Chapter 8

1. Mayfawny purchases a whole life insurance policy.

There are three ways that Mayfawny’s policy can terminate:
   a. Death (1)
   b. Diagnosis of a critical illness (2); and
   c. Lapse (3).

The policy pays a death benefit of 10,000 at the moment of death. The policy will also pay a critical illness benefit of 20,000 if Mayfawny is diagnosed with a critical illness. Only one benefit will be paid.

There is no benefit paid upon lapse.

You are also given:
   i. \( \mu_s^{(1)} = 0.01 \)
   ii. \( \mu_s^{(2)} = 0.015 \)
   iii. \( \mu_s^{(3)} = 0.06 \)
   iv. \( \delta = 0.035 \)

Mayfawny pays a net premium continuously for her lifetime as long as the policy is in force. The net premium is determined using the equivalence principle.

Calculate the net premium paid by Mayfawny.
Solution:

\[ PVP = PVB \implies P\overline{a}_x^{00} = 10,000\overline{A}_x^{01} + 20,000\overline{A}_x^{02} + 0\overline{A}_x^{03} \]

\[ \overline{a}_x^{00} = \int_0^\infty v^i \cdot p_x^{(r)} \cdot dt = \int_0^\infty e^{-0.035t} \cdot e^{-0.015t} \cdot dt = \int_0^\infty e^{-0.12t} \cdot dt = \frac{1}{0.12} \]

\[ \overline{A}_x^{01} = \int_0^\infty v^i \cdot p_x^{(r)} \cdot \mu_x^{(1)} \cdot dt = \int_0^\infty e^{-0.035t} \cdot e^{-0.085t} \cdot 0.01 \cdot dt = \frac{0.01}{0.12} \]

\[ \overline{A}_x^{02} = \int_0^\infty v^i \cdot p_x^{(r)} \cdot \mu_x^{(2)} \cdot dt = \int_0^\infty e^{-0.035t} \cdot e^{-0.085t} \cdot 0.015 \cdot dt = \frac{0.015}{0.12} \]

\[ P\overline{a}_x^{00} = 10,000\overline{A}_x^{01} + 20,000\overline{A}_x^{02} + 0\overline{A}_x^{03} \implies P = \frac{10,000\overline{A}_x^{01} + 20,000\overline{A}_x^{02} + 0\overline{A}_x^{03}}{\overline{a}_x^{00}} = \]

\[ \frac{10,000 \left( \frac{0.01}{0.12} \right) + 20,000 \left( \frac{0.015}{0.12} \right) + 0}{\left( \frac{0.12}{0.12} \right)} = 400 \]
2. Jeff is receiving a salary paid continuously for as long as he is in employed at Purdue. Jeff can leave employment through death (1), retirement (2), or disability (3). Once Jeff leaves employment, the salary stops.

You are given:

i. The salary pays at an annual rate of 70,000 per year.
ii. $\delta = 0.05$
iii. Jeff is currently age 59.
iv. Jeff will retire at age 65 if he is still teaching. He will not retire prior to age 65.

v. $\mu_{59(1)} = 0.01 + 0.001t$
vi. $\mu_{59(3)} = 0.03 - 0.001t$

Calculate the present value of Jeff’s future earnings while employed at Purdue.

Solution:

$$PV = 70,000 \int_0^6 v' \cdot p_{59}^{(r)} \cdot dt$$

The limits on the integral are determined by Jeff’s retirement date.

$$v' = e^{-\delta t} = e^{-0.05t}$$

$$p_{59}^{(r)} = e^{-\int_0^t \mu_{59(1)} \cdot dt} = e^{-\int_0^t (0.01+0.001t) \cdot dt} = e^{-\int_0^t 0.01 \cdot dt + \int_0^t 0.001t \cdot dt} = e^{-0.01 \cdot t + 0.0005 \cdot t^2} = e^{-0.005t^2}$$

$$\therefore PV = 70,000 \int_0^6 e^{-0.05t} \cdot e^{-0.005t^2} \cdot dt = 70,000 \int_0^6 e^{-0.09t^2} \cdot dt =$$

