

PRODUCTS OF COMPACT SPACES AND THE AXIOM OF CHOICE

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ABSTRACT. We study the Tychonoff Compactness Theorem for several different definitions of a compact space.

1. INTRODUCTION AND DEFINITIONS

In this paper we study products of compact spaces. The Tychonoff Compactness Theorem, the product of compact spaces is compact, is one of the oldest such theorems. In 1930, Tychonoff [ty] gave a proof of this product theorem for the closed unit interval $[0, 1]$, but his proof was easily generalized to arbitrary compact spaces, (see [c]). In 1950, Kelley [k] proved that the Tychonoff Compactness Theorem implies the axiom of choice, AC. (The Tychonoff Compactness Theorem is P(A,A) below.)

In the following definition, we define the product topology by defining its basis. Then the topology itself is the set of all unions of basis elements.

Product Topology. Let $\{X_i : i \in k\}$ be a set of topological spaces and let $X = \prod_{i \in k} X_i$ be their product. A *basis for the product topology on X* is all sets of the form $\prod_{i \in k} U_i$ where U_i is open in X_i and for all but a finite number of coordinates, $U_i = X_i$.

We shall use the forms of compactness which were defined in [dhhrs]. (The reader is referred to [dhhrs] for the reasons we chose these forms of compactness and for background information about them.) We list these definitions below. Our proofs, unless stated otherwise, will be in ZF.

Definitions:

1. A space is *compact A* if every open covering has a finite subcovering. (Equivalently, every filter has an accumulation point.)
2. A space is *compact B* if every ultrafilter has an accumulation point.
3. A space is *compact C* if there is a subbase for the topology such that every open covering by elements of the subbase has a finite subcovering.

4. A space is *compact D* if every covering by elements of a nest (a set that is linearly ordered by inclusion) of open sets has a finite subcovering.
5. A space is *compact E* if every sequence has a convergent subsequence.
6. A space is *compact F* if every infinite subset has a complete accumulation point.
7. A space is *compact G* if every infinite subset has an accumulation point.
8. A space is *compact H* if every countable open covering has a finite subcovering.
9. AC(WO) is the axiom of choice for well ordered families of sets.
10. AC_{WO} is the axiom of choice for families of well orderable sets.

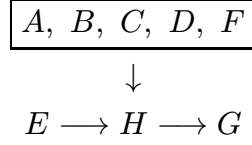
Let $P(X,Y)$ be the statement, Products of compact X spaces are compact Y . The following are results from [ho], [he], [hkrs], [dhhrs], or are known.

1. Each of the following is equivalent to AC in general topological spaces: $P(A,A)$, $P(A,C)$, $P(C,C)$, and $P(F,F)$. (See [he] and [ho].)
2. $P(B,B)$ does not imply AC. (In Blass' model ($\mathcal{M}15$ in [hr]) all ultrafilters are principal so all topological spaces are B compact. By the same argument, $P(X,B)$ doesn't imply AC if X is replaced by any of the definitions of compact above.)
3. For T_1 spaces: $P(B,B)$ iff every ultra-filter is principal or AC.
4. For T_2 spaces: $P(B,B)$ is provable in ZF and $P(A,A)$ iff BPI.
5. $P(X,E)$ is not provable in ZFC, for $X = A, B, C, D, E, F, G$, or H . (In fact, the example 2^{2^ω} with the product topology and the discrete topology on 2 shows that $\neg P(X,E)$ is provable in ZF for $X = A, B, C, D, F, G$, and H , [dhhrs].)
6. $P(E,X)$ is not provable in ZFC, for $X = A, B, C, D, E$, or F . (Take as a counterexample (ω_1, \leq) .) In addition, $P(E,G)$ is not provable in ZF^0 , (The set of atoms, A , in the model $\mathcal{N}1$ in [hr], is a counterexample [dhhrs].) $P(E,H)$ is not provable in ZF. (In the model $\mathcal{M}1$ in [hr] there is an infinite Dedekind finite subset of the reals which is compact E , because all sequences have a finite range, but not compact H . See [dhhrs].)
7. $P(B,X)$ is not provable in ZF for $X = A, C, D, E, F, G$, or H . (In the model $\mathcal{M}2$ in [hr], ω with the discrete topology is a counterexample. [tr].)
8. Neither $P(G,X)$ nor $P(H,X)$ are provable in ZFC for $X = A, B, C, D$, or F . (The counterexample is (ω_1, \leq) , see [dhhrs].)
9. Neither $P(D,D)$ nor $P(G,G)$ is provable in ZF^0 . (In the model $\mathcal{N}2$ in [hr], take 2^A with the product topology and the discrete topology on 2.)
10. Each of $P(X,F)$ implies AC, for $X = A, B, C, D, E, F, G$, and H . (See [he] and [ho].)
11. $ZFC \vdash P(X,Y)$, where $X, Y = A, B, C, D$, or F . (See [dhhrs].)

