1. You are given the following mortality table:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$q_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.20</td>
</tr>
<tr>
<td>101</td>
<td>0.30</td>
</tr>
<tr>
<td>102</td>
<td>0.45</td>
</tr>
<tr>
<td>103</td>
<td>0.70</td>
</tr>
<tr>
<td>104</td>
<td>1.00</td>
</tr>
</tbody>
</table>

You are given the following term structure of interest rates:

<table>
<thead>
<tr>
<th>$t$</th>
<th>$y_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.50%</td>
</tr>
<tr>
<td>2</td>
<td>4.00%</td>
</tr>
<tr>
<td>3</td>
<td>4.40%</td>
</tr>
<tr>
<td>4</td>
<td>4.75%</td>
</tr>
<tr>
<td>5</td>
<td>5.00%</td>
</tr>
</tbody>
</table>

Using this information, calculate:

a. $1000A_{100}$

**Solution:**

$$1000A_{100} = [1000v(1) \cdot q_{100} + v(2) \cdot p_{100} \cdot q_{101} + v(3) \cdot p_{100} \cdot p_{101} \cdot q_{102} + v(4) \cdot p_{100} \cdot p_{101} \cdot p_{102} \cdot q_{103} + v(5) \cdot p_{100} \cdot p_{101} \cdot p_{102} \cdot p_{103} \cdot q_{104}]$$

$$= 1000(1.035)^{-1}(0.2) + (1.04)^{-2}(0.8)(0.3) + (1.044)^{-3}(0.8)(0.7)(0.45)$$
$$+ (1.0475)^{-4}(0.8)(0.7)(0.55)(0.7) + (1.05)^{-5}(0.8)(0.7)(0.55)(0.3)(1)]$$

$$= 888.06$$
b. \( \ddot{a}_{100} \)

**Solution:**

\[
\ddot{a}_{100} = 1 + v(1) \cdot p_{100} + v(2) \cdot p_{100} + v(3) \cdot p_{100} + v(4) \cdot p_{100}
\]

\[
= 1 + (1.035)^{-1}(0.8) + (1.04)^{-2}(0.8)(0.7) + (1.044)^{-3}(0.8)(0.7)(0.55) + (1.0475)^{-4}(0.8)(0.7)(0.55)(0.3)
\]

\[
= 2.63812
\]

c. The annual premium that would be paid for a fully discrete whole life of 1000 on (100).

**Solution:**

\[
P \ddot{a}_{100} = 1000A_{100} \implies P \frac{1000A_{100}}{\ddot{a}_{100}} = \frac{888.06}{2.63812} = 336.63
\]

d. The one year forward rates - \( f(0,1); f(1,2); f(2,3); f(3,4); \text{ and } f(4,5) \).

**Solution:**

\[
f(0,1) = y_1 = 3.5\%
\]

\[
f(1,2) = \frac{(1 + y_2)^2}{(1 + y_1)} - 1 = \frac{1.04^2}{1.035} - 1 = 4.5024\%
\]

\[
f(2,3) = \frac{(1 + y_3)^3}{(1 + y_2)^2} - 1 = \frac{1.044^3}{1.04^2} - 1 = 5.2046\%
\]

\[
f(3,4) = \frac{(1 + y_4)^4}{(1 + y_3)^3} - 1 = \frac{1.0475^4}{1.044^3} - 1 = 5.8071\%
\]

\[
f(4,5) = \frac{(1 + y_5)^5}{(1 + y_4)^4} - 1 = \frac{1.05^5}{1.0475^4} - 1 = 6.0060\%
\]
e. The reserve at time $t=0, 1, 2, 3, 4, \text{ and } 5$.

