MA 16100
Study Guide - Exam # 2

1 Average rate of change of $y = f(x)$ over the interval $[x_1, x_2]$: $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$ (this is also the average velocity). Definition of derivative of $y = f(x)$ at a: $f'(a) = \lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$ or, equivalently, $f'(a) = \lim_{x\to a} \dfrac{f(x) - f(a)}{x - a}$; interpretation of derivative:

$$f'(a) = \begin{cases} \text{slope of tangent line the graph of } y = f(x) \text{ at } a \\ \text{velocity at time } a \\ \text{(instantaneous) rate of change of } f \text{ at } a \end{cases}$$

2 Derivative as a function: $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$; differentiable functions (i.e., $f'(x)$ exists);

higher order derivatives: y'' or equivalently $\dfrac{d^2y}{dx^2}$; y''' or equivalently $\dfrac{d^3y}{dx^3}$; etc...

3 **Basic Differentiation Rules**: If f and g are differentiable functions, and c is a constant:

(a) $\dfrac{d(c)}{dx} = 0$ (b) $\dfrac{d(u+v)}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$ (c) $\dfrac{d(u-v)}{dx} = \dfrac{du}{dx} - \dfrac{dv}{dx}$

(c) `Power Rule`: $\dfrac{d(x^n)}{dx} = nx^{n-1}$

(d) `Product Rule` : $\dfrac{d(uv)}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$ `Quotient Rule` : $\dfrac{d\left(\frac{u}{v}\right)}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

4 *Special Trig Limits* : $\boxed{\lim_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1}$ $\boxed{\lim_{\theta\to 0} \dfrac{\theta}{\sin\theta} = 1}$ $\boxed{\lim_{\theta\to 0} \dfrac{\cos\theta - 1}{\theta} = 0}$.

Hence also $\lim_{\theta\to 0} \dfrac{\sin(k\theta)}{(k\theta)} = 1$ and $\lim_{\theta\to 0} \dfrac{(k\theta)}{\sin(k\theta)} = 1$. Note that $\sin k\theta \neq k\sin\theta$.

5 **CHAIN RULE**: If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $f \circ g$ is differentiable at x and its derivative is

$$(f\circ g)'(x) = \Big\{f(g(x))\Big\}' = f'(g(x))\, g'(x)$$

i.e., if $y = f(u)$ and $u = g(x)$, then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$.

6 **Implicit Differentiation**: If an equation defines one variable as a function of the other (independent) variable, then to find the derivative of the function w.r.t. the independent variable:

`Step 1 -` Differentiate both sides of equation w.r.t. independent variable
`Step 2 -` Solve for the desired derivative

7 **Logarithmic Differentiation**

`Step 1` : Take natural log of both sides of $y = h(x)$; simplify using Law of Logarithms
`Step 2` : Differentiate implicitly w.r.t x
`Step 3` : Solve the resulting equation for $\dfrac{dy}{dx}$

8 Inverse Trig Functions - Note, for example, $\sin^{-1} x$ is same as $\arcsin x$, but $\sin^{-1} x \neq (\sin x)^{-1}$

(a) Definitions:

$y = \sin^{-1} x \iff \sin y = x \quad (-\frac{\pi}{2} \le x \le \frac{\pi}{2})$
$y = \cos^{-1} x \iff \cos y = x \quad (0 \le x \le \pi)$
$y = \tan^{-1} x \iff \tan y = x \quad (-\frac{\pi}{2} < x < \frac{\pi}{2})$

(b) Basic Derivatives:

$$\frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d(\tan^{-1} x)}{dx} = \frac{1}{1+x^2} \qquad \frac{d(\cos^{-1} x)}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

If u is a differentiable function of x, then by the Chain Rule, $\dfrac{d(\sin^{-1} u)}{dx} = \dfrac{1}{\sqrt{1-u^2}} \dfrac{du}{dx}$, etc.

9 Hyperbolic Trig Functions

(a) Definitions:

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

(b) Basic Derivatives:

$$\frac{d(\cosh x)}{dx} = \sinh x \qquad \frac{d(\sinh x)}{dx} = \cosh x \qquad \frac{d(\tanh x)}{dx} = \text{sech}^2 x$$

If u is a differentiable function of x, then by the Chain Rule, $\dfrac{d(\cosh u)}{dx} = (\sinh u) \dfrac{du}{dx}$, etc.

(c) Basic Identities:

$$\cosh(-x) = \cosh x \qquad \sinh(-x) = -\sinh x \qquad \cosh^2 x - \sinh^2 x = 1$$

10 APPLICATIONS

`Model 1` - **Exponential Growth/Decay**: $\dfrac{dy}{dt} = k\,y$ where $k =$ `relative growth/decay rate`

(If the rate of change of y is proportional to y, then the above differential equation holds.)