In ZF, the relations between the above forms of compactness are shown in the following arrow diagram.

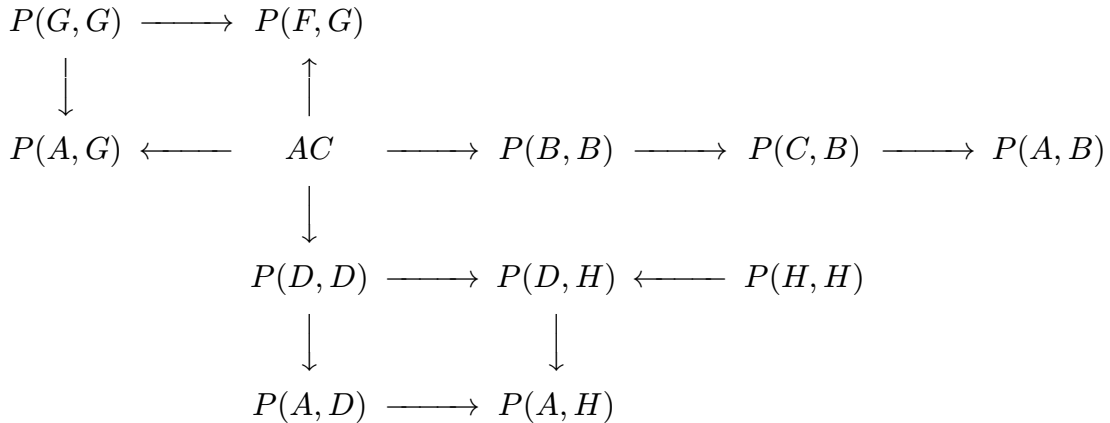
$$\begin{array}{ccccccc}
 H & \longleftarrow & D & \longleftarrow & A & \longrightarrow & C & \longrightarrow & B \\
 & & & & & & & & \\
 & & & & & \downarrow & & & \\
 & & & & & F & \longrightarrow & G &
 \end{array}$$

We note that in ZFC, compact A, B, C, D, and F are all equivalent, compact A implies compact G and H, and compact E implies compact G and H. The relationships are given in the following diagram.



2. RELATIONSHIPS BETWEEN THE FORMS P(X,Y)

It is clear that P(X,Y) implies that compact X spaces are compact Y. Therefore, it follows that if compact X implies compact Y, then P(Y,X) implies P(X,X) and P(Y,Y) and that each of P(X,X) and P(Y,Y) implies P(X,Y). Also, if compact X implies compact Y, then P(Y,Z) implies P(X,Z) and P(Z,X) implies P(Z,Y). From the introduction, we have that P(A,A) is equivalent to AC and that AC implies that compact A, B, C, D and F are equivalent. Also, it follows from [he] and [ho] that AC is equivalent to each of P(A,A), P(A,C), P(C,C), and P(F,F). Using these results and the arrow diagram above for the forms of compactness, we obtain the following relationships between the forms.



We shall show below (Theorem 1 and its Corollaries) that the negations of P(G,G) and P(H,H) are provable in ZF; P(F,G) implies AC_{WO}; and each of P(A,G) and P(A,H) implies AC.

