Chapter 10 Homework

1. Kristen is exact age 30 and has a current salary of 52,000. Kristen’s salary is assumed to increase continually.

   The salary scale function is \((1.0375)^{y-20}\) for \(y > 20\).

   a. What will Kristen’s salary rate be at age 35?

   **Solution:**

   \[
   52,000(1.0375)^5 = 62,509
   \]

   b. What will Kristen’s salary rate be at age 39.5?

   **Solution:**

   \[
   52,000(1.0375)^{9.5} = 73,772
   \]

   c. What salary will Kristen earn from age 39 to 40?

   **Solution:**

   \[
   52,000 \int_{9}^{10} (1.0375)^t \, dt = 52,000 \left[ \frac{(1.0375)^t}{\ln(1.0375)} \right]_{9}^{10} = \]

   \[
   52,000 \left[ \frac{(1.0375)^{10} - (1.0375)^9}{\ln(1.0375)} \right] = 73,776
   \]

2. Kristen is exact age 30 and has a current salary of 52,000.

   Kristen’s salary is assumed to increase by 5% each year on the date 8 months after her birthdate.

   a. What will Kristen’s salary rate be at age 35?

   **Solution:**

   \[
   52,000(1.05)^5 = 66,366
   \]
b. What will Kristen’s salary rate be at age 39.5?

Solution:

\[ 52,000(1.05)^9 = 80,669 \]

c. What salary will Kristen earn from age 39 to 40?

Solution:

\[ 52,000(1.05)^9 = 80,669 \] is the salary for the first 8 months of the year.

\[ 52,000(1.05)^{10} = 84,703 \] is the salary for the first 8 months of the year.

\[ \frac{(80,669) \times 8}{12} + \frac{(84,703) \times 4}{12} = 82,014 \]

3. Ningzhu is exact age 35. He earned a salary of 75,000 during the year prior to his 35th birthday.

Ningzhu’s salary will increase by 3.5% each year on his birthday.

The retirement benefit for Ningzhu will be based on his average salary during the five years prior to retirement. Assuming that Ningzhu will retire on his 65th birthday, calculate his five year final average salary.

Solution:

Salary from 60 to 61 = 75,000(1.035)^{61-35}
Salary from 61 to 62 = 75,000(1.035)^{62-35}
Salary from 62 to 63 = 75,000(1.035)^{63-35}
Salary from 63 to 64 = 75,000(1.035)^{64-35}
Salary from 64 to 65 = 75,000(1.035)^{65-35}

\[ 5 – Year – Average = \frac{75,000(1.035)^{61-35}[1 + 1.035 + 1.035^2 + 1.035^3 + 1.035^4]}{5} = \]

196,746
4. Kexin’s current salary is 150,000. She is exact age 46.

Her pension will be based on her career average salary between now and her retirement which is assumed to occur on her 67th birthdate.

The salary scale is \( (1.045)^t \) and salaries are assumed to increase continuously.

Calculate Kexin’s career average salary.

**Solution:**

Mid-Year Rule

\[
\frac{150,000(1.045^{0.5} + 1.045^{1.5} + \ldots + 1.045^{20.5})}{67 - 46} = 150,000 \left[ \frac{1.045^{0.5} - 1.045^{21.5}}{1 - 1.045} \right] = 246,678
\]

Exact Integration

\[
\int_{0}^{21} (1.045)^t \, dt = \frac{150,000 \left[ 1.045^{21} - 1.045^0 \right]}{\ln(1.045)} = 246,698
\]
5. Roberts Corporation targets a 65% replacement ratio with their Defined Contribution plan. Given the following assumptions, calculate the contribution rate that must be paid to meet the 65% replacement ratio.

   a. A new employee is exactly 30 years old and is assumed to retire at age 68.

   b. The benefit to be paid at retirement is assumed to be a survivor annuity with annual payments at the start of each year. Assume that the new employee’s spouse is ten years younger than that employee.

   c. After retirement, assume that mortality follows the Illustrative Life Table and the Plan will earn 6%.

   d. Prior to retirement, assume that contributions will be made annually at the end of each year. Further, assume that contributions will earn 7.5%. Upon termination prior to retirement, the accumulated contributions will be paid out to the employee or the employee’s family.

   e. Salaries are assumed to increase annually at 3% on the employee’s birthday.