**Solution:**

\[ V = 0 \]

\[ V = 0 \]

\[ V = \frac{[V + P][1 + f(0,1)] - S_{q100}}{p_{100}} = \frac{[0 + 336.63][1.035] - (1000)(0.2)}{1 - 0.2} = 185.51 \]

\[ V = \frac{[V + P][1 + f(1,2)] - S_{q101}}{p_{101}} = \frac{[185.51 + 336.63][1.045024] - (1000)(0.3)}{1 - 0.3} = 350.93 \]

\[ V = \frac{[V + P][1 + f(2,3)] - S_{q102}}{p_{102}} = \frac{[350.93 + 336.63][1.052046] - (1000)(0.45)}{1 - 0.45} = 496.98 \]

\[ V = \frac{[V + P][1 + f(3,4)] - S_{q103}}{p_{103}} = \frac{[496.98 + 336.63][1.058071] - (1000)(0.7)}{1 - 0.7} = 606.74 \]

f. $f(2,5)$

**Solution:**

\[ f(2,5) = \left[ \frac{(1 + y_2)^2}{(1 + y_2)^2} \right]^{\frac{1}{2}} - 1 = \left[ \frac{1.05^5}{1.04^5} \right]^{\frac{1}{2}} - 1 = 5.672\% \]
2. You are given the following spot interest rate curve:

<table>
<thead>
<tr>
<th>Time in Years</th>
<th>Spot Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>1.2%</td>
</tr>
<tr>
<td>1.0</td>
<td>1.5%</td>
</tr>
<tr>
<td>1.5</td>
<td>1.9%</td>
</tr>
<tr>
<td>2.0</td>
<td>2.2%</td>
</tr>
<tr>
<td>2.5</td>
<td>2.4%</td>
</tr>
<tr>
<td>3.0</td>
<td>2.5%</td>
</tr>
</tbody>
</table>

Sorge Life Insurance Company issues a 3 year temporary life annuity due to (80) which pays 1000 annually.

You are also given that mortality follows the Illustrative Life Table.

Let $Y$ be the present value random variable for this annuity.


Solution:

$$E[Y] = 1000[1 + p_{80}v_{0.015} + p_{80}p_{81}v_{0.022}^2]$$

$$= 1000 \left[1 + \frac{(1-0.0803)}{1.015} + \frac{(1-0.0803)(1-0.08764)}{(1.022)^2}\right] = 2709.47$$
b. Calculate the $Var[Y]$.

Solution:

$Y$ can take on 3 values

1000 if (80) dies in the first year $\text{Prob}=0.0803$
1000+1000(1.015)$^{-1}$ = 1985.22 if (80) dies in year 2 $\text{Prob} = (1-0.0803)(0.08764)$
1000+1000(1.015)$^{-1}$ + 1000(1.022)$^{-2}$ = 2942.63 if (80) lives 2 years $\text{Prob} = (1-0.0803)(1-0.08764)$

$E[Y^2] = (1000)^2(0.0803) + (1985.22)^2(1-0.0803)(0.08764) + (2942.63)^2(1-0.0803)(1-0.08764)$

$= 7,663,767.44$

$Var[Y] = E[Y^2] - (E[Y])^2 = 7,663,767.44 - (2709.47)^2 = 322,540$

The difference is rounding. If you carry all decimals everywhere, you get the answer given.
3. Let the random variable $D(N)$ be the number of deaths that occur in the next year if there are $N$ lives insured. Assumes that the probability of death in the next year is 10%.

Calculate:

a. $E[D(N)]$

**Solution:**

$$E[D(N)] = Nq_s = 0.1N$$

b. $Var[D(N)]$

**Solution:**

$$Var[D(N)] = Nq_sp_s = N(0.1)(1-0.1) = 0.09N$$

c. $N$ so that standard deviation $D(N)$ is equal to 10% of $E[D(N)]$

**Solution:**

Standard Deviation = $\sqrt{0.09N} = 0.3\sqrt{N}$

$$E[D(N)] = 0.1N$$

$$0.3\sqrt{N} = (0.1)(0.1N) \implies N = 900$$

d. If $N = 1000$, the probability that the number of deaths will exceed 110. Assume that normal distribution holds.

