1. The Purdue Life Insurance Company has two assets and two liabilities.

The assets are:
   a. A 5 year par value bond with a maturity value of 100,000. The bond pays annual coupons at a rate of 6%; and
   b. A zero coupon bond which matures in 10 years for 250,000.

The liabilities are:
   c. An annuity immediate with payments of 35,000 at the end of each year for 5 years; and
   d. An perpetuity immediate with annual payments of 5,000.

Calculate the owners equity (surplus) of the Purdue Life Insurance Company assuming an interest rate of 8%.

Solution:

\[
\text{Asset a} = 6000a_{\overline{5}|i} + (100,000)v^5 = 92,014.58 \\
\text{Asset b} = (250,000)v^{10} = 115,798.37 \\
\text{Liability c} = (35,000)a_{\overline{5}|i} = 139,744.85 \\
\text{Liability d} = \frac{5,000}{0.08} = 62,500.00
\]

Owners Equity = a + b - c - d = 92,014.58 + 115,798.37 - 139,744.85 - 62,500.00 = 5568

Calculate the owners equity (surplus) of the Purdue Life Insurance Company assuming an interest rate of 12%.

Solution:

\[
\text{Asset a} = 6000a_{\overline{5}|i} + (100,000)v^5 = 78,371.34 \\
\text{Asset b} = (250,000)v^{10} = 80,493.30 \\
\text{Liability c} = (35,000)a_{\overline{5}|i} = 126,167.17 \\
\text{Liability d} = \frac{5,000}{0.12} = 41,666.67
\]

Owners Equity = a + b - c - d = 78,371.34 + 80,493.30 - 126,167.17 - 41,666.67 = -8869
2. John must pay Kristen 10,000 at the end of 1 year. He also must pay Ahmad 30,000 at the end of year 2.

John wants to exactly match his liabilities by purchasing the following two bonds:

a. Bond A is a one year zero coupon bond maturing for 1000.

b. Bond B is a two year bond with annual coupons of 200 and a maturity value of 1000.

Calculate the amount of each bond that John should buy.

**Solution:**

John needs to make the cash flows going out equal to the cash flows coming in. To do this, we can set up two equations where \(a\) is the amount of bond A and \(b\) is the amount of bond B. We know that at the end of year 1 John will pay 10,000 and he will receive 1000 from every bond A and 200 from every bond B. Then at the end of year two when he needs to pay 30000, he will be receiving 1200 for every bond B.

\[
10000 = 1000a + 200b \\
30000 = 1200b 
\]

Simply solve this system algebraically.

\[
30000 = 1200b \Rightarrow b = 25 \\
10000 = 1000a + 200(25) \Rightarrow a = 5 
\]
3. Trena has agreed to pay Jayme the following payments:
   a. 30,000 at the end of 1 years;
   b. 50,000 at the end of 2 years; and
   c. 10,000 at the end of 3 years.

You are given the following three bonds:

   a. A one year bond with annual coupons of 100 and a maturity value of 2000 with a price of 2000.
   b. A two year bond with annual coupons of 80 and a maturity value of 1000 with a price of 1020.
   c. A three year bond with annual coupons of 50 and a maturity value of 1800 with a price of 1580.

Determine the amount of the two year bond that should be purchased to exactly match the cash flows of Trena’s liability. (Assume that you can purchase partial bonds.)

Solution:

<table>
<thead>
<tr>
<th>Cash Flows</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
</tr>
<tr>
<td>Benefit</td>
</tr>
<tr>
<td>Bond A</td>
</tr>
<tr>
<td>Bond B</td>
</tr>
<tr>
<td>Bond C</td>
</tr>
</tbody>
</table>

\[
A(2100) + B(80) + C(50) = 30,000
\]
\[
B(1080) + C(50) = 50,000
\]
\[
C(1850) = 10,000
\]

\[
C = \frac{10,000}{1850} = 5.405405405 \text{ of Bond C}
\]

\[
B = \frac{50,000 - 50(5.405405405)}{1080} = 46.04604605 \text{ of Bond B}
\]

\[
A = \frac{30,000 - 50(5.405405405) - 80(46.04604605)}{2100} = 12.40287907 \text{ of Bond C}
\]

We want to purchase 46.05 of the two year bond.
Rivera Insurance Company has committed to paying 10,000 at the end of one year and 40,000 at the end of two years. It’s Chief Financial Officer, Miguel, wants to exactly match this obligation using the following two bonds:

- Bond A is a one year bond which matures at par of 1000 and pays an annual dividend at a rate of 6%. This bond can be bought to yield 6% annually.
- Bond B is a two year bond which matures at par of 1000 and pays an annual dividend at a rate of 10%. This bond can be bought to yield 7% annually.