$$70,000 \left[ \frac{e^{-0.09t^2}}{-0.09} \right]_0^6 = 70,000 \left[ \frac{e^{-0.09\cdot 6^2}}{-0.09} \right] = 324,529.14$$
3. You are given the following table where decrement (1) is death, decrement (2) is lapse, and decrement (3) is diagnosis of critical illness:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$q^{(1)}_x$</th>
<th>$q^{(2)}_x$</th>
<th>$q^{(3)}_x$</th>
<th>$p^{(1)}_x$</th>
<th>$r^{(1)}_x$</th>
<th>$d^{(1)}_x$</th>
<th>$d^{(2)}_x$</th>
<th>$d^{(3)}_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>55</td>
<td>0.02</td>
<td>0.15</td>
<td>0.010</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>56</td>
<td>0.03</td>
<td>0.06</td>
<td>0.015</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>57</td>
<td>0.04</td>
<td>0.04</td>
<td>0.020</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>58</td>
<td>0.05</td>
<td>0.03</td>
<td>0.025</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>59</td>
<td>0.06</td>
<td>0.02</td>
<td>0.030</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a. Complete the table using a radix of 10,000.

See answers at end.

b. Calculate:

   i. $3p^{(2)}_{55}$

   **Solution:**

   $$3p^{(2)}_{55} = \frac{r^{(2)}_{56}}{r^{(2)}_{55}} = \frac{6605.1}{10,000} = 0.66051$$

   ii. $2q^{(2)}_{56}$

   **Solution:**

   $$2q^{(2)}_{56} = \frac{d^{(2)}_{56} + d^{(2)}_{57}}{r^{(2)}_{56}} = \frac{492 + 293.56}{8200} = 0.0958$$

   iii. $q^{(3)}_{55}$

   **Solution:**

   $$q^{(3)}_{55} = \frac{d^{(3)}_{56} + d^{(3)}_{57}}{r^{(3)}_{55}} = \frac{123 + 146.78}{10,000} = 0.026978$$
iv. The probability that a person age 55 will decrement from death or critical illness before age 60.

Solution:

\[
d_{55}^{(1)} + d_{56}^{(1)} + \ldots + d_{59}^{(1)} + d_{55}^{(2)} + d_{56}^{(3)} + \ldots + d_{59}^{(3)} = \frac{2136.763305}{10,000} = 0.21368
\]

c. Assuming uniform distribution of each decrement between integer ages, calculate:

i. \( q_{55}^{(2)} \)

Solution:

\[
q_{55}^{(2)} = 0.25(q_{55}^{(2)}) = 0.25\left(\frac{1500}{10,000}\right) = 0.0375
\]

ii. \( p_{56}^{(z)} \)

Solution:

\[
p_{56}^{(z)} = 1 - 0.5 q_{56}^{(z)} = 1 - (0.5)(q_{56}^{(z)}) = 1 - (0.5)(0.105) = 0.9475
\]

iii. \( p_{56.8}^{(z)} \)

Solution:

\[
p_{56.8}^{(z)} = \frac{p_{57.3}^{(z)}}{p_{56.8}^{(z)}} = \frac{(0.7)p_{57}^{(z)} + (0.3)p_{58}^{(z)}}{(0.2)p_{56}^{(z)} + (0.8)p_{57}^{(z)}} = \frac{(0.7)(7339) + (0.3)(6605.1)}{(0.2)(8200) + (0.8)(7339)} = 0.94776
\]

iv. \( q_{55.6}^{(1)} \)

Solution:

\[
q_{55.6}^{(1)} = \frac{0.4 d_{55.6}^{(1)} + 0.1 d_{56}^{(1)}}{l_{55.6}^{(z)}} = \frac{(0.4)(200) + (0.1)(246)}{(0.4)(10,000) + (0.6)(8200)} = 0.01173
\]
d. Assuming a constant force of decrement for each decrement between integer ages, calculate:

i. \(0.25 q_{55}^{(2)}\)

Solution:

\[
0.25 q_{55}^{(2)} = \frac{q_{55}^{(2)}}{q_{55}^{(1)}} \left( 1 - (p_{55}^{(2)})^{0.25} \right) = \frac{0.15}{0.18} \left( 1 - (0.82)^{0.25} \right) = 0.04034
\]

ii. \(0.5 p_{56}^{(2)}\)

Solution:

\[
0.5 p_{56}^{(2)} = (p_{56}^{(2)})^{0.5} = (0.895)^{0.5} = 0.94604
\]

iii. \(0.5 P_{56.8}^{(2)}\)

