Spring 2015

MA 16100
Study Guide - Exam # 3 (updated)

1 **Related Rates Word Problems Method**:

1. Read problem carefully several times to understand what is asked.
2. Draw a picture (if possible) and label.
3. Write down the given rate; write down the desired rate.
4. Find an equation relating the variables.
5. Use Chain Rule to differentiate equation w.r.t to time and solve for desired rate.

2 The *Linear Approximation* (or tangent line approximation) to a function $f(x)$ at $x = a$ is the function $L(x) = f(a) + f'(a)(x - a)$; Approximation formula $\boxed{f(x) \approx f(a) + f'(a)(x - a)}$ for x near a; if $y = f(x)$, the differential of y is $dy = f'(x)dx$.

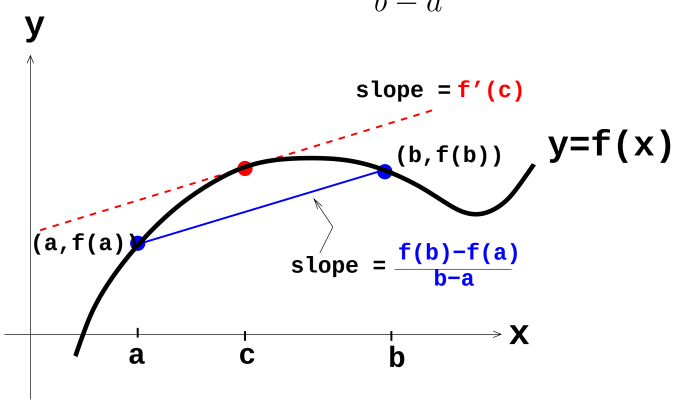
3 Definitions of absolute maximum, absolute minimum, local/relative maxmium, and local/relative minimum; c is a *critical number* of f if c is in the domain of f and either $f'(c) = 0$ or $f'(c)$ DNE.

4 **Extreme Value Theorem**: If $f(x)$ is continuous on a closed interval $[a, b]$, then f always has an absolute maximum value and an absolute minimum value on $[a, b]$.

5 **Method to Find Absolute Max/Min of $f(x)$ over Closed Interval $[a, b]$**:
- (i) Find all admissible critical numbers in (a, b);
- (ii) Find endpoints of interval;
- (iii) Make table of values of $f(x)$ at the points found in (i) and (ii).
  The largest value = abs max value of f and the smallest value = abs min value of f.

6 **Rolle's Theorem**: If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), and $f(a) = f(b)$, then $f'(c) = 0$ for some $c \in (a, b)$.

7 **Mean Value Theorem**: If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then there is a number c, where $a < c < b$, such that $\dfrac{f(b) - f(a)}{b - a} = f'(c)$:



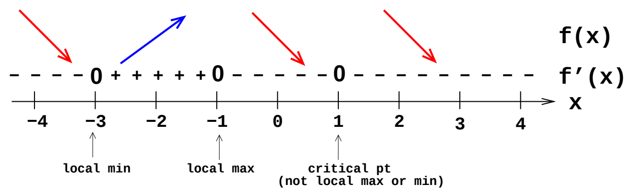
i.e., $\boxed{f(b) - f(a) = f'(c)\,(b - a)}$

If something about f' is known, then something about the sizes of $f(a)$ and $f(b)$ can be found.

8 Fact: (Useful for integration theory later)
- (a) If $f'(x) = 0$ for all $x \in I$, then $f(x) = C$ for all $x \in I$.
- (b) If $f'(x) = g'(x)$ for all $x \in I$, then $f(x) = g(x) + C$ for all $x \in I$.

$\boxed{9}$ Increasing functions: $f'(x) > 0 \Longrightarrow f \nearrow$; decreasing functions: $f'(x) < 0 \Longrightarrow f \searrow$.

Increasing/Decreasing



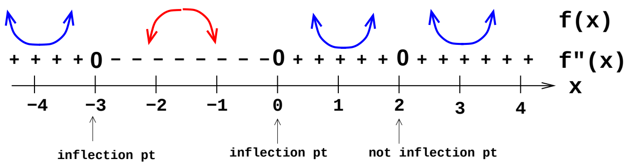
$\boxed{10}$ **First Derivative Test**: Suppose c is a critical number of a continuous function f.

(a) If f' changes from $+$ to $-$ at $c \Longrightarrow f$ has local max at c
(b) If f' changes from $-$ to $+$ at $c \Longrightarrow f$ has local min at c
(c) If f' does not change sign at $c \Longrightarrow f$ has neither local max nor local min at c

(*Displaying this information on a number line is much more efficient, see above figure.*)

$\boxed{11}$ f concave up: $f''(x) > 0 \Longrightarrow f \bigcup$; and f concave down: $f''(x) < 0 \Longrightarrow f \bigcap$;
inflection point (i.e. point where concavity changes).

Concave Up/Down



(*Displaying this information on a number line is much more efficient, see above figure.*)

$\boxed{12}$ **Second Derivative Test**: Suppose f'' is continuous near critical number c and $f'(c) = 0$.

(a) If $f''(c) > 0 \implies f$ has a local min at c.
(b) If $f''(c) < 0 \implies f$ has a local max at c.

