

# VON RIMSCHA'S TRANSITIVITY CONDITIONS

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ABSTRACT. In Zermelo-Fraenkel set theory with the axiom of choice every set has the same cardinal number as some ordinal. Von Rimscha has weakened this condition to “Every set has the same cardinal number as some transitive set.” In set theory without the axiom of choice, we study the deductive strength of this and similar statements introduced by von Rimscha.

We shall use the standard notation and terminology of set theory. In particular we recall that a set  $x$  is *transitive* if for every  $y \in x$  and every  $t \in y$ ,  $t \in x$ . The transitive closure,  $\text{TC}(x)$ , of a set  $x$  is the smallest transitive set  $z$  such that  $x \subseteq z$ . In addition, for any set  $x$ , we will use  $\text{TC}'(x)$  to stand for  $\text{TC}(x) \cup \{x\}$ .

Transitive sets play an important role in set theory. The ordinal numbers, for example, are transitive sets and under the assumption of the axiom of choice (which we shall denote by AC) given any set  $x$ , there is a bijection from  $x$  to some ordinal. In [R] von Rimscha introduces a similar statement,  $\text{Tr}$ , weakened by replacing the word “ordinal” with the words “transitive set”.

**Tr.**  $\forall x \exists u \exists f$  such that  $u$  is transitive and  $f$  is a bijection from  $x$  onto  $u$ .

Two strengthenings of  $\text{Tr}$  are also considered in [R]:

**Tr'**.  $\forall x \exists u \exists f$  such that  $u$  is transitive,  $u \subseteq \text{TC}(x)$ , and  $f$  is a bijection from  $x$  onto  $u$ .

**Tr''**.  $\forall x \exists u \exists f$  such that  $u$  is transitive,  $f$  is a bijection from  $x$  onto  $u$ , and  $\forall s \in x$ ,  $f(s) \in \text{TC}'$ .

Working in Zermelo-Fraenkel set theory (ZF), without AC, von Rimscha shows that AC implies  $\text{Tr}''$ ,  $\text{Tr}''$  implies  $\text{Tr}'$ ,  $\text{Tr}'$  implies  $\text{Tr}$ , and  $\text{Tr}$  implies that every Dedekind finite set is finite. We prefer to work in  $\text{ZF}^0$ , where ZF is modified to allow the existence of atoms. The same proofs hold with minor modifications. Von

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Rimscha also asks whether any of the implications are reversible. In this paper we answer several of these questions.

We shall prove two theorems:

**Theorem 1.** *(ZF<sup>0</sup>) Tr' implies AC.*

**Theorem 2.** *Tr is true in the Mostowski linear order model of ZF<sup>0</sup>.*

Note that, since ZF<sup>0</sup> is a weaker theory than ZF, this gives us the fact that Tr', Tr'' and AC are equivalent in ZF. Also, since there are infinite, Dedekind finite sets in the Mostowski linear order model, we see that in the theory ZF<sup>0</sup>, Tr does not imply that all Dedekind finite sets are finite. We have not answered the two questions “Does Tr imply AC in ZF?” and “Does ‘Every Dedekind finite set is finite’ imply Tr in ZF?” Our conjecture is that in both cases the answer is “no”.

*Proof of Theorem 1.* Let  $x$  be a set. Assuming Tr', we will show that  $x$  can be well ordered. Assume  $x \neq \emptyset$  and  $x$  is not an atom. Let  $A = \{a : a \text{ is an atom and } a \in \text{TC}(x)\}$ . Let  $\kappa$  be the least ordinal such that there is no function from  $A$  onto  $\kappa$ . Let  $\gamma$  be the least ordinal such that  $|\gamma| \not\leq |x \times \kappa|$ . We now define  $\{t_\gamma : t \in \text{TC}(\{x\})\}$  by  $\in$ -induction as follows. For each  $t \in \text{TC}(\{x\})$  define

$$t_\gamma = \begin{cases} \gamma & t = \emptyset \\ \{t, \gamma\} & t \in A \\ \{s_\gamma : s \in t\} & t \neq \emptyset \wedge t \notin A. \end{cases}$$

Using  $\in$ -induction it is easy to show that for every  $t, r \in \text{TC}(\{x\})$  that if  $t_\gamma = r_\gamma$  then  $t = r$ . Consequently, for every  $t \in \text{TC}(\{x\})$  if  $t \neq \emptyset$  and  $t \notin A$  then  $|t_\gamma| = |t|$ . Specifically,  $|x_\gamma| = |x|$ . So it is sufficient to well order  $x_\gamma$ .