Determine the contribution rate that the Defined Contribution Plan will need to make each year to achieve the targeted 65% replacement ratio.
Solution:

Final Salary = $S(1.03)^{37}$

Funds needed = $(0.65)\left[S(1.03)^{37}\right]\left[\bar{a}_{68:58}\right] = (0.65)\left[S(1.03)^{37}\right]\left[\bar{a}_{68} + \bar{a}_{58} - \bar{a}_{68:58}\right] = (0.65)\left[S(1.03)^{37}\right][9.1066 + 11.6133 - 8.0938]$

Contributions are $x\%$ of salary increased with interest =

$x(S)\frac{(1.075)^{37} - (1.03)^{38}(1.075)^{-1}}{1 - 1.03/1.075}$

Set contributions equal to funds needed and solve for $x$

$x(S)\frac{(1.075)^{37} - (1.03)^{38}(1.075)^{-1}}{1 - 1.03/1.075} = (0.65)\left[S(1.03)^{37}\right][9.1066 + 11.6133 - 8.0938]$

$x = \frac{(0.65)\left[S(1.03)^{37}\right][9.1066 + 11.6133 - 8.0938]}{(1.075)^{37} - (1.03)^{38}(1.075)^{-1}} = \frac{24.49965}{278.65522} = 8.8\%$
6. Using the Illustrative Service Table, calculate the following:
   a. Probability that an employee age 30 will withdraw from employment after age 55.

   **Solution:**
   \[
   25\% q^{(w)}_{30} = \frac{d^{(w)}_{55} + d^{(w)}_{56} + d^{(w)}_{57} + d^{(w)}_{58} + d^{(w)}_{59}}{l^{(x)}_{30}} = \frac{213 + 182 + 178 + 148 + 120}{100,000} = 0.00841
   \]

   b. Probability that a person age 40 still be employed at age 60.

   **Solution:**
   \[
   20\% p^{(x)}_{40} = \frac{l^{(x)}_{60}}{l^{(x)}_{40}} = \frac{23,856}{36,943} = 0.64575
   \]

   c. For an employee that is currently exact age 65, calculate the expected future working life of this person. Assume that decrements are uniformly distributed with in each year of age in the multiple decrement table. This is the same as the complete future lifetime except that it is time spent working.

   **Solution:**
   Remember that under UDD, the complete expectation=curtate expectation +0.5

   Curtate Expectation=$\sum_{t=1}^{\infty} t\% p^{(x)}_{65} = \frac{l^{(x)}_{66} + l^{(x)}_{67} + ...}{l^{(x)}_{65}} = \frac{6594 + 5145 + 3504 + 2040 + 987}{11,246} = 1.6246$

   Complete Expectation = 1.6246 + 0.5 = 2.12
7. Weichen began working on January 1, 1990 at age 30 with a salary of 40,000.

You are given:

i. The annual retirement benefit is 2% of the final five year average salary for each year of service.

ii. Weichen retires on January 1, 2025.

iii. Assume that every January 1, Weichen receives a 5% increase in salary.

a. Calculate Weichen’s annual retirement benefit.

Solution:

Weichen will have worked 35 years so at 2% per year, he will get 70% of the final 5 year average salary.

Salary from 60 to 61 = 40,000(1.05)^{60-30}
Salary from 61 to 62 = 40,000(1.05)^{61-30}
Salary from 62 to 63 = 40,000(1.05)^{62-30}
Salary from 63 to 64 = 40,000(1.05)^{63-30}
Salary from 64 to 65 = 40,000(1.05)^{64-30}

\[
\text{Annual Retirement Benefit} = (0.70) \frac{40,000(1.05)^{30} + \ldots + 40,000(1.05)^{34}}{5} = \frac{40,000(1.05)^{30} - 40,000(1.05)^{35}}{(0.70) \frac{1-1.05}{5}} = 133,736
\]

b. Calculate his replacement ratio.

Solution:

\[
R = \frac{133,736}{40,000(1.05)^{64-30}} = 0.636
\]
c. If the annual retirement benefit was 2.5% of career average salary, determine his annual retirement benefit.