Theorem 1. *Each of P(A,G) and P(A,H) implies AC.*

Proof. Let $X = \{X_i : i \in I\}$ be a collection of non-empty sets. We will show how each of the above product theorems implies that X has a choice function. We assume without loss of generality that for every $i \in I$, $X_i \cap \omega = \emptyset$. For each $i \in \omega$, let $Y_i = X_i \cup \omega$ and let

\mathcal{T}_i be the topology consisting of the complements in Y_i of finite subsets of ω together with all subsets of ω . The topology \mathcal{T}_i is \mathbb{A} compact since any neighborhood of a point in X_i is cofinite. By using one of the product theorems mentioned in the hypothesis we conclude that $\prod_{i \in I} Y_i$ (with the product topology) is \mathbb{G} or \mathbb{H} compact.

Assume first that $\prod_{i \in I} Y_i$ is \mathbb{G} compact. For each $n \in \omega$, let $f_n \in \prod_{i \in I} Y_i$ be the function defined by $f_n(i) = n$ for all $i \in I$. We argue by contradiction that any accumulation point g of the set $F = \{f_n : n \in \omega\}$ will be a choice function for X . If for some $j \in I$, $g(j) \notin X_j$ then $g(j) \in \omega$ in which case $\{f \in \prod_{i \in I} Y_i : f(j) = g(j)\}$ is a neighborhood of g containing at most one element of F , namely $f_{g(j)}$. Since each neighborhood of g must contain some point of F different from g we conclude that $g \neq f_{g(j)}$. Therefore, for some $k \in I$, $g(k) \neq g(j)$. But then $\{f \in \prod_{i \in I} Y_i : f(j) = g(j) \text{ and } f(k) \neq g(j)\}$ is a neighborhood of g containing no points of F contradicting our assumption that g is an accumulation point of F .

Finally, if $\prod_{i \in I} Y_i$ is \mathbb{H} compact, the absence of a choice function for X would imply that $\{\{f \in \prod_{i \in I} Y_i : (\exists i \in I) f(i) = n\} : n \in \omega\}$ is an open cover for $\prod_{i \in I} Y_i$ with no finite subcover. \square

Corollary 1. *Each of $P(F, G)$ and $P(F, H)$ implies the axiom of choice for families of sets for which each set in the family has no infinite, Dedekind finite subset. (In particular, they imply AC_{WO} .)*

Proof. Let X be a collection of non-empty sets as in the proof of the theorem and assume in addition that no element of X has an infinite, Dedekind finite subset. We will show (as in the theorem) that both $P(F, G)$ and $P(F, H)$ implies that X has a choice function by considering $\prod_{i \in I} Y_i$ where Y_i and \mathcal{T}_i are defined as before. We may assume without loss of generality that for each $i \in I$, X_i is infinite, replacing X_i by $X_i \times \omega$ if necessary. (If X_i has no infinite, Dedekind finite subset then neither does $X_i \times \omega$.) In view of the proof of the theorem we only have to argue that for each $i \in I$, the topology \mathcal{T}_i is F compact. Let $Z \subset Y_i$ be infinite. We claim that any $x \in X_i$ is a complete accumulation point for Z . This follows because any neighborhood N of x is the complement (in $Y_i = X_i \cup \omega$) of a finite subset of ω and therefore Z and $Z \cap N$ differ by at most a finite set. Since Z is Dedekind infinite it follows that $|Z| = |Z \cap N|$. \square

Corollary 2. *The negation of each of $P(G, G)$ and $P(H, H)$ is provable in ZF.*

Proof. In ZF, $P(G, G)$ implies $P(A, G)$ which implies AC; and $P(H, H)$ implies $P(A, H)$ which implies AC. In [no] it is shown that AC implies $\neg P(G, G)$ and $\neg P(H, H)$. Therefore, $\neg P(G, G)$ and $\neg P(H, H)$ are provable in ZF. \square

Corollary 3. *Each of the following implies AC: $P(A, D)$, $P(D, D)$, and $P(X, H)$ where $X = B, C, D$, or G .*

Proof. $P(D, D) \rightarrow P(A, D) \rightarrow P(A, H)$; $P(X, H) \rightarrow P(A, H)$, for $X = B, C, D$, or G . \square

Corollary 4. *Each of the following implies AC: $P(X, G)$, where $X = B, C, D$, or H .*