By Tr', there is a transitive set  $u \subseteq \text{TC}(x_\gamma \times \kappa)$  and a one to one function  $f$  from  $x_\gamma \times \kappa$  onto  $u$ . Since  $|x_\gamma| = |x|$ ,  $|x_\gamma \times \kappa| = |x \times \kappa|$ . But  $|x \times \kappa| \not\leq |\gamma|$ , so  $|x_\gamma \times \kappa| \not\leq |\gamma|$ . Since  $u$  is transitive,  $\text{TC}(u) = u$  and so  $|\text{TC}(u)| = |x_\gamma \times \kappa| \not\leq |\gamma|$ . Let  $z \in u$ . Then  $|\text{TC}(z)| \not\leq |\gamma|$ . Since  $u \subseteq \text{TC}(x_\gamma \times \kappa)$ ,

$$z \in \text{TC}(\{\{t_\gamma\}, \{t_\gamma, \alpha\} : t \in x \wedge \alpha \in \kappa\}). (\{\{t_\gamma\}, \{t_\gamma, \alpha\}\} = \langle t_\gamma, \alpha \rangle.)$$

However,

$$\begin{aligned} \text{TC}(x_\gamma \times \kappa) &= \{\{\{t_\gamma\}, \{t_\gamma, \alpha\}\} : t \in x \wedge \alpha \in \kappa\} \cup \\ &\quad \bigcup_{t \in x \wedge \alpha \in \kappa} \{\{t_\gamma\}, \{t_\gamma, \alpha\}\} \cup \text{TC}(x_\gamma) \cup \kappa. \end{aligned}$$

Notice that for every  $t \in \text{TC}(\{x\})$ ,  $\gamma \subseteq \text{TC}(t_\gamma)$ . So for every  $t \in \text{TC}(\{x\})$ ,  $z \neq t_\gamma$  and  $\text{TC}(t_\gamma) \not\subseteq \text{TC}(z)$ . Thus,  $z \in \text{TC}(x_\gamma) \cup \kappa$ . Let  $\delta = \max\{\gamma + 1, \kappa\}$ . We will show that  $z \in \delta \cup A$ . Suppose  $z \in \text{TC}(x_\gamma)$ . By our construction of  $x_\gamma$ ,  $z \in s_\gamma$  for some  $s \in \text{TC}(\{x\})$ . Since  $z \notin \{t_\gamma : t \in \text{TC}(\{x\})\}$ ,  $s \in A$  or  $s = \emptyset$ . If  $s = \emptyset$  then  $z \in \emptyset_\gamma = \gamma \subseteq \delta$ . If  $s \in A$  then  $z \in s_\gamma = \{s, \gamma\} \subseteq \delta \cup A$ . Thus  $z \in \delta \cup A$  and so  $u \subseteq \delta \cup A$ .

Now we define a function  $g$  from  $A$  into  $\kappa$ . For each  $a \in A$  let

$$g(a) = \begin{cases} 0 & a \notin f[x_\gamma \times \kappa] \\ \beta & a \in f[x_\gamma \times \{\beta\}]. \end{cases}$$

Since  $f$  is a one to one function,  $g$  is well defined. By our initial choice of  $\kappa$ ,  $g$  is not an onto function. Let  $\alpha \in \kappa$  be the least ordinal such that  $\alpha \notin g[A]$ . Then,  $A \cap f[x_\gamma \times \{\alpha\}] = \emptyset$ . Thus,  $f[x_\gamma \times \{\alpha\}] \subseteq \delta$ . So  $x_\gamma$ , and consequently,  $x$ , can be well ordered.  $\square$