**Solution:**

$$E[D(1000)] = 100$$

$$Var[D(1000)] = 90$$

$$\Phi = \frac{110 - 100}{\sqrt{90}} = 1.054 \implies Pr(D > 110) = 1 - 0.8531 = 0.1469$$
4. Let the random variable \( D(N) \) be the number of deaths that occur in the next year if there are \( N \) lives insured.

Assumes that the probability of death in the next year is distributed as follows:

<table>
<thead>
<tr>
<th>Probability of Death</th>
<th>Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>8%</td>
<td>0.25</td>
</tr>
<tr>
<td>10%</td>
<td>0.50</td>
</tr>
<tr>
<td>12%</td>
<td>0.25</td>
</tr>
</tbody>
</table>

Calculate:

a. \( E[D(N)] \)

Solution:

\[
E[D(N)] = (0.08N) \times 0.25 + (0.1N) \times 0.5 + (0.12N) \times 0.25 = 0.1N
\]

b. \( Var[D(N)] \)

Solution:

\[
Var[D(N)] = E[Var(D \mid i)] + Var[E(D \mid i)]
\]
\[
E[Var(D \mid i)] = (Np_iq_i)i_i + (Np_2q_2)i_2 + (Np_3q_3)i_3
\]
\[
= N(0.08)(0.92)0.25 + N(0.1)(0.9)0.50 + N(0.12)(0.88)0.25 = 0.0898N
\]
\[
Var[E(D \mid i)] = Var[Np_i] = N^2(E(p_i^2) - [E(p_i)]^2)
\]
\[
= N^2 \{[(0.08^2)(0.25) + (0.1^2)(0.50) + (0.12^2)(0.25)] - [(0.08)(0.25) + (0.1)(0.50) + (0.12)(0.25)]^2 \} = 0.0002N^2
\]
\[
\therefore Var[D(N)] = 0.0898N + 0.0002N^2
\]

c. If \( N = 1000 \), the probability that the number of deaths will exceed 110. Assume that normal distribution holds.

Solution:

\[
P(D > 110) = P(Z > \frac{110 - 0.1 \times 1000}{\sqrt{0.0898 \times 1000 + 0.0002 \times 1000^2}}) = 1 - \Phi(0.59) = 1 - 0.7224 = 0.2776
\]
5. A one year term insurance on \((x)\) provides a death benefit of 100,000 at the end of the year of death.

You are given that \(q_x = 0.0525\). The annual premium is determined assuming \(i = 5\%\).

da. (5 points) Calculate the annual premium for this policy.

**Solution:**

\[
PVP = PVB
\]

\[
P = 100,000q_x = (100,000) \frac{0.0525}{1.05} = 5000
\]

The actual interest rate is a random variable with a distribution of:

<table>
<thead>
<tr>
<th>Rate</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.04</td>
<td>0.30</td>
</tr>
<tr>
<td>0.05</td>
<td>0.50</td>
</tr>
<tr>
<td>0.06</td>
<td>0.20</td>
</tr>
</tbody>
</table>

Let \(L_0\) be the loss-at-issue random variable for this policy using the premium developed in Part a.
b. Calculate $E[L_0]$

**Solution:**

\[
i = 0.04 \text{& Die} \implies L_0 = \frac{100,000}{1.04} - 5000 = 91,153.85 \implies \Pr = (0.0525)(0.3)
\]

\[
i = 0.04 \text{& Live} \implies L_0 = -5000 = -5000 \implies \Pr = (1 - 0.0525)(0.3)
\]

\[
i = 0.05 \text{& Die} \implies L_0 = \frac{100,000}{1.05} - 5000 = 90,238.10 \implies \Pr = (0.0525)(0.5)
\]

\[
i = 0.05 \text{& Live} \implies L_0 = -5000 = -5000 \implies \Pr = (1 - 0.0525)(0.5)
\]

