tau_i, i \in I\}$ of X.*

Proof. ($\rightarrow$). Fix $\{(X_i, \tau_i) : i \in I\}$ a family of non-empty compact A spaces and let $X = \prod_{i\in I} X_i$. We will show that X is compact A with respect to the standard subbase $\mathcal{H}$. Let $\mathcal{U} \subset \mathcal{H}$ be an open cover of X. For every $i \in I$ put $\mathcal{U}_i = \mathcal{U}|\{\pi_i^{-1}(O) : O \in \tau_i\}$. If some $\mathcal{U}_i$ covers X then it is easy to see that $\mathcal{U}_i$, hence $\mathcal{U}$, has a finite subcover. So assume than no $\mathcal{U}_i$ covers X and arrive at a contradiction. Put $G = \{G_i : i \in I\}$, where $G_i = X_i \setminus \bigcup\{O \in \tau_i : \pi_i^{-1}(O) \in \mathcal{U}_i\}$. Clearly each G_i with the subspace topology is an compact A T_2 space. Thus, by Theorem 19, G has a choice function f. It is straightforward to see that f is not covered by $\mathcal{U}$, contradicting the fact that $\mathcal{U}$ covers X.

($\leftarrow$). By Theorem 19, it suffices to prove form 343. Fix $\{(X_i, \tau_i) : i \in I\}$ a family of non-empty compact A T_2 spaces. Let $Y_i = X_i \cup \{*_i\}$, $*_i \notin X_i$, carry the disjoint union topology of X_i and the discrete space $\{*_i\}$. Clearly Y_i is an compact A T_2 space. Hence, by $P^*_{T_2}$(A, C), $Y = \prod_{i\in I} Y_i$, the Tychonoff product of the family $\{Y_i : i \in I\}$, is compact C with respect to the standard subbase $\mathcal{H}$. If $\prod_{i\in I} X_i = \emptyset$, then $\mathcal{U} = \{\pi_i^{-1}(*_i) : i \in I\} \subset \mathcal{H}$ is an open cover of Y. It follows that $\mathcal{U}$ has a finite subcover, which is a contradiction. $\square$

Theorem 21. *CSC(T_2) implies $P_{T_2}^{WO}$(A, A), where $P_{T_2}^{WO}$(A, A) stands for: the Tychonoff product of every well ordered family $\{(X_i, \tau_i) : i \in \aleph\}$ of compact A T_2 spaces is compact A. In particular, CSC(T_2) implies form 154.*

Proof. Fix $\{(X_i, \tau_i) : i \in k\}$ a well ordered family of compact A T_2 spaces and let $X = \prod_{i\in k} X_i$ be their Tychonoff product. Let $G = \{G_j : j \in J\}$ be a family of closed sets having the f.i.p. We will show that $\cap G \neq \emptyset$. Let f be a choice function of the family of all non-empty closed sets of the one point compactification of the disjoint topological union of the family $\{(X_i, \tau_i) : i \in k\}$. It follows that the restriction h of f to the family $\{H : \emptyset \neq H \subset X_i, (X \setminus H) \in \tau_i, i \in k\}$ is a choice function. Now we can complete the proof of the theorem by mimicing the proof of Theorem 20. $\square$

The status of the relationship between forms 343 and 154 is indicated as unknown in Table 1 of [hr]. By Theorems 19 and 21 we have the following:

Corollary. *Form 343 implies form 154.*

6 An Independence Result

Our last result shows that P(F,G) does not imply AC in ZF^0. We do this by showing that P(F,G) is true in Mostowski's linearly ordered model, $\mathcal{N}$. (This is the model $\mathcal{N}3$ in [hr].) The set of atoms A is ordered so that it is isomorphic to the set of rational numbers. The group G is the group of order preserving isomorphisms and the ideal of supports is the set of all finite subsets of A. If E is a finite subset of A, we let $G_E = \{\phi \in G : \phi \text{ fixes } E\}$. Then G_E is a subgroup of G. (Note that if $\phi \in G_E$ then ϕ fixes E pointwise.) We will also use the following facts about $\mathcal{N}$.

(1) Every element x of $\mathcal{N}$ has a smallest support supp(x) with the property that $\forall\phi \in G$, $\phi(x) = x \Leftrightarrow \phi(\text{supp}(x)) = \text{supp}(x)$.