Proof. $P(X, G) \rightarrow P(A, G)$, for $X = B, C, D$, or H . \square

Corollary 5. *Each of $P(F,A)$ and $P(F,D)$ implies AC_{WO} .*

Proof. $P(F,X) \rightarrow P(F,H)$, for $X = A$ or D . \square

Using the preceding results, results from [he], and [ho], the fact that if $X \rightarrow Y$, then $P(Z,X) \rightarrow P(Z,Y)$ and $P(Y,Z) \rightarrow P(X,Z)$, and that AC implies that compact A, B, C, D , and F are equivalent, we get the following results:

(1) Each of the following statements is equivalent to AC :

$P(A,A), P(A,C), P(A,D), P(A,F), P(A,G), P(A,H), P(B,A), P(B,C), P(B,D), P(B,F), P(B,G), P(B,H), P(C,A), P(C,C), P(C,D), P(C,F), P(C,G), P(C,H), P(D,A), P(D,C), P(D,D), P(D,F), P(D,G), P(D,H)$, and $P(F,F)$.

$$(2) \quad AC \rightarrow P(F,A) \rightarrow P(F,C) \rightarrow P(F,B)$$

$$(3) \quad AC \rightarrow P(F,A) \rightarrow P(F,G) \rightarrow AC_{WO}$$

$$(4) \quad AC \rightarrow P(F,A) \rightarrow P(F,D) \rightarrow P(F,H) \rightarrow AC_{WO}$$

$$(5) \quad AC \rightarrow P(B,B) \rightarrow P(C,B) \rightarrow P(A,B) \leftarrow P(D,B) \leftarrow AC$$

It is shown in [he] that $BPI + P(A,B)$ implies $P(A,A)$. The statement AC_{WO} is false in the model $\mathcal{M}4$ in [hr]. It is shown in [dhhrs] that $P(F,B)$ is not provable in ZF ; $P(X,Y)$ is not provable in ZFC for $X = G$ or H , and $Y = A, B, C, D, E$, or F ; and that $P(G,H)$ is not provable in ZFC . We show in Corollary 2 to Theorem 1 above that $\neg P(G,G)$ and $\neg P(H,H)$ are provable in ZF . Thus, it follows that none of the product theorems listed above are provable in ZF .

We end this section by giving a relationship between $AC(WO)$ and a product theorem.

Theorem 2. *$AC(WO)$ iff in Tychonoff products of compact A spaces, well ordered basic open covers have finite subcovers.*

Proof. (\leftarrow). Let $A = \{A_i : i \in \mathbb{N}\}$ be a well-ordered family of disjoint non-empty sets. Let X be the Tychonoff product of the spaces $Y_i = A_i \cup b_i$, $b_i \notin \bigcup_{i \in \mathbb{N}} A_i$, taken with the topology in which Y_i, A_i and all finite subsets of A_i are the closed sets. If A has no choice set, then for every $x \in X$ there exists $i \in \mathbb{N}$ with $x(i) = b_i$. Thus, $U = \{\pi_i^{-1}(b_i) : i \in \mathbb{N}\}$ is a well ordered open cover of X . Hence U must have a finite subcover, which is impossible. Thus, A has a choice set, as required.

(\rightarrow). Let $\{(X_i, T_i) : i \in k\}$ be a family of compact A spaces and let $X = \prod_{i \in k} X_i$ be their Tychonoff product. Let $U = \{U_i : i \in \aleph\}$ be an open basic cover of X . For each $i \in \aleph$, let $\text{supp}(U_i) = \{j \in \aleph : \pi_j(U_i) \neq X_j\}$. Since $\text{supp}(U_i)$ is finite for each $i \in \aleph$, it follows by AC(WO) that $s = \bigcup\{\text{supp}(U_i) : i \in \aleph\}$ is well ordered. For every $j \in k$, let R_j be the topology on X_j generated by the well ordered subbase $C_j = \{\pi_j(U_i) : i \in \aleph\}$. Clearly the base B_j consisting of all finite intersections of members of C_j together with X_j , is well ordered and consequently, in view of AC(WO), $B = \bigcup\{B_j : j \in s\}$ is well ordered. Furthermore, the restrictions of the elements of U to the Tychonoff product $Y = \prod_{i \in k} (X_i, R_i)$ form an (well ordered) open cover, denoted by U' , of Y . It is straightforward to see that U has a finite subcover in X iff U' has a finite subcover in Y . We show that U' has a finite subcover in Y . Assume to the contrary that U' has no finite subcover. Without loss of generality we may assume that for every $i \in \aleph$, $U'_i \setminus \bigcup\{U'_j : j \in i\} \neq \emptyset$ where for $i \in \aleph$, U'_i denotes the restriction of U_i to Y . Let $S = \{x_n : n \in \aleph\}$ be a choice function for the family $\{U'_i \setminus \bigcup\{U'_j : j \in i\} : i \in \aleph\}$.