Calculate the amount of each bond that Rivera should purchase.

Calculate the cost of Rivera to exactly match this obligation.

**Solution:**

\[ 10000 = 1060a + 100b \]
\[ 40000 = 1100b \]

Solving,
\[ a = 6.0034305 \]
\[ b = 36.363636 \]

Now, let’s find the price of each bond in order to find the total cost to match.

Price of Bond 1:
\[ N = 1 \quad I/Y = 6\% \quad PMT = 60 \quad FV = 1000 \quad CPT \quad PV = 1000 \]

Note: We could also know this intuitively because the coupon rate is the same as the yield.

Price of Bond 2:
\[ N = 2 \quad I/Y = 7\% \quad PMT = 100 \quad FV = 1000 \quad CPT \quad PV = 1054.240545 \]

Now we can find the total price
\[ 6.0034305(1000) + 36.363636(1054.240545) = 44339.45 \]
5. Wang Life Insurance Company issues a three year annuity that pays 40,000 at the end of each year. Wang uses the following three bonds to absolutely match the cash flows under this annuity:

a. A zero coupon bond which matures in one year for 1000.
b. A two year bond which matures for 1200 and pays an annual coupon of 100. This bond is priced using an annual yield of 7%.
c. A three year bond which matures for 2000 and pays annual coupons of 75. This bond has a price of 1,750.

It cost Wang 104,000 to purchase all three bonds to absolutely match this annuity.

Calculate the one year spot interest rate.

Solution:

Absolute matching means that the cash flow from the bonds at any given time should be exactly the same as the cash flow from the annuity. In this case, we need the bonds to produce a cash flow of $40,000 at times 1, 2, and 3. In order to do this we will need to buy a certain amount of each bond. These values will be “x, y, and z”.

Sometimes it is easiest to set up a table that summarizes cash flows:

<table>
<thead>
<tr>
<th>Bond</th>
<th>Amount</th>
<th>Price</th>
<th>CF at t=1</th>
<th>CF at t=2</th>
<th>CF at t=3</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>X</td>
<td>1000/(1+r₁)</td>
<td>1000</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>Y</td>
<td>1228.928291</td>
<td>100</td>
<td>1300</td>
<td>0</td>
</tr>
<tr>
<td>C</td>
<td>Z</td>
<td>1750</td>
<td>75</td>
<td>75</td>
<td>2075</td>
</tr>
</tbody>
</table>

In the table above, the Price for bond B was found using the calculator.

\[ N = 2 \quad PMT = 100 \quad I/Y = 7 \quad FV = 1200 \quad CPT \quad PV = 1228.928291 \]

Now we can set up a few equations in order to solve for the unknown variables.

First, since there is only one bond with a cash flow at time 3 we can find z very easily:

\[ 2075Z = 40000 \]
\[ Z = 19.277 \]

Then we can find y:

\[ 1300Y + 75Z = 40000 \]
\[ Y = \frac{40000 - 75(19.277)}{1300} = 29.657 \]

Finally we can find x.
\[1000X + 100Y + 75Z = 40000\]
\[X = \frac{40000 - 75(19.277) - 100(29.657)}{1000} = 35.5885\]

Now that we know the amount of each bond we are purchasing and we are given our total cost, we just need to find the price of the one year bond which will in turn give us the one year spot rate.

\[104000 = \frac{1000}{1 + r_1} X + 1228.928291Y + 1750Z\]

Note that the price of the first bond = \(\frac{1000}{1 + r_1}\).

Now just plug in the values we found above to get \(r_1 = 0.05233\).

6. Ace is receiving an annuity immediate with level annual payments of 500 for 18 years.

Calculate the Macaulay duration and the Modified duration at an annual effective interest rate of 6%.

