Chapter 4 Section 5

1. A 30 year annuity immediate pays 50 each quarter of the first year. It pays 100 each quarter of the second year. The payments continue to increase annually so that the payments in each quarter are 50 higher than the previous year.

   a. Calculate the present value of this annuity at an annual effective interest rate of 10%.

   **Solution:**

   Use the formula that doesn’t follow the rules

   $n = 30$

   $i = 0.1$

   $i^{(4)} = (1.1)^{0.25} - 1 = 0.024113689$

   FirstPayment = 50

   $\left( Ia \right)^{(m)} = \frac{\ddot{a}}{i^{(m)}} - nV_i^n = 50 \left[ \frac{\ddot{a}}{i^{(m)}} - nV_i^n \right] = 50 \left[ \frac{(1.1) \left( \frac{1 - (1.1)^{-30}}{0.1} \right) - 30 (1.1)^{-30}}{0.024113689} \right] = 17,936.59$
b. Calculate the present value of this annuity at a nominal rate of 10% compounded quarterly.

Solution:

Now we need to redefine our interest rates:

\[ i^{(4)} = 0.1 \Rightarrow \frac{i^{(4)}}{4} = 0.025 \]

\[ (1 + i) = (1.025)^4 = 1.103812891 \Rightarrow i = 0.103812891 \]

Then use the formula as we did in part a.

\[ n = 30 \]

\[ \text{FirstPayment} = 50 \]

\[ (Ia)^{(m)}_{\overline{n}|i} = \frac{\bar{a}^m_n - mv^n_i}{i^{(m)}} = 50 \left( \frac{\bar{a}^m_n - mv^n_i}{i^{(m)}} \right) = 50 \left[ \frac{(1.1038)\left(1-\frac{10}{0.025}\right)}{0.025} - 30(1.1038)^{-30} \right] = 17,067.43 \]

2. An annuity due makes monthly payments for 15 years. The first payment is 10. Each subsequent payment is 10 larger than the previous payment. In other words, the payment at the start of the second month is 20 and the payment at the start of the third month is 30, etc.

Calculate the present value of this annuity using a nominal interest rate of 12% compounded monthly.

Solution:

Now we need to use the “P and Q” formula from our notes. Be careful to use the correct interest rate!

\[ P = 10; Q = 10; n = 12 \times 15 = 180 \]

\[ \frac{i^{(12)}}{12} = 0.01 \]

\[ PV = (I_{P,Q} \bar{a}^m_n) = \left(1 + \frac{i^{(12)}}{12}\right) (I_{P,Q} a^m_n) \]

\[ = 1.01 \left[ Pa^m_n + \frac{Q}{i^{(12)}(a^m_n - nv^n)} \right] \]

\[ = 1.01 \left[ 10 \left( \frac{1 - (1.01)^{-180}}{0.01} \right) + 10 \left[ \frac{1 - (1.01)^{-180}}{0.01} \right]^{-180} (1.01)^{-180} \right] = 54,675.21 \]
3. Alicia is making deposits into her retirement account at the end of every month. During the first year, she deposits 100 each month. During the second year, Alicia deposits 200 each month. During the third year, the payments are 300. The payments continue in the same pattern until she deposits 3000 per month during the 30th year.

If Alicia earns an annual effective interest rate of 7%, how much will she have accumulated after 30 years.

**Solution:**

Use the formula that doesn’t follow the rules.

\( n = 30; \)

\( i = 0.07; \Rightarrow \frac{t^{(12)}}{12} = (1.07)^{1/12} - 1 = 0.005654145 \)

FirstPayment = 100

\[
(Ia)^{(m)}_{n} = \frac{\hat{a}_{n} - n v_{i}^{n}}{i^{(m)}} = 100 \left( \frac{\hat{a}_{n} - n v_{i}^{n}}{i^{(m)}} \right) = 100 \left[ \frac{(1.07) \left( 1 - 1.07^{-30} \right) - 30(1.07)^{-30}}{0.005654145} \right] = 165,129.4856
\]

We need to find the accumulated value, so we multiply our answer by \((1 + i)^{n}\).

