**Fall 2011**

# MA 16200
## Study Guide - Exam # 3

**(1)** Sequences; limits of sequences; Limit Laws for Sequences; Squeeze Theorem; monotone sequences; bounded sequences; Monotone Sequence Theorem.

**(2)** Infinite series $\sum_{n=1}^{\infty} a_n$; sequence of partial sums $s_n = \sum_{k=1}^{n} a_k$; the series $\sum_{n=1}^{\infty} a_n$ converges to $s$ if $s_n \to s$.

**(3)** **Special Series**:

(a) `Geometric Series`: $\sum_{n=1}^{\infty} ar^{n-1} = a(1 + r + r^2 + r^3 + \cdots) = \frac{a}{1-r}$, if $|r| < 1$ (converges).
The Geometric Series diverges if $|r| \geq 1$.

(b) `p-Series`: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$; diverges when $p \leq 1$.

**(4)** List of Convergence Tests for $\sum_{n=1}^{\infty} a_n$.

(0) **Divergence Test**
(1) **Integral Test**
(2) **Comparison Test**
(3) **Limit Comparison Test**
(4) **Alternating Series Test**
(5) **Ratio Test**
(6) **Root Test**

(Useful inequality: $\ln x < x^m$, for any $m > \frac{1}{e} > 0.37$.)

**(5)** Strategy for Convergence/Divergence of Infinite Series.

Usually consider the form of the series $\sum a_n$:

(i) If $\lim_{n\to\infty} a_n \neq 0$ or DNE, the use **Divergence Test**.

(ii) If series is a $p$-Series $\sum \frac{1}{n^p}$, use *p-Series* conclusions; if series is a Geometric series $\sum a\, r^{n-1}$, use *Geometric Series* conclusions.

(iii) If $\sum a_n$ looks like a $p$-Series or Geometric series, use **Comparison Test** or **Limit Comparison Test**.

(iv) If $a_n$ involves factorials, use **Ratio Test**.

(v) If $a_n$ involves $n^{th}$ powers, use **Root Test**.

(vi) If $\sum a_n$ is an alternating series, use **Alternating Series Test** (or use **Ratio/Root Test** to show absolute or conditional convergence).

(vii) The last resort is usually to use the **Integral Test**.

# Convergence Tests

**0** **DIVERGENCE TEST**: If $\lim_{n\to\infty} a_n \neq 0$ or DNE $\Longrightarrow \sum_{n=1}^{\infty} a_n$ diverges

**1** **INTEGRAL TEST**: Let $f$ be continuous, decreasing, and positive on $[1, \infty)$ and $f(n) = a_n$.

(i) If $\int_1^{\infty} f(x)\,dx$ converges $\Longrightarrow \sum_{n=1}^{\infty} a_n$ converges.

(ii If $\int_1^{\infty} f(x)\,dx$ diverges $\Longrightarrow \sum_{n=1}^{\infty} a_n$ diverges.

(i.e., the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ converges.)

**2** **COMPARISON TEST**: Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms $(a_n,\, b_n > 0)$.

(i) If $a_n \leq b_n$ for all $n$ and $\sum b_n$ converges $\Longrightarrow \sum a_n$ converges.

(ii) If $a_n \geq b_n$ for all $n$ and $\sum b_n$ diverges $\Longrightarrow \sum a_n$ diverges.

**3** **LIMIT COMPARISON TEST**: Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms $(a_n,\, b_n > 0)$. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where $0 < c < \infty$, then either both series converge or both diverge.

**4** **ALTERNATING SERIES TEST**: Given an alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$.

If $\begin{cases} b_n > 0 \\ \{b_n\} \text{ is a decreasing sequence} \\ \lim_{n\to\infty} b_n = 0 \end{cases}$, then the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges.

**5** **RATIO TEST**: Let $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$.

(i) If $L < 1 \Longrightarrow \sum_{n=1}^{\infty} a_n$ converges.

(ii) If $L > 1 \Longrightarrow \sum_{n=1}^{\infty} a_n$ diverges.

(If $L = 1$, then test is inconclusive.)

**6** **ROOT TEST**: Let $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$.

(i) If $L < 1 \Longrightarrow \sum_{n=1}^{\infty} a_n$ converges.

(ii) If $L > 1 \Longrightarrow \sum_{n=1}^{\infty} a_n$ diverges.

(If $L = 1$, then test is inconclusive.)

**(6)** Alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$; estimating alternating series: if $s=\sum_{n=1}^{\infty}(-1)^{n-1}b_n$, then $|s-s_n|\le b_{n+1}$ Conditional convergence (i.e.,$\sum a_n$ converges, but $\sum |a_n|$ diverges); Absolute convergence (i.e. $\sum |a_n|$ converges). NOTE: absolute convergence $\Longrightarrow$ convergence.

**(7)** Power series about $a$: $\sum_{n=0}^{\infty} c_n(x-a)^n$; **Radius of Convergence** and **Interval of Convergence**; usually use Ratio Test to determine **ROC** and **IOC** (for IOC don't forget to check the endpoints!)

**(8)** Operations on power series; differentiation, integration of power series.

**(9)** **Taylor Series about *a***: $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n\,!}\,(x-a)^n.$

**(10)** **Maclaurin Series**: $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n\,!}\,x^n.$

**(11)** ***n*^th^-degree Taylor polynomial**: $T_n=\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k\,!}\,(x-a)^k\approx f(x),$ near $x=a$.

**(12)** **Binomial Series**: $(1+x)^k=\sum_{n=0}^{\infty}\binom{k}{n}x^k$, where $\binom{k}{n}=\frac{n\,!}{k\,!\,(n-k)\,!}$.

**(13)** **Useful Maclaurin Series**

$(a)$ $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n=1+x+x^2+x^3+\cdots,$ valid for $-1<x<1$

$(b)$ $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n\,!}=1+x+\frac{x^2}{2\,!}+\frac{x^3}{3\,!}+\cdots,$ valid for $-\infty<x<\infty$

$(c)$ $\sin x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)\,!}=x-\frac{x^3}{3\,!}+\frac{x^5}{5\,!}-\frac{x^7}{7\,!}+\cdots,$ valid for $-\infty<x<\infty$

$(d)$ $\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)\,!}=1-\frac{x^2}{2\,!}+\frac{x^4}{4\,!}-\frac{x^6}{6\,!}+\cdots,$ valid for $-\infty<x<\infty$

$(e)$ $\tan^{-1}x=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)}=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots,$ valid for $-1<x<1$

$(f)$ $(1+x)^k=\sum_{n=0}^{\infty}\binom{k}{n}x^k=1+kx+\frac{k(k-1)}{2\,!}\,x^2+\frac{k(k-1)(k-2)}{3\,!}\,x^3+\cdots,$ for $-1<x<1$