Let \mathcal{F} be a maximal filter in $\{\pi_j^{-1}(O) \cap S : (\exists j \in k)(O \in B_j \text{ and } |\pi_j^{-1}(O) \cap S| = \aleph)\}$. (This can be done via an easy transfinite induction on B , because B is well-ordered.) For every $j \in k$, let $Q_j = \{x \in X_j : \forall O_x \in B_j, x \in O_x \rightarrow \pi_j^{-1}(O_x) \cap S \in \mathcal{F}\}$

Claim: $Q_j \neq \emptyset$ for all $j \in k$.

Proof of the claim: If not, then every $x \in X_j$ has a neighborhood O_x such that $|\pi_j^{-1}(O_x) \cap S| < \aleph$ for some member $F_x \in \mathcal{F}$. It follows, by the compactness of X_j , that some finite number of these neighborhoods, say $O_{x_1}, O_{x_2}, \dots, O_{x_v}$ covers X_j . Thus,

$$S = (\pi_j^{-1}(O_{x_1}) \cap S) \cup (\pi_j^{-1}(O_{x_2}) \cap S) \cup \dots \cup (\pi_j^{-1}(O_{x_v}) \cap S) \in \mathcal{F}.$$

On the other hand, $|\pi_j^{-1}(O_{x_j}) \cap F_{x_j}| < \aleph$ for all $j = 1, 2, \dots, v$ implies $[(\pi_j^{-1}(O_{x_1}) \cap S) \cup \pi_j^{-1}(O_{x_2}) \cap S] \cup \dots \cup (\pi_j^{-1}(O_{x_v}) \cap S) \cap (\bigcap\{F_{x_j} : j \leq v\}) \subseteq \bigcup\{(\pi_j^{-1}(O_{x_j}) \cap F_{x_j}) : j \leq v\}$. Since $|\bigcup\{(\pi_j^{-1}(O_{x_j}) \cap F_{x_j}) : j \leq v\}| < \aleph$, it follows that $|\bigcap\{F_{x_j} : j \leq v\}| < \aleph$, contradicting the fact that \mathcal{F} is a filter and establishing the proof of the claim.

Let $Q = \{Q_j : j \in k\}$ and let q be a choice function of Q . Clearly every neighborhood of q meets S in a set of size \aleph . Since U' is a cover of Y , it follows that $q \in U_j$ for some $j \in \aleph$. Thus, U'_j includes \aleph many members of S . This is a contradiction because, by the construction of S , U'_j misses the tail $\{x_n : n > j\}$. This contradiction establishes the proof of the theorem. \square

3. PROJECTION THEOREMS

We shall let $\text{Pr}(X)$ be the statement: Projections in Tychonoff products of compact X spaces are closed, where “X” is any one of the definitions of compact given in section 1.

It is shown in [no] that in ZFC the product of two compact G spaces may not be compact G, as follows. Using a well-ordering of $\beta\mathbb{N}$ (the Stone-Ćech compactification of a countable, discrete space \mathbb{N}), two subspaces A_1 and A_2 of $\beta\mathbb{N}$ are constructed such that both are compact G and $A_1 \cap A_2 = \mathbb{N}$. In the product $A_1 \times A_2$, the set $\mathbb{N}' = \{\langle n, n \rangle : n \in \mathbb{N}\}$

is an infinite set with no limit point, and thus $A_1 \times A_2$ is not compact G. This example also shows that $\text{Pr}(G)$ is false in ZFC, since \mathbb{N}' is closed in $A_1 \times A_2$, while its projection \mathbb{N} in A_1 or A_2 is not closed. Furthermore, all the spaces involved are T_1 , so this example shows that $\text{P}(H,H)$ and $\text{Pr}(H)$ are also false in ZFC. (For T_1 spaces, compact G and H are equivalent in ZFC. See [dhhrs].)