- If $k > 0$, this is the law of *Natural Growth* (for example, population growth).
- If $k < 0$, this is the law of *Natural Decay* (for example, radioactive decay).

All solutions to this differential equation have the form $y(t) = y(0)\,e^{kt}$.

(Usually need **two** pieces of information to determine both constants $y(0)$ and k, unless they are given explicitly.)

Half-life = time it takes for radioactive substance to lose half its mass.)

`Model 2` - **Newton's Law of Cooling** : If $T(t) =$ temperature of an object at time t and $T_s =$ temperature of its surrounding environment, then the rate of change of $T(t)$ is proportional to the difference between $T(t)$ and T_s :

$$\frac{dT}{dt} = k\,(T(t) - T_s)$$

The solution to this particular differential equation is always $T(t) = T_s + Ce^{kt}$

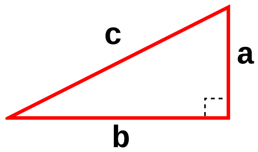
(Usually need **two** pieces of information to determine both constants C and k, unless they are given explicitly.)

Model 3 - **Related Rates** (Method to Solve) :

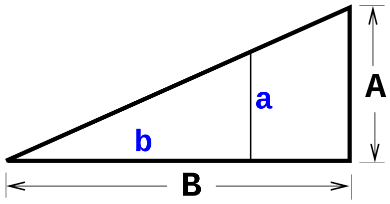
1. Read problem carefully several times to understand what is asked.
2. Draw a picture (if possible) and label.
3. Write down the given rate; write down the desired rate.
4. Find an equation relating the variables.
5. Use Chain Rule to differentiate equation w.r.t to time and solve for desired rate.

Useful Formulas for Related Rates

(i) **Pythagorean Theorem**: $c^2 = a^2 + b^2$

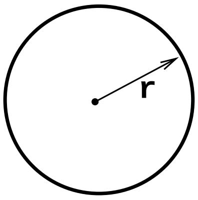


(ii) **Similar Triangles**: $\frac{a}{b} = \frac{A}{B}$



(iii) **Formulas from Geometry**:

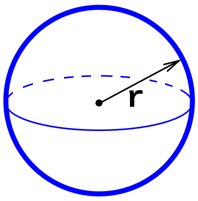
Circle of radius r



$A = \pi r^2$

$C = 2\pi r$

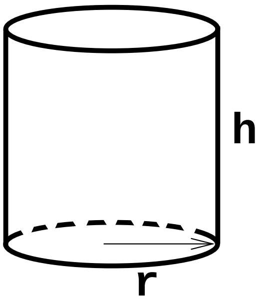
Sphere of radius r



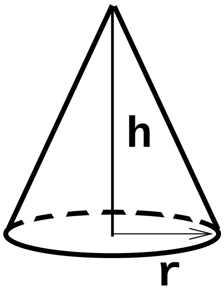
$V = \frac{4}{3}\pi r^3$

$S = 4\pi r^2$ (surface area of sphere)

Cylinders and Cones:



$V = \pi r^2 h$



$V = \frac{1}{3}\pi r^2 h$

Additional Differentiation Formulas

(u is a differentiable function of x)

$\dfrac{d\left(u^{n}\right)}{dx}=nu^{n-1}\,\dfrac{du}{dx}$	$\dfrac{d\left(e^{u}\right)}{dx}=e^{u}\,\dfrac{du}{dx}$	$\dfrac{d\left(a^{u}\right)}{dx}=a^{u}\left(\ln a\right)\dfrac{du}{dx}$

$\dfrac{d\left(\ln u\right)}{dx}=\dfrac{1}{u}\,\dfrac{du}{dx}$	$\dfrac{d\left(\log_a u\right)}{dx}=\dfrac{1}{u\ln a}\,\dfrac{du}{dx}$

$\dfrac{d\left(\sin u\right)}{dx}=\left(\cos u\right)\dfrac{du}{dx}$	$\dfrac{d\left(\cos u\right)}{dx}=\left(-\sin u\right)\dfrac{du}{dx}$	$\dfrac{d\left(\tan u\right)}{dx}=\left(\sec^{2}u\right)\dfrac{du}{dx}$

$\dfrac{d\left(\csc u\right)}{dx}=\left(-\csc u\cot u\right)\dfrac{du}{dx}$	$\dfrac{d\left(\sec u\right)}{dx}=\left(\sec u\tan u\right)\dfrac{du}{dx}$	$\dfrac{d\left(\cot u\right)}{dx}=\left(-\csc^{2}u\right)\dfrac{du}{dx}$