(2) If E and $F \neq \emptyset$ are disjoint finite subsets of A, then the set $S = \{\phi(F) : \phi \in G_E\}$ is an infinite, Dedekind finite set in the model.
(If we choose $t \in F$, then the set $\{\phi(t) : \phi \in G_E\}$ is infinite. By (1), since supp(F) = F, if ϕ and ψ are in G_E and $\phi(t) \neq \psi(t)$ then $\phi(F) \neq \psi(F)$. It follows that S is infinite. If S were Dedekind infinite, then there would be a finite subset G of A which was a support of infinitely many elements of S. But the only finite subsets of A supported by G are the finitely many subsets of G. It follows that S is Dedekind finite.)

(3) If $E = \{e_0, e_1, \ldots, e_n\}$ is any finite subset of A and if $\mathcal{W} = \{(-\infty, e_0), (e_0, e_1), \ldots, (e_n, \infty)\}$ is the set of open intervals determined by E, then for any two subsets F and F' of A both disjoint from E, if $|F \cap W| = |F' \cap W|$ for all $W \in \mathcal{W}$, then there is a $\phi \in G_E$ such that $\phi(F) = F'$.
(The proof of (3) uses the following fact about the ordering on A: If $B = \{b_1, b_2, \ldots, b_k\}$ and $C = \{c_1, c_2, \ldots, c_k\}$ are two subsets of A both with the same number of elements and $b_1 < b_2 < \cdots < b_k$ and $c_1 < c_2 < \cdots c_k$ then there is an order isomorphism ϕ

of A such that for all i, $1 \leq i \leq k$, $\phi(b_i) = c_i$). In order to see that this is true we may assume that A is the set of rational numbers with the usual ordering and let ϕ be the union of the functions ϕ_1, $\phi_2, \ldots, \phi_{k-1}$ where ϕ_1 is the linear function whose domain is $(-\infty, b_2]$ and whose graph goes through the points (b_1, c_1) and (b_2, c_2), for $2 \leq i \ < k-1$, ϕ_i is the linear function whose domain is $[b_i, b_{i+1})$ and whose graph goes through the points (b_i, c_i) and (b_{i+1}, c_{i+1}) and ϕ_{k-1} is the linear function whose domain is $[b_{k-1}, \infty)$ whose graph goes through the points (b_{k-1}, c_{k-1}) and (b_k, c_k). Now (3) follows by taking $B = E \cup F$ and $C = E \cup F'$.

Theorem 22. *P(F,G) is true in the Mostowski linearly ordered model, $\mathcal{N}$.*

Proof. Assume that $\{(X_i, T_i) : i \in I\} \in \mathcal{N}$ is a collection of ordered pairs each of which is an compact F topological space in $\mathcal{N}$ and let $X = \left(\prod_{i \in I} X_i\right)^{\mathcal{N}}$ where the superscript $\mathcal{N}$ means that the product is taken in the model. Assume that $Z \in \mathcal{N}$ is an infinite subset of X. Let E be a support of the ordered pair (X, Z). For each $i \in I$, let $\pi_i : X \to X_i$ be the projection map. We first claim that it suffices to prove that

$$(\#) \qquad \begin{gathered} Z \subseteq X, \text{ has a cluster point (in } \mathcal{N}\text{) under the assumption} \\ (\forall i \in I)(|\pi_i(Z)| > 1) \end{gathered}$$

(Note that f is a *cluster point* of Z if every neighborhood of f contains a point in Z different from f.)

To prove the claim, let $I' = \{i \in I : |\pi_i(Z)| > 1\}$, and let $\pi_{I'} : X \to \prod_{i \in I'} X_i$ be the projection map. Then $(\forall i \in I')(|\pi_i(\pi_{I'}(Z))| > 1)$. It is also easy to see that $\pi_{I'}(Z)$ is infinite since Z is. Therefore, if we assume (#), $\pi_{I'}(Z)$ has a cluster point in $\prod_{i \in I'} X_i$, call it f'. If we then define $f \in X$ by

$$f(i) = \begin{cases} f'(i) & i \in I' \\ \text{the unique element of } \pi_i(Z) & \text{otherwise} \end{cases}$$

we can show that f is a cluster point of Z.

For the remainder of the proof we shall assume

$$(*) \qquad (\forall i \in I)(|\pi_i(Z)| > 1)$$

and we shall show that Z has a cluster point.