Theorem 3. *Pr(A) implies P(A,A).*

Proof. We first note that it suffices to prove the following lemma:

Lemma. *Assume X is a topological space and that for every compact A space Y, the projection $\pi_Y : Y \times X \rightarrow Y$ is a closed map, then X is compact A.*

Using the lemma the theorem can be proved as follows. Assume $\text{Pr}(A)$ and assume that $\{X_i : i \in I\}$ is a family of topological spaces each of which is compact A. Let Y be any compact A topological space. Then by $\text{Pr}(A)$ the projection $\pi_Y : Y \times (\prod_{i \in I} X_i) \rightarrow Y$ is a closed map. Therefore, by the lemma, $\prod_{i \in I} X_i$ is compact A.

Now we turn to the proof of the lemma. Our proof is a variation of Herrlich's proof in [he] of the assertion that a topological space X is compact A if and only if for every topological space Y , the projection $\pi_Y : Y \times X \rightarrow Y$ is closed. Assume the hypotheses and suppose that X is not compact A. Then there is an filter \mathcal{F} on X with no accumulation point. To arrive at a contradiction we will construct a compact A topological space Y and a closed set \mathcal{C} in $Y \times X$ such that the projection of \mathcal{C} onto Y is not closed.

Let $Y = \mathcal{F} \cup \{\infty\}$ (where ∞ is some object not in \mathcal{F}) with the topology whose basis sets are the sets $N_U = \{D \in \mathcal{F} : D \cap U = \emptyset\} \cup \{\infty\}$ for each open set U in the topology on X such that the complement of U is in \mathcal{F} . Note that $N_{U_1} \cap N_{U_2} = N_{U_1 \cup U_2}$ which is also a basis set. We also note that the only neighborhood of X in this topology is $N_\emptyset = \mathcal{F} \cup \{\infty\} = Y$ since $X \in N_U$ implies $X \cap U = \emptyset$ which means $U = \emptyset$. It follows that the topology just defined is (trivially) compact.

Let $\mathcal{C} = \bigcup_{F \in \mathcal{F}} \{F\} \times \bar{F}$, where \bar{F} denotes the closure of F in the topology on X . The projection $\pi_Y(\mathcal{C}) = \mathcal{F}$ of \mathcal{C} onto Y is not closed since ∞ is in the closure of \mathcal{F} but $\infty \notin \mathcal{F}$. We complete the proof by showing that \mathcal{C} is closed in $Y \times X$. Let $(z, x) \in (Y \times X) \setminus \mathcal{C}$. We need to find a neighborhood of (z, x) which is disjoint from \mathcal{C} .

We first consider the case where $z = F \in \mathcal{F}$. Since $(z, x) = (F, x) \notin \mathcal{C}$, $x \notin \bar{F}$. Let U be the open set $X \setminus \bar{F}$. Then N_U is open in the topology on Y , $x \in U$ and $U \cap \bar{F} = \emptyset$. It follows that $F \in N_U$ and, hence, (F, x) is in the open set $N_U \times U$. It is also the case that $N_U \times U$ is disjoint from \mathcal{C} for if $(F', x') \in \mathcal{C} \cap (N_U \times U)$, it follows from the definition of \mathcal{C} that $x' \in \bar{F'}$. Since x' would also have to be in U , we would have $x' \in \bar{F'} \cap U$. But $F' \cap U = \emptyset$ since $F' \in N_U$. Since U is open, we also have $\bar{F'} \cap U = \emptyset$. This is a contradiction.