**Solution:**

Use for formula for Macaulay Duration first:

\[MacDur = \frac{\sum C_t v^t}{\sum Ct v^t} = \frac{500v + 500(2)v^2 + \ldots + 500(18)v^{18}}{500v + 500v^2 + \ldots + 500v^{18}}\]

\[\left[500a_{\overline{18}|} + \frac{500}{i}(a_{\overline{18}|} - 18v^{18})\right] = \left[\frac{(500)10.82760 + \frac{500}{0.06}(10.82760 - 6.30618824)}{(500)10.82760}\right] = 7.9597\]

Now that we have the Macaulay Duration, it is easy to find the modified duration.

\[Modified = (MacDur)v = \frac{7.9597}{1.06} = 7.5092\]
7. Li Life Insurance Company is paying Cora an annuity due of 234 per year for the next 10 years. Calculate the Modified duration of Cora’s annuity at an annual effective interest rate of 10%.

**Solution:**

\[
MacDur = \frac{\sum C_n \cdot v^n}{\sum C_n \cdot v^n} = \frac{234 \cdot v^0 + 234 \cdot v^1 + \ldots + 234 \cdot v^9}{234 \cdot v^0 + 234 \cdot v^1 + \ldots + 234 \cdot v^9}
\]

\[
= \left[ \frac{234a_{\overline{10}|} + 234}{i} \left( a_{\overline{10}|} - 9v^9 \right) \right]_{234\bar{a}_{\overline{10}|}}
\]

\[
= \left[ \frac{(234)(5.759023816) + \frac{234}{0.10} (5.759023816 - 3.816878565)}{6.759023816} \right] = 3.725460513
\]

Now that we have the Macaulay Duration, it is easy to find the modified duration.

\[
Modified = (MacDur)v = \frac{3.725460513}{1.1} = 3.3868
\]
8. A 20 year bond has annual coupons of 400. The bond matures for 13,000.

Calculate the Macaulay Duration of this bond at an annual effective interest rate of 5.5%.

Solution:

\[
MacDur = \frac{\sum C_i (t) v^t}{\sum C_i v^t} = \frac{400(1) v + 400(2) v^2 + \ldots + 400(20) v^{20} + 13,000(20) v^{20}}{400 v + 400 v^2 + \ldots + 400 v^{20} + 13,000 v^{20}}
\]

Numerator: P&Q Formula

\[
= 400 a_{\overline{20}|} + \frac{400}{0.055} (a_{\overline{20}|} - 20 v^{20}) + 13,000(20) v^{20}
\]

\[
= 400(11.95038248) + 7272.727272(11.95038248 - 6.854579267) + 89,109.53047
\]

\[
= 130,950.0705
\]

Denominator: Financial Calculator

N=20
I/Y=5.5
PMT=400
FV=13000
CPT PV=9235.629577

\[
MacDur = \frac{130,950.0705}{9235.629517} = 14.17879206
\]
9. Mayfawny owns an 8 year bond with a par value of 1000. The bond matures for par and pays semi-annual coupons at a rate of 6% convertible semi-annually.

Calculate the Modified duration of this bond at an annual effective interest rate of 8.16%.

Solution:

Use your calculator to find the price of the bond.

\[
PMT = 30 | \ FV = 1000 | \ N = 16 | \ I/Y = 4 \ CPT | PV = 883.48
\]

Modified Duration =

\[
\frac{\sum C_i v^t}{\sum C_i v^t} = \frac{30(0.5)v^{0.5} + 30(1)v + 30(1.5)v^{1.5} + \ldots + 30(8)v^8 + 1000(8)v^8}{883.48} \left( \frac{1}{1.0816} \right)
\]

\[
\frac{\sum C_i v^t}{\sum C_i v^t} = \frac{15(1.0816)^{-0.5} + 30(1.0816)^{-1} + 45(1.0816)^{-1.5} + \ldots + 240(1.0816)^{-8} + 1000(8)v^8}{883.48} \left( \frac{1}{1.0816} \right) =
\]

\[
\frac{\sum C_i v^t}{\sum C_i v^t} = \frac{15(1.04)^{-1} + 30(1.04)^{-2} + 45(1.04)^{-3} + \ldots + 240(1.04)^{-16} + 1000(8)v^8}{883.48} \left( \frac{1}{1.0816} \right)
\]

\[
15a_{16|0.04} + \frac{15}{0.04} \left( a_{16|0.04} - 16 \left( \frac{1}{1.04} \right)^{16} \right) + 8000(1.0816)^{-8} = 883.48(1.0816) = 5.8732
\]
10. A five year bond matures for 20,000. The bond pays coupons of:

a. 3000 at the end of the first year,
b. 1500 at the end of the second year,
c. 1000 at the end of the third year,
d. 750 at the end of the fourth year, and
e. 600 at the end of the fifth year.