Lemma. *If for any $i \in I$, there is an element $t \in X_i$ for which supp$(t) \not\subseteq$ supp(X_i), then Z has a cluster point.*

Proof. Assume i satisfies the hypothesis, then the set $W = \{\phi(t) : \phi \in G_{\text{supp}(X_i)}\}$ is an infinite, Dedekind finite subset of X_i since it can be put in a one to one correspondence in the model with the set $\{\phi(\text{supp}(t) \setminus \text{supp}(X_i)) : \phi \in G_{\text{supp}(X_i)}\}$. (This latter set is infinite and Dedekind finite in the model by (2).)

By our assumption $(*)$, there are two elements f and g of Z such that $f(i) \neq g(i)$. Since X_i is compact F, the set $W' = W \cup \{f(i), g(i)\}$ has a complete accumulation point $z \in X_i$. We assume without loss of generality that $z \neq f(i)$. We first note that $f(i)$ is in every neighborhood of z. For assume that N is a neighborhood of z in X_i. Then since W' is Dedekind finite and $|W' \cap N| = |W'|$ we have $W' \cap N = W'$. Hence, $f(i) \in N$. Define $h \in X$ by

$$h(j) = \begin{cases} z & \text{if } j = i \\ f(j) & \text{otherwise} \end{cases}.$$

Then $f \neq h$ and f is in every neighborhood of h since $f(i)$ is in every neighborhood of $z = h(i)$. Therefore, h is an accumulation point of Z.

Using the lemma, we will assume for the remainder of the proof that

$$(**) \qquad (\forall i \in I)(\forall t \in X_i)(\mathrm{supp}(t) \subseteq \mathrm{supp}(X_i)).$$

Lemma. *If for some $f \in Z$, supp$(f) \nsubseteq E$ then Z has a cluster point.*

Proof. Assume that f_0 is an element of Z whose support is not contained in E. We may also assume that the function $i \mapsto X_i$ is one to one and therefore, for any $\phi \in G_E$, $\phi(i) = i \Leftrightarrow \phi(X_i) = X_i$.

Let $E = \{e_0, e_1, \ldots, e_n\}$ and let $\mathcal{W}$ be the set of open intervals determined by E. That is, $\mathcal{W} = \{(-\infty, e_0), (e_0, e_1), \ldots, (e_n, \infty)\}$. (See (3) above.) For the proof of the lemma we will need to consider the G_E orbits of I. For each $i \in I$, let $ob(i) = \{\phi(i) : \phi \in G_E\}$ and let Ob$= \{ob(i) : i \in I\}$. For each $Q \in$ Ob choose $i_Q \in Q$ so that

(†) for every open interval W determined by E, every element of $\mathrm{supp}(f_0) \cap W$ is less than every element of $\mathrm{supp}(i_Q) \cap W$.

The function $Q \mapsto i_Q$ may not be in $\mathcal{N}$, however the set f_1 defined by

$$f_1 = \{(\phi(i_Q), \phi(f_0(i_Q))) : \phi \in G_E \text{ and } Q \in \mathrm{Ob}\}$$

is in the model since it has support E. We also make the following claims about f_1.

Claim A. *f_1 is a function.*

*T*o prove claim A, assume that $(\phi_1(i_Q), \phi_1(f_0(i_Q)))$ and $(\phi_2(i_P), \phi_2(f_0(i_P)))$ are in f_1 where ϕ_1 and ϕ_2 are in G_E and Q and P are in Ob and assume that $\phi_1(i_Q) = \phi_2(i_P)$. Then $i_Q = \phi_1^{-1}\phi_2(i_P)$. From this we may draw two conclusions: First that i_Q and i_P are in the same orbit and, hence, $i_Q = i_P$. Secondly (using $(**)$) that $f_0(i_Q) = \phi_1^{-1}\phi_2(f_0(i_P)) = \phi_1^{-1}\phi_2(f_0(i_Q))$ and therefore, $\phi_1(f_0(i_Q)) = \phi_2(f_0(i_P))$. Claim A follows.

Claim B. $f_1 \in \prod_{i \in I} X_i$

*T*his claim follows from the facts that the domain of f_1 is $\{\phi(i_Q) : \phi \in G_E \text{ and } Q \in \mathrm{Ob}\} = I$ and for each $\phi(i_Q)$ in the domain of f_1, $f_1(\phi(i_Q)) = \phi(f_0(i_Q)) \in X_{\phi(i_Q)}$.

Claim C. *f_1 is a cluster point of Z*

*W*e first note that proving claim C will complete the proof of the lemma. To prove the claim, let N be a basic neighborhood of f_1. Then $N = \prod_{i\in I} K_i$ where for some finite subset I_0 of I, $K_i = X_i$ for all $i \notin I_0$. We will show that N contains an element f_2 of Z different from f_1.