*Proof of Theorem 2.* We begin with a brief description of the Mostowski linear order model  $\mathcal{M}$ . Details may be found in [J] or in [H/R]. The construction of  $\mathcal{M}$  begins with a model  $\mathcal{M}'$  of  $\text{ZF}^0 + \text{AC}$  in which the set of atoms  $A$  is countable. We choose an ordering  $\leq$  of  $A$  so that  $(A, \leq)$  is order isomorphic to the rational numbers (with their usual ordering). Let  $G$  be the group of all order automorphisms of  $(A, \leq)$ . For each  $\phi \in G$  there is a unique automorphism of the model  $\mathcal{M}'$  which extends  $\phi$ . This extension will also be called  $\phi$ . For each  $x \in \mathcal{M}'$ , let  $\text{fix}(x) = \{\phi \in G : (\forall t \in x)(\phi(t) = t)\}$  and let  $\text{sym}(x) = \{\phi \in G : \phi(x) = x\}$ . Then  $\mathcal{M} = \{x \in \mathcal{M}' : (\forall t \in \text{TC}'(x))(\exists E \subseteq A)(E \text{ is finite and } \text{fix}(E) \subseteq \text{sym}(t))\}$ . If  $E$  and  $F$  are finite subsets of  $A$  we will also use  $\text{fix}_E(F)$  for the group  $\text{fix}(E \cup F)$ . If  $x \in \mathcal{M}$  and  $E$  and  $F$  are finite subsets of  $A$  such that  $\text{fix}_E(F) \subseteq \text{sym}(x)$  then we say that  $F$  is a support of  $x$  relative to  $E$ . If  $F$  is a support of  $x$  relative to  $\emptyset$  then we will simply say that  $F$  is a support of  $x$ . In our argument below we shall use the following fact about  $\mathcal{M}$  (see [J]):

**Support Lemma.** *If  $x \in \mathcal{M}$  and  $E$  is any finite subset of  $A$  then  $x$  has a (unique) minimum support  $\text{supp}_E(x)$  relative to  $E$ . This minimum support has the following properties:*

- (i)  $\text{supp}_E(x) \cap E = \emptyset$
- (ii) For all  $\phi \in \text{fix}(E)$ ,  $\phi \in \text{fix}(\text{supp}_E(x))$  if and only if  $\phi(x) = x$
- (iii) If  $E \subseteq E'$ , then  $\text{supp}_{E'}(x) = \text{supp}_E(x) \setminus E'$
- (iv) If  $\phi \in \text{fix}(E)$ , then  $\text{supp}_E(\phi(x)) = \phi(\text{supp}_E(x))$ .

Assume  $x \in \mathcal{M}$ . We shall construct a one to one function  $G$  which is in  $\mathcal{M}$ , whose domain is  $x$  and whose range is a transitive set. This will suffice to prove the theorem. Let  $E = \{a_0 a_1, \dots, a_n\}$  be a support of  $x$  and assume  $a_0 < a_1 < \dots < a_n$ . For ease of exposition we adjoin a largest and smallest element ( $\infty$  and  $-\infty$  respectively) to the set of atoms  $A$ . In addition if  $a$  and  $b$  are in  $A$  with  $a < b$ , we will use the notation  $(a, b)$  for the open interval from  $a$  to  $b$ . Let  $I_0 = (-\infty, a_0), I_1 = (a_0, a_1), \dots, I_{n+1} = (a_n, \infty)$ . Before defining the function  $G$  it will be necessary to prove

**Lemma.** *We may choose  $E' \supseteq E$  so that for each  $j, j = 0, \dots, n+1$ , if there is a  $t \in x$  such that  $\text{supp}_{E'}(t) \cap I_j \neq \emptyset$ , then there is a  $t' \in x$  such that  $\text{supp}_{E'}(t') \subseteq I_j$  and  $|\text{supp}_{E'}(t')| = 1$ .*