\[
i = 0.06 \text{& Die} \implies L_0 = \frac{100,000}{1.06} - 5000 = 89,339.62 \implies \Pr = (0.0525)(0.2)
\]

\[
i = 0.06 \text{& Live} \implies L_0 = -5000 = -5000 \implies \Pr = (1 - 0.0525)(0.2)
\]

\[
E[L_0] = 91,153.85(0.0525)(0.3) + 89,339.62(0.0525)(0.2) - 5000[(1 - 0.0525)(0.3) + (1 - 0.0525)(0.2)]
\]

\[
= 4.99
\]

The values at 5% can be ignored since they contribute 0 as $E[L_0]$ is 0 at 5% interest.

c. Calculate the standard deviation of $L_0$

**Solution:**

We will use the information from above

\[
E[L_0^2] = (91,153.85)^2(0.0525)(0.3) + (90,238.10)^2(0.0525)(0.5)
\]

\[
+ (89,339.62)^2(0.0525)(0.2) + (-5000)^2(1 - 0.0525) = 452,112,605.3
\]

\[
Var[L_0] = E[L_0^2] - (E[L_0])^2 = 452,112,605.3 - (4.99)^2 = 452,112,580.4
\]

Standard Deviation $= \sqrt{452,112,580.4} = 21,263$

You could do this as $Var[E[L_0 \mid i]] + E[Var[L_0 \mid i]]$ and you would get the same answer.
6. You are given the following mortality table:

<table>
<thead>
<tr>
<th>x</th>
<th>l_x</th>
<th>q_x</th>
<th>p_x</th>
</tr>
</thead>
<tbody>
<tr>
<td>90</td>
<td>1000</td>
<td>0.10</td>
<td>0.90</td>
</tr>
<tr>
<td>91</td>
<td>900</td>
<td>0.20</td>
<td>0.80</td>
</tr>
<tr>
<td>92</td>
<td>720</td>
<td>0.40</td>
<td>0.60</td>
</tr>
<tr>
<td>93</td>
<td>432</td>
<td>0.50</td>
<td>0.50</td>
</tr>
<tr>
<td>94</td>
<td>216</td>
<td>1.00</td>
<td>0.00</td>
</tr>
<tr>
<td>95</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a. Calculate the net annual premium for a fully discrete whole life of 100,000 on (91) at 4%.

**Solution:**

\[
PVP = PVB
\]

\[
P = \left[ 1 + (0.8) \left( \frac{1}{1.04} \right) + (0.8)(0.6) \left( \frac{1}{1.04} \right)^2 + (0.8)(0.6)(0.5) \left( \frac{1}{1.04} \right)^3 \right]
\]

\[
= 100,000 \left[ 0.2 \left( \frac{1}{1.04} \right) + (0.8)(0.4) \left( \frac{1}{1.04} \right)^2 + (0.8)(0.6)(0.5) \left( \frac{1}{1.04} \right)^3 + (0.8)(0.6)(0.5) \left( \frac{1}{1.04} \right)^4 \right]
\]

\[
2.426P = 90,667.78
\]

\[
P = 37,367.56
\]
b. Assume that the actual interest rate is distributed as follows:

<table>
<thead>
<tr>
<th>Interest Rate</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.5%</td>
<td>0.3</td>
</tr>
<tr>
<td>4.0%</td>
<td>0.4</td>
</tr>
<tr>
<td>4.5%</td>
<td>0.3</td>
</tr>
</tbody>
</table>

Calculate $E[L_0]$ and $Var[L_0]$ where $L_0$ is the loss at issue random variable.