For each $j \in I_0$, let Q_j be the unique element of Ob such that $j \in Q_j$ and let $\phi_j \in G_E$ be chosen so that $\phi_j(i_{Q_j}) = j$. Choose an $\eta \in G_E$ so that

(‡) for each of the intervals W determined by E and every element a of $\text{supp}(f_0)$ in W, $\eta(a) < c$ for every element $c \in \text{supp}(j) \cap W$.

Let $f_2 = \eta(f_0)$. Then $f_2 \neq f_1$ since E is a support of f_1 but $\text{supp}(f_2) = \eta(\text{supp}(f_0)) \neq E$ since $\text{supp}(f_0) \neq E$ and η fixes E.

We also note that $f_2 = \eta(f_0) \in Z$ since $f_0 \in Z$, Z has support E and η fixes E pointwise. Therefore, to complete the proof of the claim, it suffices to show that $f_2 \in N$. We will do this by arguing that $\forall j \in I_0$, $f_2(j) = f_1(j)$.

Assume that $j \in I_0$ then, using the notation introduced above, $\phi_j(i_{Q_j}) = j$ where $\phi_j \in G_E$ where $j \in Q_j$. It follows that for each open interval W determined by E, $|\text{supp}(i_{Q_j}) \cap W| = |\text{supp}(j) \cap W|$. Similarly $|\text{supp}(f_0) \cap W| = |\text{supp}(\eta(f_0)) \cap W|$. By (†), $\text{supp}(f_0) \cap \text{supp}(i_{Q_j}) \cap W = \emptyset$ and by (‡), $\text{supp}(\eta(f_0)) \cap \text{supp}(j) \cap W = \emptyset$ hence,

$$|(\text{supp}(f_0) \cup \text{supp}(i_{Q_j})) \cap W| = |(\text{supp}(\eta(f_0)) \cup \text{supp}(j)) \cap W|.$$

Therefore, letting $F = \text{supp}(f_0) \cup \text{supp}(i_{Q_j})$ and $F' = \text{supp}(\eta(f_0)) \cup \text{supp}(j)$ we conclude by property 3 (in our list of properties of $\mathcal{N}$) that there is a $\beta \in G_E$ such that $\beta(F) = F'$. Since β is order preserving (and using (†) and (‡)) we conclude that $\beta(\text{supp}(f_0)) = \text{supp}(\eta(f_0))$ and $\eta(\text{supp}(i_{Q_j})) = \text{supp}(j)$. Therefore, $\beta(f_0) = \eta(f_0)$ and $\beta(i_{Q_j}) = j$. Hence,

$$f_2(j) = (\eta(f_0))(j) = (\beta(f_0))(j) = \beta(f_0(\beta^{-1}(j))) = \beta(f_0(i_{Q_j})) = f_1(j).$$

This completes the proof of claim C and therefore, the lemma is proved.

We therefore assume for the remainder of the proof that

$$(***) \qquad \forall f \in Z,\ \text{supp}(f) \subseteq E.$$

It follows from (∗∗) that each X_i is well orderable in $\mathcal{N}$ and therefore, the power set of X_i in $\mathcal{N}$ (which we denote by $\mathcal{P}(X_i)^{\mathcal{N}}$) is the same as the power set of X_i in the ground model. Similarly $(\mathcal{P}(\mathcal{P}(X_i)))^{\mathcal{N}}$ is the same as $\mathcal{P}(\mathcal{P}(X_i))$ in the ground model. We denote the ground model (that is, the model of ZF+AC from which $\mathcal{N}$ was constructed) by $\mathcal{N}_0$. It follows that (X_i, T_i) is an compact F topological space in $\mathcal{N}_0$. Therefore, since $\mathcal{N}_0$ is a model of AC, $\prod_{i\in I} X_i$ (taken in $\mathcal{N}_0$) is compact G in $\mathcal{N}_0$. Since $Z \subseteq \left(\prod_{i\in I} X_i\right)^{\mathcal{N}} \subseteq \prod_{i\in I} X_i$, Z has a cluster point in $\prod_{i\in I} X_i$. Let h be such a cluster point. If $h \in \mathcal{N}$, then it is not hard to verify that h is a cluster point of Z in $\mathcal{N}$ in which case the proof is complete. Assume therefore, that $h \notin \mathcal{N}$. Then there is a permutation $\sigma \in G_E$ and a $j \in I$ such that $\sigma(h(j)) \neq h(\sigma(j))$. (Otherwise h has support E.) One consequence of this assumption is that $\sigma(j) \neq j$. For if $\sigma(j) = j$ then by $(**)$, $\sigma(h(j)) = h(j)$ and, hence, $\sigma(h(j)) = h(j) = h(\sigma(j))$.