*Proof.* From  $E$  we shall construct a support  $E'$  of  $x$  which satisfies the conditions of the lemma. For each  $j, j = 0, \dots, n+1$ , choose an element  $t_j \in x$  such that  $\text{supp}_E(t_j) \cap I_j \neq \emptyset$  if such an element exists. Otherwise let  $t_j = \emptyset$ . For each  $i, 0 \leq i \leq n+1$ , choose a (finite) subset  $F_i \subseteq I_i$  so that

$$|F_i| = \max_{0 \leq j \leq n+1} |\text{supp}_E(t_j) \cap I_i|.$$

Let  $E' = E \cup \left( \bigcup_{0 \leq i \leq n+1} F_i \right)$ . (Note that if for each  $j, j = 0, \dots, n+1$ ,  $\text{supp}_E(t_j) \cap I_j = \emptyset$  for all  $t_j \in x$ , then the support of each element of  $x$  is a subset of  $E$ . Thus,  $x$  can be well ordered so the theorem follows.)

We must argue that for each interval  $I$  determined by  $E'$ , if there is a  $t$  in  $x$  such that  $\text{supp}_{E'}(t) \cap I \neq \emptyset$ , then there is a  $t' \in x$  such that  $\text{supp}_{E'}(t') \subseteq I_j$  and  $|\text{supp}_{E'}(t')| = 1$ . To simplify the argument we will choose one specific interval  $I$ . The argument for the other intervals determined by  $E'$  is similar. Assume that  $F_1 = \{b_0, \dots, b_k\}$  where  $a_0 < b_0 < b_1 < \dots < b_k < a_1$ . We will give the argument for the interval  $I = (a_0, b_0)$ . If there is a  $t_1 \in x$  such that  $\text{supp}_{E'}(t_1) \cap I \neq \emptyset$  then  $\text{supp}_E(t_1) \cap I_1 \neq \emptyset$ . (Where, as above,  $I_1 = (a_0, a_1)$ .) Let  $\text{supp}_E(t_1) \cap I_1 = \{c_0, c_1, \dots, c_m\}$  where  $c_0 < c_1 < \dots < c_m$ . By the choice of  $F_1$ ,  $|\text{supp}_E(t_1) \cap I_1| \leq |F_1|$  and similarly  $|\text{supp}_E(t_1) \cap I_i| \leq |F_i|$  for  $0 \leq i \leq n+1$ . We may therefore choose a permutation  $\phi \in \text{fix}(E)$  so that

$$\begin{aligned} \phi(c_0) &\in (a_0, b_0) \setminus F_1, \\ \phi(\{c_1, \dots, c_m\}) &\subseteq F_1, \text{ and} \\ \forall i, i \neq 1, \phi(\text{supp}_E(t_1) \cap I_i) &\subseteq F_i. \end{aligned}$$

Then by the support lemma  $\text{supp}_{E'}(\phi(t_1)) = \text{supp}_E(\phi(t_1)) \setminus E' = \phi(\text{supp}_E(t_1)) \setminus E' = \{c_0\}$ . Therefore  $t' = \phi(t_1)$  satisfies the required condition.  $\square$

The definition of the function  $G$  will involve several components.