Calculate the Macaulay Duration of this bond at 5%.

Solution:

Macaulay Duration= 

\[
D(i, \infty) = \frac{\sum C_i v^t}{\sum C_i v^{t'}} = \frac{3000v + 1500(2)v^2 + 1000(3)v^3 + 750(4)v^4 + 600(5)v^5 + 20,000(5)v^5}{3000v + 1500v^2 + 1000v^3 + 750v^4 + 600v^5 + 20,000v^5}
\]

\[
= \frac{3000(v + v^2 + v^3 + v^4 + v^5) + 100,000v^5}{3000v + 1500v^2 + 1000v^3 + 750v^4 + 600v^5 + 20,000v^5} = 4.1824
\]

11. James has a loan of 10,000 which is to be repaid with 10 level annual payments at an annual effective interest rate of 12%. Calculate the Macaulay duration of the loan using the 12% interest rate.

Solution:

First use your calculator to find the payments:

\[
PV = 1000 \quad N = 10 \quad I/Y = 12 \quad CPT \quad PMT = 1769.84
\]

\[
\sum C_i v^t = \frac{1769.84v + 1769.84(2)v^2 + \cdots + 1769.84(10)v^{10}}{10,000}
\]

\[
\sum C_i v^{t'} = 1769.84(Ia)_{10} = 1769.84 \left( \frac{(1.12)q_{10} - 10v^{10}}{0.12} \right) = 45,846.48734
\]

\[
\sum C_i v^t = \frac{45,846.48734}{10,000} = 4.58
\]
12. Trena has agreed to pay Jayme the following payments:
   a. 30,000 at the end of 2 years;
   b. 50,000 at the end of 5 years; and
   c. 10,000 at the end of 8 years.

   You are given that \( v = 0.9 \).

   Calculate the Modified Duration of Trena’s liability.

   **Solution:**

   \[
   ModDur = \frac{v \sum C_i (t) v^t}{\sum C_i v^t}
   \]

   \[
   = (0.9) \left[ \frac{30,000(2)(0.9)^2 + 50,000(5)(0.9)^5 + 10,000(8)(0.9)^8}{30,000(0.9)^2 + 50,000(0.9)^5 + 10,000(0.9)^8} \right]
   \]

   \[
   = (0.9) \left( \frac{230,659.8768}{58,129.1721} \right) = 3.571251432
   \]

13. Ivey Investments owns a preferred stock which pays a quarterly dividend of $5 per quarter with the next dividend paid in 3 months. Calculate the modified duration of this stock at an annual effective rate of 8.243216%.

   **Solution:**

   | First, let’s find the proper i: 1.08243216^{0.25} = 1.02 |

   \[
   \frac{\sum C_i t v^t}{\sum C_i v^t} = \frac{5(0.25) v^{0.25} + 5(0.5) v^{0.5} + \cdots}{\left( \frac{5}{0.02} \right)} (v) = \frac{1.25 \left( \frac{1}{0.02} + \frac{1}{0.02^2} \right)}{250} \left( \frac{1}{1.08243216} \right) = 11.78
   \]
14. The Macaulay Duration of a perpetuity immediate with level annual payments is 26.

Determine the interest rate that was used to calculate the Macaulay Duration.

Solution:

\[
\sum C_i t^i = \frac{P(la)}{P(1+i)} = \frac{P(1+i) \left( \frac{1}{i} + \frac{1}{i^2} \right) i}{P(1+i)} = \left( 1 + \frac{1}{i} \right) = 26
\]

\[1 + \frac{1}{i} = 26\]

\[i = \frac{1}{25} = 0.04\]

15. Sparks-Norris Asset Partners (SNAP) manages the following portfolio of bonds:

<table>
<thead>
<tr>
<th>Bond</th>
<th>Price</th>
<th>Macaulay Duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5,000</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>10,000</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>20,000</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>15,000</td>
<td>2</td>
</tr>
</tbody>
</table>

The duration is calculated at an annual effective interest rate of 7%.

Calculate the modified duration of SNAP’s portfolio.