Solution:

\[ Benefit = (0.025)(40,000)(1 + 1.05 + 1.05^2 + ... + 1.05^{34}) = \]

\[ (0.025)(40,000) \left( \frac{1 - 1.05^{35}}{1 - 1.05} \right) = 90,320 \]
8. JT is a participant in a career average pension plan and he is currently exact age 50.

You are given:

i. The annual benefit upon retirement at age 65 is 1.5% of total earnings during JT’s years of service. JT will retire at age 65 and his benefit will be paid monthly with the first payment at age 65. The retirement benefit will be based solely on JT’s life.

ii. JT’s current salary after the increase on his birthday is 82,000. Future salary increases will occur on birthdays and be equal to 3.5%.

iii. JT’s total past earnings are 700,000.

iv. Death is the only decrement prior to retirement and \( P_{50} = 0.9 \).

v. \( \overline{a}_{65}^{(12)} \) is determined using the Illustrative Life Table and 6% interest. Further, assume that deaths are uniformly distributed between integral ages.

Using an interest rate of 6%, calculate the actuarial present value at age 50 of JT’s annual retirement benefit.

**Solution:**

First determine the amount of the benefit:

\[
(0.015) \times \text{Lifetime earnings} = (0.015) \times (\text{Past Earnings} + \text{Future Earnings}) =
\]

\[
(0.015) \times [700,000 + (82,000)(1 + 1.035^1 + 1.035^2 + ... + 1.035^{14})] =
\]

\[
(0.015) \times \left[ 700,000 + 82,000 \left( \frac{1 - 1.035^{15}}{1 - 1.035} \right) \right] = 34,233.69
\]

\[
PV_{65} = (34,233.69) \overline{a}_{65}^{(12)} = (34,233.69) \left[ \alpha(12) \overline{a}_{65} - \beta(12) \right] =
\]

\[
(34,233.69) \left[ (1.00028)(9.8969) - (0.46812) \right] = 322,876.80
\]

\[
PV_{50} = (PV_{65})(P_{50})^{y_{15}} = (322,876.80)(0.90)(1.06)^{-15} = 121,253
\]
9. Wanling is currently exact age 50. She just received a salary increase and her salary is 112,000. She is in defined benefit plan and will retire at exact age 65. She has been in the plan since exact age 30. The retirement benefit at age 65 is 1% of her final three year average salary.

Wanling will receive salary increases at a rate of 2.5% on each birthday.

Wanling wants to retire with a total annual retirement income of 100,000. She knows that the defined benefit plan will not provide an annual benefit that is that large. Therefore, she wants to make contributions into a defined contribution plan so that she will total annual retirement income at 65 will be 100,000. Retirement benefits will commence at exact age 65 and be paid annual at the beginning of the year.

a. If \( \ddot{a}_{65} = 10 \), calculate the amount that Wanling must have in the defined contribution plan at age 65 to achieve her annual benefit of 100,000.

**Solution:**

Benefit from DB Plan =

\[
(0.01)(65-30) \frac{112,000(1.025)^{12} + 112,000(1.025)^{13} + 112,000(1.025)^{14}}{3} = 54,048.62
\]

Amount in the DC Plan = \((100,000 - 54,048.62)(\ddot{a}_{65}) = 459,514\)

b. Assume that Wanling makes contributions that are \( x\% \) of salary into the defined contribution plan at the end of each year for the next 15 years. Further assume that she earns 7% in the defined contribution plan. Determine \( x \) in order to fund the amount needed in Part a.

**Solution:**

\[
x(112,000)[1.07^{14} + (1.025)(1.07)^{13} + ... + (1.025)^{14}(1.07)^{0}] = 459,514
\]

\[
x(112,000) \left[ \frac{1.07^{14} - (1.025)^{15}(1.07)^{-1}}{1 - 1.025/1.07} \right] = 459,514
\]

\[x = 14.09\%\]
Answers

1.
   a. 62,509
   b. 73,771
   c. 73,776

2.
   a. 66,367
   b. 80,669
   c. 82,014

3. 196,746

4. 246,678 using midyear rule or 246,698 using exact integration.

5. 8.8%

6.
   a. 0.00841
   b. 0.64575
   c. 2.12 years

7.
   a. 133,736
   b. 0.636
   c. 90,320

8. 121,253

9.
   a. 459,514
   b. 15.5%