- For each finite subset  $F$  of  $A$  we define a function  $W_F$  from ordinals into  $\mathcal{M}$  by recursion on ordinals:  $W_F(0) = F$ ,  $W_F(\alpha + 1) = W_F(\alpha) \cup \{W_F(\alpha)\}$ , and for limit ordinals  $\lambda$ ,  $W_F(\lambda) = \bigcup_{\alpha < \lambda} W_F(\alpha)$ . It is easy to verify that
  - a) for any ordinal  $\alpha$ ,  $W_F(\alpha)$  is a transitive set
  - b)  $W_F(\alpha) = F \cup \{W_F(\beta) : \beta < \alpha\}$
  - c) If  $F \neq H$  are finite subsets of  $A$ , then  $W_F$  and  $W_H$  have disjoint ranges
  - d)  $W_F$  is one to one.
- For each  $t \in x$  we define the  $E$  orbit of  $t$ ,  $Ob(t)$ , to be  $Ob(t) = \{\phi(t) : \phi \in \text{fix}(E)\}$ . The set  $\{Ob(t) : t \in x\}$  is a partition of  $x$ .
- Let  $J = \{j : 0 \leq j \leq n \text{ and } (\exists t \in x)(\text{supp}_E(t) \cap I_j \neq \emptyset)\}$ . Assuming that  $E$  satisfies the condition of the lemma, we choose for each  $j \in J$ , an element  $t_j$  such that  $\text{supp}_E(t_j) \subseteq I_j$  and  $|\text{supp}_E(t_j)| = 1$ . Let  $\text{supp}_E(t_j) = \{d_j\}$ .
- Let  $\mathcal{C}$  be the set consisting of the orbits other than  $Ob(t_j)$  for  $j \in J$ , that is,  $\mathcal{C} = \{Ob(t) : t \in x \wedge (\forall j \in J)(t \notin Ob(t_j))\}$ . ( $\mathcal{C}$  can be well ordered in  $\mathcal{M}$  because all  $\phi \in \text{fix}(E)$  leave  $\mathcal{C}$  point-wise fixed.) We partition  $\mathcal{C}$  using the equivalence relation  $Ob(t) \equiv Ob(s)$  if and only if  $\exists \phi \in \text{fix}(E)$  such that  $\phi(\text{supp}_E(t)) = \text{supp}_E(s)$ . For each  $\equiv$ -equivalence class  $D \in \mathcal{C}$ , we let  $g_D$  be a one to one function from  $D$  onto an ordinal.

We begin our definition of the function  $G$  by defining  $G$  on  $\bigcup_{j \in J} Ob(t_j)$ : For  $j \in J$  and  $\phi \in \text{fix}(E)$ ,  $G(\phi(t_j)) = \phi(d_j)$ . (In other words, for  $s \in \bigcup_{j \in J} Ob(t_j)$ ,  $G(s)$  is the single element of  $\text{supp}_E(s)$ .) Using the support lemma, we see that  $G$  restricted to  $\bigcup_{j \in J} Ob(t_j)$  is one to one. Further the range of this restriction of  $G$  is  $\bigcup_{j \in J} I_j$ . For  $t \notin \bigcup_{j \in J} Ob(t_j)$ , assume  $t \in D$ , where  $D$  is an  $\equiv$ -equivalence class in  $\mathcal{C}$ . Define  $G(t) = W_{\text{supp}_E(t)}(g_D(Ob(t)))$ .

We first argue that  $G$  is in  $\mathcal{M}$ . It will suffice to show that  $G$  has support  $E$  and to show that  $G$  has support  $E$  it suffices to show that for any  $\phi \in \text{fix}(E)$ , and any  $t$  in  $x$ ,  $G(\phi(t)) = \phi(G(t))$ . This is clear for  $t \in \bigcup_{j \in J} Ob(t_j)$ . For  $t \in \bigcup \mathcal{C}$ ,  $G(\phi(t)) = W_{\text{supp}_E(\phi(t))}(g_D(Ob(t)))$  since  $Ob(\phi(t)) = Ob(t)$ . On the other hand  $\phi(G(t)) = \phi(W_{\text{supp}_E(t)}(g_D(Ob(t)))) = W_{\phi(\text{supp}_E(t))}(g_D(Ob(t)))$  (since  $g_D(Ob(t))$  is an ordinal hence fixed by  $\phi$ )  $= W_{\text{supp}_E(\phi(t))}(g_D(Ob(t)))$  by the support lemma.