Solution:

\[
0.5 P_{56.8}^{(2)} = \frac{P_{57.3}^{(2)}}{P_{56.8}^{(2)}} = \frac{P_{57.3}^{(2)}}{P_{56}^{(2)}} \left( p_{56}^{(2)} \right) = \frac{(7339)(0.9)^{0.3}}{(8200)(0.895)^{0.8}} = 0.94763
\]

iv. \(0.5 q_{55.6}^{(1)}\)

Solution:

\[
0.5 q_{55.6}^{(1)} = \frac{0.4 q_{55.6}^{(1)} + 0.1 d_{56}^{(1)}}{P_{55}^{(1)}} = \frac{\Delta_{55}^{(1)} - 0.4 d_{55}^{(1)} + 0.1 d_{56}^{(1)}}{\left( P_{55}^{(1)} \right)^{0.6}} = \frac{200 - (10,000 \left( \frac{0.02}{0.18} \right) (1 - (0.82)^{0.6}) + (8200 \left( \frac{0.03}{0.105} \right) (1 - (0.895)^{0.1})}{(10,000)(0.82)^{0.6}} = \frac{101.1184537}{8877.45156} = 0.011390
\]
4. A fully discrete 3 year term pays a benefit of 1000 upon any death. It pays an additional 1000 (for a total of 2000) upon death from accident. You are given:

<table>
<thead>
<tr>
<th>x</th>
<th>$q_x^{(1)}$</th>
<th>$q_x^{(2)}$</th>
<th>$y_x^{(1)}$</th>
<th>$d_x^{(1)}$</th>
<th>$d_x^{(2)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.030</td>
<td>0.010</td>
<td>1000</td>
<td>30</td>
<td>10</td>
</tr>
<tr>
<td>21</td>
<td>0.025</td>
<td>0.020</td>
<td>960</td>
<td>24</td>
<td>19.2</td>
</tr>
<tr>
<td>22</td>
<td>0.020</td>
<td>0.030</td>
<td>916.8</td>
<td>18.336</td>
<td>27.504</td>
</tr>
</tbody>
</table>

Decrement (1) is death from accidental causes while decrement (2) is death from non-accidental causes.

The annual effective interest rate is 10%.

a. Calculate the level annual net premium for this insurance.

**Solution:**
Columns above in yellow were added to facilitate the calculation.

\[ P_{VP} = P_{V} \]

\[ P(1000 + 960v + 916.8v^2) = 2000(30v + 24v^2 + 18.336v^3) + 1000(10v + 19.2v^2 + 27.504v^3) \]

\[ P = \frac{2000(30v + 24v^2 + 18.336v^3) + 1000(10v + 19.2v^2 + 27.504v^3)}{(1000 + 960v + 916.8v^2)} = 63.64 \]
b. Calculate the net premium reserve at the end of year 0, 1, 2, and 3.

**Solution:**

By Definition $V = 0$ and $V = 0$

We will use the recursive formula to get the other two reserves.

$$1_V = \frac{(0 + 63.64)(1.1) - (2000)(0.03) - (1000)(0.01)}{1 - 0.03 - 0.01} = 0.00$$

$$2_V = \frac{(1 + 63.64)(1.1) - (2000)(0.025) - (1000)(0.02)}{1 - 0.025 - 0.02} = 0.00$$

Do not erroneously draw the conclusion that all reserves are zero. It just so happens here. A poorly developed question. :)

5. You are given:
   a. $q_x^{(1)} = 0.200$
   b. $q_x^{(2)} = 0.080$
   c. $q_x^{(3)} = 0.125$

Assuming that each decrement is uniformly distributed over each year of age in the associated single decrement table, calculate $q_x^{(1)}$.

**Solution:**

$$q_x^{(1)} = q_x^{(1)} \left[ 1 - \frac{1}{2} \left[ q_x^{(2)} + q_x^{(3)} \right] + \frac{1}{3} \left[ q_x^{(2)} \cdot q_x^{(3)} \right] \right] =$$

$$(0.2) \left[ 1 - \frac{1}{2} [0.08 + 0.125] + \frac{1}{3} [0.08 \cdot 0.125] \right] = 0.180167$$
6. You are given:
   a. \( q_x^{(1)} = 0.200 \)
   b. \( q_x^{(2)} = 0.080 \)
   c. \( q_x^{(3)} = 0.125 \)

Assuming that each decrement in the multiple decrement table is uniformly distributed over each year of age, calculate \( q_x^{(1)} \).