Now, as in the proof of the previous lemma, we choose one representative i_Q from each $Q \in \mathrm{Ob}$ but rather than require (†) we require only that $i_{ob(j)} = j$. Again as in the proof of the previous lemma (see the definition of f_1) we define

$$h_1 = \{(\phi(i_Q), \phi(h(i_Q))) : \phi \in G_E \text{ and } Q \in \mathrm{Ob}\}.$$

The proof that h_1 is in $\left(\prod_{i\in I} X_i\right)^{\mathcal{N}}$ is almost identical to the proof that $f_1 \in \left(\prod_{i\in I} X_i\right)^{\mathcal{N}}$. We also note

$$(\heartsuit) \qquad \forall Q \in \mathrm{Ob},\ h_1(i_Q) = h(i_Q).$$

We define $h_2 \in \prod_{i\in I} X_i$ by

$$h_2(i) = \begin{cases} h_1(i) & i \neq \sigma(j) \\ h(i) & i = \sigma(j) \end{cases}.$$

It is clear that $h_2 \in \mathcal{N}$ since h_1 is. We will show that h_2 is a cluster point of Z.

First note that
$\sigma(h_2(j)) = \sigma(h_1(j))$ since $j \neq \sigma(j)$ and
$\sigma(h_1(j)) = \sigma(h(j))$ since $j = i_{ob(j)}$ by $(\heartsuit)$ and
$\sigma(h(j)) \neq h(\sigma(j)) = h_2(\sigma(j))$ so
$\sigma(h_2(j)) \neq h_2(\sigma(j))$

and since $\sigma \in G_E$ it follows that $\mathrm{supp}(h_2) \not\subseteq E$. By $(***)$ $h_2 \notin Z$. Therefore, to show that h_2 is a cluster point of Z it suffices to show that every neighborhood of h_2 contains an element of Z.

Let N be a neighborhood of h_2, then $N = \prod_{i\in I} K_i$ where for some finite subset I_0 of I, $K_i = X_i$ for all $i \notin I_0$. For each $k \in I_0$, choose $\eta_k \in G_E$ so that $\eta_k(k) = i_{ob(k)}$. For each $Q \in \mathrm{Ob}$, let $N_Q = \bigcap\{\eta_k(K_k) : k \in Q \wedge k \neq \sigma(j)\}$. Since K_k is a neighborhood of $h_2(k) = h_1(k)$ for all $k \in Q$ such that $k \neq \sigma(j)$ and since η fixes h_1, $\eta_k(K_k)$ is a neighborhood of $h_1(i_Q)$. Further, since $K_k = X_k$ for all but finitely many k, $\eta_k(K_k) = X_{i_Q}$ for all but finitely many k in Q. It follows that N_Q is a neighborhood of $h_1(i_Q) = h_2(i_Q)$. Let $M = \prod_{i\in I} M_i$ where

$$M_i = \begin{cases} N_{ob(i)} & \text{if } i \in I_0, i = i_{ob(i)}, i \neq \sigma(j) \\ K_{\sigma(j)} & \text{if } i = \sigma(j) \\ X_i & \text{otherwise} \end{cases}.$$

Then M is a neighborhood of h and therefore there is an element f of Z such that $f \in M$. We complete the proof by showing that $f \in N$.

It follows from the choice of f that $f(j) \in K_j$ and we also have for $i \notin I_0$, $f(i) \in X_i = K_i$. Suppose $i \in I_0$ and $i \neq \sigma(j)$. Then $f(i_{ob(i)}) \in N_{ob(i)} \subseteq \eta_i(K_i)$. So $\eta_i^{-1}(f(i_{ob(i)})) \in K_i$. That is, $\eta_i^{-1}(f)(\eta_i^{-1}(i_{ob(i)})) \in K_i$. But by $(***)$, $\eta_i^{-1}(f) = f$ and by the definition of η_i, $\eta_i^{-1}(i_{ob(i)}) = i$. Therefore, $f(i) \in K_i$ and, hence, f is in N. $\square$

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