Solution:

\[
D_{PORT} = \sum D_i' P_i'
\]

where \(D_i'\) is Modified Duration.

In order to get Modified Duration from Macaulay Duration, we multiply Macaulay by \(v\).

\[
\sum D_i' P_i' = \frac{10(5000) + 6(10000) + 8(20000) + 2(15000)}{5000 + 10000 + 20000 + 15000} \cdot v = 5.607
\]
16. An annuity immediate pays 100 at the end of each year for 5 years. Calculate the Macaulay convexity and the Modified convexity of this annuity at an annual effective rate of 6%.

**Solution:**

**Modified Convexity**

\[
\sum_{t=1}^{5} C_t (t)(t+1) v^t \quad \sum_{t=1}^{5} C_t \cdot v^t
\]

\[
(1.06)^2 \left( \frac{(100)(1)(2)(1.06)^{-1} + (100)(2)(3)(1.06)^{-2} + (100)(3)(4)(1.06)^{-3} + (100)(4)(5)(1.06)^{-4} + (100)(5)(6)(1.06)^{-5}}{100(1.06)^{-1} + 100(1.06)^{-2} + 100(1.06)^{-3} + 100(1.06)^{-4} + 100(1.06)^{-5}} \right)
\]

= 11.7392

**Macaulay Convexity:**

Note: As you can see above, the 100 in the numerator and denominator will cancel since all cash flows are 100. Therefore, the 100's have not been included in the formulas below.

\[
= \left( \frac{1(1.06)^{-1} + 2^2(1.06)^{-2} + 3^2(1.06)^{-3} + 4^2(1.06)^{-4} + 5^2(1.06)^{-5}}{(1.06)^{-1} + (1.06)^{-2} + (1.06)^{-3} + (1.06)^{-4} + (1.06)^{-5}} \right)
\]

\[
= \left( \frac{1(1.06)^{-1} + 4(1.06)^{-2} + 9(1.06)^{-3} + 16(1.06)^{-4} + 25(1.06)^{-5}}{(1.06)^{-1} + (1.06)^{-2} + (1.06)^{-3} + (1.06)^{-4} + (1.06)^{-5}} \right)
\]

= 10.306543
17. Calculate the modified convexity for a 3 year bond with annual coupons of 300 and a maturity value of 5000 using an annual effective interest rate of 8%.

**Solution:**

\[
ModConv = \frac{MacCon + MacDur}{(1+i)^2}
\]

\[
MacDur = \frac{300(1)v + 300(2)v^2 + 5300(3)v^3}{300v + 300v^2 + 5300v^3} = 2.82615046
\]

\[
MacConv = \frac{300(1)v + 300(4)v^2 + 5300(9)v^3}{300v + 300v^2 + 5300v^3} = 8.260224443
\]

\[
ModConv = \frac{2.838615046 + 8.260224443}{(1.08)^2} = 9.506892566
\]

*Or*

\[
ModConv = v^2 \sum_{t} C_t (t)(t+1)v^t =
\]

\[
(1.08)^{-2} \left( \frac{(300)(1)(2)(1.08)^{-1} + (300)(2)(3)(1.08)^{-2} + (5300)(3)(4)(1.08)^{-3}}{300(1.08)^{-1} + 300(1.08)^{-2} + 5300(1.08)^{-3}} \right)
\]

\[= 9.507\]
18. A 3 year bond has annual coupons.

The coupon at the end of the first year is 100.

The coupon at the end of the second year is 300.

The coupon at the end of the third year is 500.

The bond matures for 700.

Calculate the modified convexity of this bond at an annual effective rate of interest of 6%.

Solution:

Modified Convexity:

\[
\frac{\sum C_t (t)(t+1)v'}{\sum C_t \cdot v'} = (1.06)^{-2} \left( \frac{100(1)(2)(1.06)^{-1} + 300(2)(3)(1.06)^{-2} + 1200(3)(4)(1.06)^{-3}}{100(1.06)^{-1} + 300(1.06)^{-2} + 1200(1.06)^{-3}} \right)
\]

\[
= \frac{12,354.21}{1368.88} = 9.025
\]
19. John is receiving an annuity due with three annual payments. The first payment is 100, the second payment is 200, and the last payment is 300.

John calculates the Macaulay Covexity of his annuity to be 2.20268 at an annual effective interest rate of \( i \).

Determine \( i \), the interest rate used by John. (Hint: Find \( v \) and then find \( i \).)