\( 165,129.4856(1.07)^{30} = 1,257,007.76 \)
4. A 25 year annuity with monthly payments makes payments of 1000 at the end of each month during the first year. The payments during the second year are 1200 per month. The payments during the third year are 1400 per month. The payments continue to increase by 200 in each year with 5800 being paid each month during the 25th year.

Calculate the present value of this annuity at an annual effective interest rate of 8%.

**Solution:**

For this problem, we want to separate this into two different annuities.

1. A 25 year annuity with monthly payments of $800 at the end of each month.
2. A 25 year annuity with payments of $200 for the first year, $400 for the second year, etc.

We can calculate the value of each annuity and then add them up. The interest rates that we will use are:

\[
\frac{i^{(12)}}{12} = (1.08)^{\frac{1}{12}} - 1 = 0.00642403
\]

1. Normal Annuity: 

\[
PV = 800a_{300} = 800 \left[ \frac{1 - (1.00642403)^{-300}}{0.00642403} \right] = 106,183.1643
\]

2. Formula doesn’t follow the rules:

\[
FirstPayment = 200
\]

\[
(Ia)^{(m)}_{n} = 200 \left( \frac{\ddot{d}_{m} - n\ddot{v}_{i}^{n}}{n^{(m)}} \right) = 200 \left[ \frac{(1.08) \left( \frac{1 - (1.08)^{-25}}{0.08} \right) - 25(1.08)^{-25}}{0.00643403} \right] = 244,895.051
\]

This gives our final answer: 

\[
106,183.1643 + 244,895.051 = 351,078.2153
\]
5. Yuchen is receiving an annuity immediate for ten year. The annuity makes monthly payments. The payments in the first year are 10,000 each month. The payments in the second year are 9000 each month. The payments in the third year are 8000 per month. This pattern continues until payments of 1000 are made each month in the 10 year.

Using an interest rate of 12% compounded monthly, calculate the present value of Yuchen’s annuity.

Solution:

Here we have decreasing payments but the payments are level over each year. We can create the same pattern of payments with level payments of 11,000 each month for 10 years minus an annuity that pays 1000 each month of the first year, 2000 each month of the second year, etc.

Therefore,

\[ PV = (11,000) \frac{d_{10}^{0.12682503} - (10)(1.12682503)^{−10}}{0.01} \]

\[ = (11,000) \left( \frac{1−1.01^{−120}}{0.01} \right) - 1000 \left( \frac{1−(1.12682503)^{−10}}{0.12682503} - (10)(1.12682503)^{−10} \right) \]

\[ = 766,705.7423 - 316,285.9162 = 450,419.83 \]
6. Matt is receiving an increasing perpetuity. The payments are made at the end of each quarter. The payments in the first year are 25. The quarterly payments in the second year are 50. The payments continue to increase each year by 25, but are level throughout the year.

Calculate the present value of this perpetuity at a nominal rate of 9% compounded monthly.

Solution:

We now switch to the perpetuity formula that breaks the same rules.

\[(Ia)^{(m)}_{\infty} = \frac{1 + i}{i^* \frac{1}{m}}.\]

\[i^{(12)} = \frac{0.09}{12} = 0.0075\]

First, let’s find the appropriate interest rates. \((1 + i)(1.0075)^{12} = 1.093806898\)

\[i^{(4)} = \frac{1.093806898^{1/4} - 1 = 0.022669172}{4}\]

Now we can find the present value:

\[\frac{1.093806898}{(0.093806898 * 0.022669172)}(25) = 12,859.09\]

Chapter 4, Section 6

7. A continuous annuity pays at a rate of 100 per year for the next 25 years.

Calculate the present value of this annuity at a force of interest of 8%.

Solution:

\[100\overline{a}_{25} = 100 \left( \frac{1 - e^{-ns}}{\delta} \right) = 100 \left( \frac{1 - e^{-25*0.08}}{0.08} \right) = 1,080.83\]
8. An increasing annuity makes payments continuously for the next 10 years. The payment rate is $1000t$ at time $t$.

Calculate the present value of this annuity at annual effective interest rate of 6%.