The second (and only remaining case) is that (z, x) (in the complement of \mathcal{C}) is (∞, x) . Since the filter \mathcal{F} has no accumulation points in X , x is not such a accumulation point. Therefore, for some $F \in \mathcal{F}$, $x \notin \bar{F}$. This means that x is in the open set $U = X \setminus \bar{F}$. $N_U \times U$ is open in the product and $(\infty, x) \in N_U \times U$. We complete the proof of the lemma by showing that $\mathcal{C} \cap (N_U \times U)$ is empty. Assume $(F', x') \in \mathcal{C} \cap (N_U \times U)$. Then, as before, it follows both that $x \in F' \cap U$ and that $F' \cap U = \emptyset$. \square

Corollary 1. $Pr(A) \rightarrow AC$.

We note that if compact X implies compact Y then $Pr(Y)$ implies $Pr(X)$. Therefore, using the arrow diagram for the definitions of compactness in section 1, we get the following arrow diagram for $Pr(X)$.

$$\begin{array}{ccccccc} & & AC & & & & \\ & & \uparrow & & & & \\ Pr(D) & \longrightarrow & Pr(A) & \longleftarrow & Pr(C) & \longleftarrow & Pr(B) \end{array}$$

Thus, we obtain:

Corollary 2. $Pr(X) \rightarrow AC$, for $X = A, B, C$, and D .

Corollary 3. $Pr(F) \rightarrow AC_{WO}$.

Proof. Using the proof of Theorem 3 with “for every compact A space Y ” replaced by “for every compact F space Y ” we obtain a proof that $Pr(F) \rightarrow P(F,A)$. Then, use Corollary 5 to Theorem 1 to obtain $P(F,A) \rightarrow AC_{WO}$. \square

As mentioned above, neither $Pr(G)$ nor $Pr(H)$ is provable in ZFC. However, it is known that $Pr(A)$ is provable in ZFC. For completeness we shall give a proof below.

Lemma 1. $Pr(X)$ is provable in ZFC, for $X = A, B, C, D$, and F .

Proof. Let $Z = \prod_{i \in I} X_i$ be a product of compact A spaces, and let C be a closed subset of Z . Fix $j \in I$, and let x_j be a point in X_j but not in the projection $\pi_j(C)$; to show that π_j is a closed map we will show that x_j is not an accumulation point of $\pi_j(C)$. Let $Y = \prod_{j \neq i \in I} X_i$. Since C is closed and $(\{x_j\} \times Y) \cap C$ is empty and Y is compact (by $P(A, A)$), it follows from the “tube lemma” (see [mu]) that there is an open neighborhood W of x_j in X_j such that $(W \times Y) \cap C$ is empty. Then $W \cap \pi_j(C)$ is also empty, so $x_j \in W$ is not an accumulation point of $\pi_j(C)$.

In ZFC, compact A, B, C, D and F are all equivalent so this completes the proof of the Lemma. \square

Thus, it follows that each of the statements $Pr(X)$, for $X = A, B, C$, and D is equivalent to AC and $AC \rightarrow Pr(F) \rightarrow AC_{WO}$.

Lemma 2. $Pr(E)$ is not provable in ZFC. (In fact, $\neg Pr(E)$ is provable in ZFC.)

Proof. Consider the spaces ω_1 and $\omega_1 + 1$ with the order topology. We know that ω_1 is compact E , since every countable sequence in ω_1 is bounded and therefore its limit points are in the space. Clearly, $\omega_1 + 1$ is also compact E , since every countable sequence in $\omega_1 + 1$ either has a subsequence contained in ω_1 or is eventually constant with value $\omega_1 \in \omega_1 + 1$.

Consider the set $C = \{\langle \alpha, \alpha \rangle : \alpha \in \omega_1\}$. We claim that C is closed in $\omega_1 \times \omega_1 + 1$: if $\alpha < \beta \leq \omega_1$, then $\langle \alpha, \beta \rangle \in [0, \alpha + 1) \times (\alpha, \omega_1]$ and if $\beta < \alpha < \omega_1$, then $\langle \alpha, \beta \rangle \in (\beta, \omega_1) \times [0, \beta + 1)$, while both open rectangles are disjoint from C .

However, the projection of C into the space $\omega_1 + 1$ is ω_1 , which is not closed in $\omega_1 + 1$. \square

Consequently, none of the statements $\text{Pr}(Y)$ for $Y = E, G,$ or $H,$ are provable in ZFC.

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