Solution:

\[
\frac{100(0) + 200v(1^2) + 300v^2(2^2)}{100 + 200v + 300v^2} = 2.20268
\]

\[
200v + 1200v^2 = 220.268 + 440.536v + 660.804v^2
\]

\[
539.196v^2 - 240.536v - 220.268 = 0
\]

\[
v = \frac{240.536 \pm \sqrt{(240.536)^2 + 4(220.268)(539.196)}}{2(539.196)}
\]

\[
v = 0.9
\]

\[
0.9 = (1 + i)^{-1}
\]

\[
i = 0.11111
\]

20. A bond has a Macaulay Duration of 4.450926 and a Macaulay Convexity of 21.19773 when calculated using an annual effective interest rate of 8%. The price of the bond is 1096.19.

A. Estimate the price of the bond if the annual interest rate increases to 8.5% using only the duration.

B. Estimate the price of the bond if the annual interest rate increases to 8.5% using both the duration and the convexity.

C. The values in this problem are based on a 5 year bond with annual coupons of 70 and a maturity value of 1200. What would the actual price be at 8.5%.
Solution:
A:

\[ P(i) = P(i_0) \left[ 1 - (ModDur)(i - i_0) + (ModConv) \left( \frac{(i - i_0)^2}{2} \right) \right] \]

If we ignore the last term, we have

\[ P(i) = P(i_0) \left[ 1 - (ModDur)(i - i_0) \right] \]

\[ ModDur = v(MacDur) = \left( \frac{4.450926}{1.08} \right) \]

\[ P(i) = 1096.19 \left[ 1 - \left( \frac{4.450926}{1.08} \right)(0.085 - 0.08) \right] = 1073.60 \]

B:

We need to find the Modified Convexity:

\[ ModConv = \frac{MacConv + MacDur}{(1 + i)^2} = \frac{21.19773 + 4.450926}{(1.08)^2} = 21.989588 \]

Now we can find the change in price:

\[ P(i) = P(i_0) \left[ 1 - (ModDur)(i - i_0) + (ModConv) \left( \frac{(i - i_0)^2}{2} \right) \right] = \]

\[ = 1096.19 \left[ 1 - \left( \frac{4.450926}{1.08} \right)(0.085 - 0.08) + (21.989588) \left( \frac{(0.085 - 0.08)^2}{2} \right) \right] \]

\[ = 1073.90 \]

C:

Use your calculator: \[ PMT = 70 \quad N = 5 \quad I/Y = 8.5 \quad FV = 1200 \quad CPT \quad PV = 1073.90 \]
21. The Wang Insurance Company has the following two assets:

   a. A zero coupon bond which matures for 10,000 in five years and has an annual yield of 8%.
   
   b. A bond with a price of 20,000 which has a Macaulay Duration of 3 and a Macaulay Convexity of 11. All values were calculated at an annual effective interest rate of 8%.

Let $D_{\text{mod}}^{\text{Port}}$ be the Modified Duration of this portfolio of assets at an interest rate of 8%.

Let $C_{\text{mod}}^{\text{Port}}$ be the Modified Convexity of this portfolio of assets at an interest rate of 8%.

Calculate $C_{\text{mod}}^{\text{Port}} - D_{\text{mod}}^{\text{Port}}$.

Solution:

\[
\text{MacDur}^{\text{A}} = \frac{10000(5)v^5}{10000v^5} = 5
\]

\[
\text{MacConv}^{\text{A}} = \frac{10000(25)v^5}{10000v^5} = 25
\]

\[
\text{MacDur}^{\text{Port}} = \frac{6805.83197}{20000 + 6805.83197} (5) + \frac{20000}{20000 + 6805.83197} (3) = 3.507787408
\]

\[
\text{ModDur}^{\text{Port}} = 3.247951304
\]

\[
\text{MacConv}^{\text{Port}} = \frac{6805.83197}{20000 + 6805.83197} (25) + \frac{20000}{20000 + 6805.83197} (11) = 14.55451186
\]

\[
\text{ModConv}^{\text{Port}} = \frac{14.55451186 + 3.507787408}{(1.08)^2} = 15.48551035
\]

\[
\text{ModConv}^{\text{Port}} - \text{ModDur}^{\text{Port}} = 12.23755904
\]
22. Jenna owns the following portfolio.