Next we argue that the range of  $G$  is a transitive set. Assume that  $y$  is in the range of  $G$ . If  $y = G(t)$  for some  $t \in \bigcup_{j \in J} Ob(t_j)$  then  $y$  is an atom. Assume  $y = W_{\text{supp}_E(t)}(g_D(Ob(t)))$  for some  $t \in \bigcup \mathcal{C}$  and assume  $z \in y$ . We will show that  $z$  is in the range of  $G$ . By b),  $z \in W_{\text{supp}_E(t)}(g_D(Ob(t)))$  implies that  $z \in \text{supp}_E(t)$  or  $z$  is  $W_{\text{supp}_E(t)}(\beta)$  for some ordinal  $\beta < g_D(Ob(t))$ . If  $z \in \text{supp}_E(t)$  we use the facts that  $\text{supp}_E(t) \subseteq \bigcup_{j \in J} I_j$  and  $\bigcup_{j \in J} I_j \subseteq \text{range}(G)$  to conclude that  $z$  is in the range of  $G$ . In the second case we use the fact that  $g_D$  is a function onto an ordinal to conclude that for some  $s \in D$ ,  $g_D(Ob(s)) = \beta$ . Since  $Ob(s) \equiv Ob(t)$  we may choose  $s$  so that  $\text{supp}_E(s) = \text{supp}_E(t)$ . Then  $G(s) = W_{\text{supp}_E(t)}(\beta) = z$ . Hence,  $z$  is in the range of  $G$ .

Finally, we argue that  $G$  is one to one. We have already noted that  $G$  restricted to  $\bigcup_{j \in J} Ob(t_j)$  is one to one. We leave to the reader the proof that  $G$  restricted to  $\bigcup_{j \in J} Ob(t_j)$  and  $G$  restricted to  $\bigcup \mathcal{C}$  have disjoint ranges. It remains to show that  $G$  restricted to  $\bigcup \mathcal{C}$  is one to one. Assume  $G(s) = G(t)$  where  $s$  and  $t$  are in  $\bigcup \mathcal{C}$ . Then

$$W_{\text{supp}_E(s)}(g_D(Ob(s))) = W_{\text{supp}_E(t)}(g_{D'}(Ob(t)))$$

Where  $D$  is the  $\equiv$ -equivalence class of  $Ob(s)$  and  $D'$  is the  $\equiv$ -equivalence class of  $Ob(t)$ . By c)  $\text{supp}_E(s) = \text{supp}_E(t)$ . Therefore,  $D = D'$  and by d)  $g_D(Ob(s)) = g_{D'}(Ob(t))$ . Since  $g_D$  is one to one,  $Ob(s) = Ob(t)$ . But by the support lemma, two elements in the same  $E$  orbit with the same support relative to  $E$  must be identical. Therefore,  $s = t$ .  $\square$

Since  $\text{Tr}$  is true in Mostowski's linearly ordered model, it follows that in  $\text{ZF}^0$  not only does  $\text{Tr}$  not imply that 'Every Dedekind finite set is finite', but it also does not imply any of the following statements:

- The  $2n = n$  Principle: For any infinite cardinal  $n$ ,  $2n = n$ . (Form 3 in [H/R].)
- The Principle of Dependent Choices: If  $S$  is a relation on a non-empty set  $A$  and  $(\forall x \in A)(\exists y \in A)(xSy)$  then there is a sequence  $a(0), a(1), a(2), \dots$  of elements of  $A$  such that  $(\forall n \in \omega)(a(n) S a(n+1))$ . (Form 43 in [H/R].)
- Every linearly ordered set can be well ordered. (Form 90 in [H/R].)
- The Partition Principle: If  $P$  is a partition of a set  $X$ , then there is a 1-1 function mapping  $P$  into  $X$ . (Form 101 in [H/R].)
- The Kinna-Wagner Selection Principle for a countable number of sets: For every countable set  $X$  there is a function  $f$  such that for all  $A \in X$ , if  $|A| > 1$ , then  $\emptyset \neq f(A) \subsetneq A$ . (Form 355 in [H/R].)

(In [H/R], Mostowski's linearly ordered model is  $\mathcal{N}3$ .)

In addition,  $\text{Tr}$  is false in the basic Cohen model ( $\mathcal{M}1$  in [H/R]), and the following statements are all true in  $\mathcal{M}1$ .

- The Boolean prime Ideal Theorem: Every Boolean algebra has a prime ideal. (Form 14 in [H/R].)
- Every set can be linearly ordered. (Form 30 in [H/R].)
- The axiom of choice for a family of well orderable sets. (Form 60 in [H/R].)

Thus, it follows that none of these forms imply  $\text{Tr}$  in  $\text{ZF}$ .

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