Note: If $f''(c) = 0$, then 2^{nd} Derivative Test cannot be used, so then use 1^{st} Derivative Test.

$\boxed{13}$ INDETERMINATE FORMS:

(a) **Indeterminate Form (Types):** $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \cdot \infty$, $\infty - \infty$, 0^0, ∞^0, 1^∞

(b) **L'Hopital's Rule**: Let f and g be differentiable and $g'(x) \neq 0$ on an open interval I containing a (except possibly at a). If $\lim_{x\to a} f(x) = 0$ and if $\lim_{x\to a} g(x) = 0$ [Type $\frac{0}{0}$]; or if $\lim_{x\to a} f(x) = \infty$ (or $-\infty$) and if $\lim_{x\to a} g(x) = \infty$ (or $-\infty$) [Type $\frac{\infty}{\infty}$], then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right side exists or is infinite.

Use algebra to convert the different Indetermine Forms in (a) *into expressions where the above formula can be used.*

Important Remark: L'Hopital's Rule is also valid for one-sided limits, $x \to a^-$, $x \to a^+$, and also for limits when $x \to \infty$ or $x \to -\infty$.

14 Curve Sketching Guidelines:

(a) `Domain` of f
(b) `Intercepts` (if any)
(c) `Symmetry`:
$f(-x) = f(x)$ for *even* function;
$f(-x) = -f(x)$ for *odd* function;
$f(x+p) = f(x)$ for *periodic* function
(d) `Asymptotes`:
$x = a$ is a *Vertical Asymptote*: if either $\lim_{x \to a^-} f(x)$ or $\lim_{x \to a^+} f(x)$ is infinite

$y = L$ is a *Horizontal Asymptote*: if either $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$.

(e) `Intervals`: where f is increasing $\nearrow$ and decreasing $\searrow$; local max and local min
(f) `Intervals`: where f is concave up $\bigcup$ and concave down $\bigcap$; inflection points

15 Optimization (Max/Min) Word Problems Method:

$\boxed{1}$ Read problem carefully several times.
$\boxed{2}$ Draw a picture (if possible) and label it.
$\boxed{3}$ Introduce notation for the quantity, say Q, to be extremized as a function of one or more variables.
$\boxed{4}$ Use information given in problem to express Q as a function of only one variable, say x. Write the domain of Q.
$\boxed{5}$ Use Max/Min methods to determine the absolute maximum value of Q or the absolute minimum of Q, whichever was asked for in problem.

16 Integration Theory:

(a) $F(x)$ is an *antiderivative* of $f(x)$, if $F'(x) = f(x)$.

(b) **Definite Integral** $\int_a^b f(x)\,dx$ is a number; gives the net area under a curve $y = f(x)$ when $a \le x \le b$; also gives net distance traveled by particle with velocity $y = f(x)$ from time $x = a$ to $x = b$; many other applications (take Calculus II).

(c) Properties of Definite Integrals.

(d) **FUNDAMENTAL THEOREM OF CALCULUS:**

$\boxed{\textbf{I}}$ If $f(x)$ is continuous on $[a,b]$ and $g(x) = \int_a^x f(t)\,dt \implies g'(x) = f(x)$.

i.e., $\boxed{\dfrac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)}$ `Leibnitz' Rule`

$\boxed{\textbf{II}}$ If $F(x)$ is any antiderivative of $f(x)$ $\implies$ $\int_a^b f(x)\,dx = F(x)\Big|_{x=a}^{x=b} = F(b) - F(a)$.

i.e., $\boxed{\int_a^b F'(x)\,dx = F(b) - F(a)}$ `Net Change Rule`

(e) **Indefinite Integral** $\int f(x)\,dx$ is a function.

Recall that $\int f(x)\,dx = F(x)$ means $F'(x) = f(x)$, i.e. the indefinite integral $\int f(x)\,dx$ is simply the most general antiderivative of $f(x)$.

(f) `Generalized Leibnitz' Rule:` $\boxed{\dfrac{d}{dx}\left(\int_a^{u(x)} f(t)\,dt\right) = f(u(x))\,\dfrac{du(x)}{dx}}$

(g) **Subsitution Rule** (Indefinite Integrals): $\boxed{\int f(g(x))\,g'(x)\,dx = \int f(u)\,du}$, $u = g(x)$.

(h) **Subsitution Rule** (Definite Integrals): $\boxed{\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du}$, $u = g(x)$.

Basic Table of Indefinite Integrals

(1) $\int k\,dx = kx + C$

(2) $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C \quad (n \neq -1)$

(3) $\int \dfrac{1}{x}\,dx = \ln|x| + C$

(4) $\int e^x\,dx = e^x + C$

(5) $\int \cos x\,dx = \sin x + C$

(6) $\int \sin x\,dx = -\cos x + C$

(7) $\int \sec^2 x\,dx = \tan x + C$

(8) $\int \sec x\,\tan x\,dx = \sec x + C$

(9) $\int \dfrac{1}{1+x^2}\,dx = \tan^{-1} x + C$

(10) $\int \dfrac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$

(11) $\int \sinh x\,dx = \cosh x + C$

(12) $\int \cosh x\,dx = \sinh x + C$

(13) $\int a^x\,dx = \dfrac{a^x}{\ln a} + C$