Solution:

\[
p^{(r)}_x = p^{(1)}_x \cdot p^{(2)}_x \cdot p^{(3)}_x = (1 - q^{(1)}_x)(1 - q^{(2)}_x)(1 - q^{(3)}_x) = (0.8)(0.92)(0.875) = 0.644
\]

\[
p^{(1)}_x = \left(p^{(r)}_x\right) \frac{q^{(1)}_x}{1 - q^{(1)}_x} \implies 0.8 = (0.644)^{\frac{q^{(1)}_x}{1 - q^{(1)}_x}} \implies q^{(1)}_x = \frac{\ln(0.8)}{\ln(0.644)} (1 - 0.644) = 0.180520
\]

7. You are given the following for a double decrement table:
   a. \( q_x^{(1)} = 0.200 \)
   b. \( q_x^{(2)} = 0.080 \)

Assuming that each decrement in the multiple decrement table is uniformly distributed over each year of age, calculate \( 0.4q_x^{(4)} \).

Solution:

\[
p^{(r)}_x = p^{(1)}_x \cdot p^{(2)}_x = (1 - q^{(1)}_x)(1 - q^{(2)}_x) = (0.8)(0.92) = 0.736
\]

\[
p^{(1)}_x = \left(p^{(r)}_x\right) \frac{q^{(1)}_x}{1 - q^{(1)}_x} \implies 0.8 = (0.736)^{\frac{q^{(1)}_x}{1 - q^{(1)}_x}} \implies q^{(1)}_x = \frac{\ln(0.8)}{\ln(0.736)} (1 - 0.736) = 0.192186173
\]

\[
0.4q^{(4)}_x = \frac{0.4d^{(1)}_{x+0.4}}{l^{(r)}_{x+0.4}} = \frac{(0.4)(0.192186173)}{1 - (0.4)(1 - 0.736)} = 0.085951 \implies \text{This assumes } l^{(r)}_x = 1
\]
8. You are given:
   a. \( q_x^{(1)} = 0.200 \)
   b. \( q_x^{(2)} = 0.080 \)

   Decrement (1) is uniformly distributed over the year. Decrement (2) occurs at time 0.6.

   Calculate \( q_x^{(2)} \).

   **Solution:**

   Worked in class.

9. For a double decrement table with \( l_{40}^T = 2000 \):

<table>
<thead>
<tr>
<th>x</th>
<th>( q_x^{(1)} )</th>
<th>( q_x^{(2)} )</th>
<th>( q_x^{(1)}' )</th>
<th>( q_x^{(2)}' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>0.24</td>
<td>0.10</td>
<td>0.25</td>
<td>y</td>
</tr>
<tr>
<td>41</td>
<td>--</td>
<td>--</td>
<td>0.20</td>
<td>2y</td>
</tr>
</tbody>
</table>

   Calculate \( l_{42}^T \).

   **Solution:**

   \[
   p_{40}^{(r)} = 1 - q_{40}^{(1)} - q_{40}^{(2)} = p_{40}^{(1)} \cdot p_{40}^{(2)} \implies 1 - 0.24 - 0.10 = 0.66 = (1 - 0.25)(1 - y)
   \]

   \[
   y = 1 - \frac{0.66}{0.75} = 0.12 \implies 2 \cdot y = 0.24
   \]

   \[
   l_{42}^{(r)} = l_{40}^{(r)} \cdot p_{40}^{(r)} \cdot p_{41}^{(r)} = l_{40}^{(r)} \cdot p_{40}^{(1)} \cdot p_{40}^{(2)} \cdot p_{41}^{(1)} \cdot p_{41}^{(2)} =
   \]

   \[
   2000(1 - 0.25)(1 - 0.12)(1 - 0.2)(1 - 0.24) = 802.56
   \]
10. For iPhones, the phone may cease service for mechanical failure or for other reasons (lost, stolen, dropped in a pitcher of beer, etc). You are given the following double decrement table:

<table>
<thead>
<tr>
<th>Year of Service</th>
<th>For an iPhone at the beginning of the year of service, probability of</th>
<th>Survival through the year of service</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mechanical Failure</td>
<td>Failure for Other Reasons</td>
</tr>
<tr>
<td>1</td>
<td>0.2</td>
<td>0.30</td>
</tr>
<tr>
<td>2</td>
<td>0.40</td>
<td>--</td>
</tr>
<tr>
<td>3</td>
<td>0.20</td>
<td>--</td>
</tr>
</tbody>
</table>

You are also given:

a. The number of iPhones that terminate for other reasons in year 3 is 40% of the number of iPhones that survive to the end of year 2.

b. The number of iPhones that terminate for other reasons in year 2 is 80% of the number of iPhones that survive to the end of year 2.