**Solution:**

\[
E[L_0] = (0.3)[E[L_0]|i = .035] + (0.4)[E[L_0]|i = .04] + (0.3)[E[L_0]|i = .045]
\]

\[
[E[L_0]|i = .035] = 100,000 \left[ (.2)(\frac{1}{1.035}) + (.8)(.4)(\frac{1}{1.035})^2 + (.8)(.6)(.5)(\frac{1}{1.035})^3 + (.8)(.6)(.5)(\frac{1}{1.035})^4 \right]
\]

\[
-37,367.56 \left[ 1 + (.8)(\frac{1}{1.035}) + (.8)(.6)(\frac{1}{1.035})^2 + (.8)(.6)(.5)(\frac{1}{1.035})^3 \right] = 731.70
\]

\[
[E[L_0]|i = .04] = 100,000 \left[ (.2)(\frac{1}{1.04}) + (.8)(.4)(\frac{1}{1.04})^2 + (.8)(.6)(.5)(\frac{1}{1.04})^3 + (.8)(.6)(.5)(\frac{1}{1.04})^4 \right]
\]

\[
-37,367.56 \left[ 1 + (.8)(\frac{1}{1.04}) + (.8)(.6)(\frac{1}{1.04})^2 + (.8)(.6)(.5)(\frac{1}{1.04})^3 \right] = 0.00
\]

\[
[E[L_0]|i = .045] = 100,000 \left[ (.2)(\frac{1}{1.045}) + (.8)(.4)(\frac{1}{1.045})^2 + (.8)(.6)(.5)(\frac{1}{1.045})^3 + (.8)(.6)(.5)(\frac{1}{1.045})^4 \right]
\]

\[
-37,367.56 \left[ 1 + (.8)(\frac{1}{1.045}) + (.8)(.6)(\frac{1}{1.045})^2 + (.8)(.6)(.5)(\frac{1}{1.045})^3 \right] = -659.34
\]

\[
E[L_0] = (731.70)(0.3) + (0.00)(0.4) + (-659.34)(0.3) = 21.709
\]
7. *You are given:

i. \( Z(n) \) is the present value random variable for an \( n \)-year term insurance with a death benefit of 1000 payable at the end of the year of death on a life age \( x \), based on the current yield curve.

ii. \( E[Z(1)] = 19.04762 \) and \( E[Z(2)] = 41.14340 \)

iii. The current one year spot interest rate is 5%.

iv. \( q_{x+1} = 0.025 \)

Calculate the current two year spot interest rate.

**Solution:**

\[
E[Z(1)] = 1000v_x q_x = 19.04762 \implies q_x = \frac{19.04762}{1000(1.05)^{-1}} = 0.02
\]

\[
E[Z(2)] = 1000(v(1)q_x + v(2)p_x q_{x+1}) = 1000[(1.05)^{-1}(0.02) + v(2)(0.98)(0.025)] = 41.14340
\]

\( v(2) = 0.90186861 \)

\[
y_2 = \left(\frac{1}{0.90186861}\right)^{\frac{1}{2}} - 1 = 5.3\%
\]
8. A pension plan for Flens Enterprises provides a death benefit equal to two times current annual salary in the year of death. The death benefit is paid at the end of the year of death. This benefit ends at retirement.

Xintong, an employee at Flens, is exact age 62. She began employment at Flens at exact age 40 with a salary of 82,000. Flens gives 3% salary increases on each employee’s birthday.

Xintong will retire at exact age 65 unless she dies prior to age 65.

You are given:

i. \( q_{62+k} = 0.05 + 0.015k \) for \( k = 0, 1, 2, 3 \)

ii. The \( k \)-year spot interest rates on Xintong’s 62\textsuperscript{nd} birthday are \( 0.040 + 0.004k \) for \( k = 1, 2, 3 \)

Calculate the actuarial present value of Xintong’s death benefit on her 62\textsuperscript{nd} birthday.

Solution:

\[
PVB = (82,000)(2)(1.03)^{-2} \left[ \frac{1}{1.044}\left(1+0.044\right) + (1.03)(0.95)(0.065) \left\{ \frac{1}{1.048} \right\} \right] = 53,595.16
\]