**Solution:**

First, let’s find the force of interest:

$$ (1.06) = e^{\delta} $$

$$ \delta = \ln(1.06) = 0.058268908 $$

Now use the formula: $$(\bar{a}_n|n = \frac{a_{|n}}{\delta}$$.

$$ \text{Answer: } 1000 \left[ \frac{1-(1.06)^{-10}}{\ln(1.06)} \right] = 34234.3414 $$

9. Tao invests money in a fund continuously over the next 22 years. The amount invested is at a rate of 4000 per year.

Calculate the amount that Tao will have after 22 years if $\delta = 0.12$.

**Solution:**

We need to find $4000\bar{a}_{|22} = 4000 \left( \frac{e^{\delta} - 1}{\delta} \right) = 4000 \left( \frac{e^{22\cdot0.12} - 1}{0.12} \right) = 433,773.4536$

10. Shinji is receiving continuous payments under a perpetuity that pays $100t$ at time $t$. Calculate the present value of this perpetuity at annual effective rate of interest of 20%.

**Solution:**

This is an increasing continuous perpetuity. Our formula is: $PV = 100 \left( \frac{1}{\delta^2} \right)$.

Our answer is: $PV = 100 \left( \frac{1}{(\ln(1.2))^2} \right) = 3,008.32$
11. A 10 year continuous annuity pays at a rate of $t^3$ at time $t$. The discount function for this annuity is $1 - 0.006t^2$. Calculate the present value of this annuity.

**Solution:** For this problem, we need to integrate. Remember that $\int_0^n f(t)v(t)\,dt$

We have,

$$PV = \int_0^{10} t^3(1 - 0.006t^2)\,dt = \int_0^{10} t^3 - 0.006t^5\,dt = \left[ \frac{t^4}{4} - \frac{0.006t^6}{6} \right]_0^{10} = \frac{10^4}{4} - \frac{0.006*10^6}{6} = 1,500$$
12. Calculate the **accumulated value** at a constant force of interest of 5% of a 20 year continuous annuity which pays at the rate of \((t+1)/2\) per period at exact moment \(t\).

**Solution:**

First, set up the equation for PV.

\[
PV = \int_0^{20} \left( \frac{t+1}{2} \right) e^{-0.05t} \, dt = 0.5 \int_0^{20} (t+1) e^{-0.05t} \, dt
\]

We are going to need to integrate by parts.

\[
\begin{align*}
  u &= t + 1 \\
  du &= 1 \\
  dv &= e^{-0.05t} \\
  v &= -20e^{-0.05t} \\
  uv - \int v \, du
\end{align*}
\]

\[
PV = 0.5 \left[ \left( t + 1 \right) (-20) e^{-0.05t} \bigg|_0^{20} + 20 \int_0^{20} e^{-0.05t} \, dt \right]
\]

\[
= 0.5 \left[ \left( t + 1 \right) (-20) e^{-0.05t} \bigg|_0^{20} + 20 \left( e^{-1} - 1 \right) \right]
\]

\[
= 59.169
\]

Now that we have the present value, we can calculate the future value.

\[
FV = 59.169 e^{0.05(20)} = 160.84
\]

Or we can do this the easy way which is

\[
PV = \int_0^{20} \left( \frac{t+1}{2} \right) e^{-0.05t} \, dt = 0.5 \int_0^{20} (t + 1) e^{-0.05t} \, dt = 0.5 \int_0^{20} tv' \, dt + 0.5 \int_0^{20} v' \, dt \quad \text{since} \quad e^{-0.05t} = v'
\]

\[
= 0.5(\bar{a})_{20|} + 0.5\bar{a}_{20|} \quad \text{since} \quad (\bar{a})_{20|} = \int_0^{20} tv' \, dt \quad \text{and} \quad \bar{a}_{20|} = \int_0^{20} v' \, dt
\]

\[
= 59.169
\]

Now that we have the present value, we can calculate the future value.

\[
FV = 59.169 e^{0.05(20)} = 160.84
\]
Answers

1.  
a. 17,936.59  
b. 17,067.43  
2. 54,675.21  
3. 1,257,007.67  
4. 351,078.21  
5. 450,419.83  
6. 12,859.09  
7. 1080.83  
8. 34,234.34  
9. 433,773.45  
10. 3008.32  
11. 1500  
12. 160.84