Calculate the probability that an iPhone will cease to function due to mechanical failure during the three year period following its entry into service.
Solution:
Let (m) be mechanical failure and (o) be failure for other reasons.

Now we will build a table assuming that we start with 1000 phones.

<table>
<thead>
<tr>
<th>Year</th>
<th>$d_1^{(m)}$</th>
<th>$d_1^{(o)}$</th>
<th>Surviving Phones</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>200$^a$</td>
<td>300$^b$</td>
<td>500$^c$</td>
</tr>
<tr>
<td>2</td>
<td>50$^f$</td>
<td>200$^d$</td>
<td>250$^e$</td>
</tr>
<tr>
<td>3</td>
<td>100$^g$</td>
<td>100$^f$</td>
<td>50$^h$</td>
</tr>
</tbody>
</table>

\( a \Rightarrow 1000(q_1^{(m)}) = (1000)(0.2) = 200 \)
\( b \Rightarrow 1000(q_1^{(o)}) = (1000)(0.3) = 300 \)
\( c \Rightarrow 1000 - 200 - 300 = 500 \)
\( d \Rightarrow 500(q_1^{(o)}) = (500)(0.4) = 200 \)
\( e \Rightarrow \text{From Given (b)} \Rightarrow 200 = (0.8)(e) \Rightarrow 200 / 0.8 = 250 \)
\( f \Rightarrow 500 - 200 - 250 = 50 \)
\( g \Rightarrow \text{From Given (a)} \Rightarrow g = (0.4)(250) = 100 \)
\( h \Rightarrow 250(0.2) = 50 \)
\( i = 250 - 100 - 50 = 100 \)

\( \sum q_x = \frac{d_1^{(m)} + d_2^{(m)} + d_3^{(m)}}{l_0^{(x)}} = \frac{200 + 50 + 100}{1000} = 0.35 \)
11. *Your actuarial student has constructed a multiple decrement table using independent mortality and lapse tables. The multiple decrement table values, where decrement $d$ is death and decrement $w$ is lapse, are as follows:

<table>
<thead>
<tr>
<th>$l_{00}^{(z)}$</th>
<th>$d_{00}^{(d)}$</th>
<th>$d_{00}^{(w)}$</th>
<th>$l_{01}^{(r)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>950,000</td>
<td>2,580</td>
<td>94,742</td>
<td>852,678</td>
</tr>
</tbody>
</table>

You discover that an incorrect value of $q_{60}^{(w)}$ was taken from the independent lapse table. The correct value is 0.05.

Decrements are uniformly distributed over each year of age in the multiple decrement table.

You correct the multiple decrement table, keeping $l_{00}^{(r)} = 950,000$.

Calculate the correct values of $d_{00}^{(w)}$. 
Solution:

\[ p_{60}^{(d)} = \left( p_{60}^{(\tau)} \right)^{q_{60}^{(d)}} \]

\[ q_{60}^{(\tau)} = q_{60}^{(d)} + q_{60}^{(w)} = 1 - \frac{l_{61}^{(\tau)}}{l_{60}^{(\tau)}} \]

\[ p_{60}^{(\tau)} = \frac{l_{61}^{(\tau)}}{l_{60}^{(\tau)}} \]

\[ p_{60}^{(d)} = \left( p_{60}^{(\tau)} \right)^{q_{60}^{(d)}} = \left( \frac{852,678}{950,000} \right)^{\left(\frac{2580/950,000}{1-852.678/950,000}\right)} = 0.997138907 \]

Note that we can use the incorrect values to derive \( p_{60}^{(d)} \) since this value was correct in the calculations.

\[ p_{60}^{(w)} = \left( p_{60}^{(\tau)} \right)^{q_{60}^{(w)}} = \left( \frac{852,678}{950,000} \right)^{\left(\frac{2580/950,000}{1-852.678/950,000}\right)} \]

\[ q_{60}^{(w)} = \frac{\ln(0.95)}{\ln \left( \frac{(0.997138907)(1-0.05)}{(0.997138907)(1-0.05)} \right) [1-(0.997138907)(1-0.05)]} = 0.049929 \]

\[ d_{60}^{(w)} = (950,000)(0.049929) = 47,433 \]