<table>
<thead>
<tr>
<th>Asset</th>
<th>Price</th>
<th>Macaulay Duration</th>
<th>Macaulay Convexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bond 1</td>
<td>25,000</td>
<td>6.0</td>
<td>40</td>
</tr>
<tr>
<td>Bond 2</td>
<td>30,000</td>
<td>4.5</td>
<td>25</td>
</tr>
<tr>
<td>Bond 3</td>
<td>45,000</td>
<td>3.0</td>
<td>12</td>
</tr>
</tbody>
</table>

The price, Macaulay Duration, and Macaulay Convexity were calculated at an annual effective rate of 5%.

Estimate the price of the portfolio at an annual effective rate of interest of 7% using both the duration and convexity.

**Solution:**

First find the Macaulay Convexity and the Macaulay Duration of the Portfolio:

\[
D_{\text{Mac}}^{\text{Portfolio}} = \left( \frac{25,000}{100,000} \right)^{(6)} + \left( \frac{30,000}{100,000} \right)^{(4.5)} + \left( \frac{45,000}{100,000} \right)^{(3)} = 4.2
\]

\[
C_{\text{Mac}}^{\text{Portfolio}} = \left( \frac{25,000}{100,000} \right)^{(40)} + \left( \frac{30,000}{100,000} \right)^{(25)} + \left( \frac{45,000}{100,000} \right)^{(12)} = 22.9
\]

Then find the Modified Convexity:

\[
\text{ModConv} = \frac{\text{MacConv} + \text{MacDur}}{(1 + i)^2} = \frac{22.9 + 4.2}{(1.05)^2} = 24.5805
\]

Now we can find the change in price:

\[
P(i) = P(i_0) \left[ 1 - (\text{ModDur})(i - i_0) + (\text{ModConv}) \left( \frac{(i - i_0)^2}{2} \right) \right]
\]

\[
= (25,000 + 35,000 + 40,000) \left[ 1 - \left( \frac{4.2}{1.05} \right)(0.07 - 0.05) + (24.5805) \left( \frac{0.07 - 0.05}{2} \right) \right]
\]

\[
= 92,491.61
\]
23. Sue wants to fully immunize a future payment of 100,000 at time 10 using the following two bonds:
   a. A zero coupon bond maturing in 5 years; and
   b. A zero coupon bond maturing in 20 years.

Determine the amount that Sue should spend on each bond at an annual effective interest rate of 10%.

**Solution:**

Present Value Matching:

Let $A =$ Present value of 5 year bond

Let $B =$ Present value of 20 year bond

$\text{PV(Assets)} = \text{PV(Liabilities)}$

\[ A + B = 100,000(1.1)^{-10} \]

Duration Matching:

$\text{Duration(Assets)} = \text{Duration(Liabilities)}$

\[ 5A + 20B = 10(100,000(1.1)^{-10}) \]

Now we have two equations with two unknowns. We can solve for $A$ and $B$.

\[ A = 100,000(1.1)^{-10} - B \]
\[ A = 38,554.32894 - B \]

\[ 5(38,554.32894 - B) + 20B = 10(38,554.32894) \]
\[ 15B = 385,543.2894 - 192,771.6447 \]
\[ B = 12,851.44 \]

\[ A = 38,554.32894 - 12,851.44 \]
\[ A = 25,702.89 \]
24. Lauren wants to fully immunize a future payment of $X$ at time $Y$ using the following two bonds:
   a. Bond A is a zero coupon bond maturing in 2 years; and
   b. Bond B is a zero coupon bond maturing in 10 years.

Lauren pays 13,622.79 for Bond A and 6,192.18 for Bond B. Determine $X$ and $Y$ if the annual effective interest rate of 5%.

**Solution:**

**Present Value Matching:**

$$x(1.05)^{-y} = 13622.79 + 6192.18$$

**Duration Matching:**

$$yx(1.05)^{-y} = 2(13622.79) + 10(6192.18)$$

Now we have two equations with two unknowns, so we can solve for $x$ and $y$:

$$x = 19814.97(1.05)^y$$

$$y(19,814.97(1.05)^y)(1.05)^{-y} = 89,167.38$$

$$y = \frac{89,167.38}{19,814.97}$$

$$y = 4.5$$

$$x = 19,814.97(1.05)^{4.5}$$

$$x = 24,680.00$$