Fall 2011

MA 16200
Study Guide - Exam # 3

(1) Sequences; limits of sequences; Limit Laws for Sequences; Squeeze Theorem; monotone sequences; bounded sequences; Monotone Sequence Theorem.

(2) Infinite series $\sum_{n=1}^{\infty} a_n$; sequence of partial sums $s_n = \sum_{k=1}^{n} a_k$; the series $\sum_{n=1}^{\infty} a_n$ converges to s if $s_n \to s$.

(3) **Special Series**:

(a) `Geometric Series`: $\sum_{n=1}^{\infty} ar^{n-1} = a(1 + r + r^2 + r^3 + \cdots) = \dfrac{a}{1-r}$, if $|r| < 1$ (converges).
The Geometric Series diverges if $|r| \geq 1$.

(b) `p-Series`: $\sum_{n=1}^{\infty} \dfrac{1}{n^p}$ converges when $p < 1$; diverges when $p \geq 1$.

(4) List of Convergence Tests for $\sum_{n=1}^{\infty} a_n$.

(0) **Divergence Test**
(1) **Integral Test**
(2) **Comparison Test**
(3) **Limit Comparison Test**
(4) **Alternating Series Test**
(5) **Ratio Test**
(6) **Root Test**

(Useful inequality: $\ln x < x^m$, for any $m > \dfrac{1}{e} > 0.37$.)

(5) Strategy for Convergence/Divergence of Infinite Series.

Usually consider the form of the series $\sum a_n$:

(i) If $\lim_{n\to\infty} a_n \neq 0$ or DNE, the use **Divergence Test**.

(ii) If series is a p-Series $\sum \dfrac{1}{n^p}$, use *p-Series* conclusions; if series is a Geometric series $\sum a\, r^{n-1}$, use *Geometric Series* conclusions.

(iii) If $\sum a_n$ looks like a p-Series or Geometric series, use **Comparison Test** or **Limit Comparison Test**.

(iv) If a_n involves factorials, use **Ratio Test**.

(v) If a_n involves n^{th} powers, use **Root Test**.

(vi) If $\sum a_n$ is an alternating series, use **Alternating Series Test** (or use **Ratio/Root Test** to show absolute or conditional convergence).

(vii) The last resort is usually to use the **Integral Test**.

Convergence Tests

0 **DIVERGENCE TEST**: If $\lim_{n\to\infty} a_n \neq 0$ or DNE $\Longrightarrow \sum_{n=1}^{\infty} a_n$ diverges

1 **INTEGRAL TEST**: Let f be continuous, decreasing, and positive on $[1, \infty)$ and $f(n) = a_n$.

(i) If $\int_1^{\infty} f(x)\,dx$ converges $\Longrightarrow \sum_{n=1}^{\infty} a_n$ converges.

(ii If $\int_1^{\infty} f(x)\,dx$ diverges $\Longrightarrow \sum_{n=1}^{\infty} a_n$ diverges.

(i.e., the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ converges.)

2 **COMPARISON TEST**: Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms ($a_n,\ b_n > 0$).

(i) If $a_n \leq b_n$ for all n and $\sum b_n$ converges $\Longrightarrow \sum a_n$ converges.

(ii) If $a_n \geq b_n$ for all n and $\sum b_n$ diverges $\Longrightarrow \sum a_n$ diverges.

3 **LIMIT COMPARISON TEST**: Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms ($a_n,\ b_n > 0$). If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where $0 < c < \infty$, then either both series converge or both diverge.

4 **ALTERNATING SERIES TEST**: Given an alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$.

If $\begin{cases} b_n > 0 \\ \{b_n\} \text{ is a decreasing sequence} \\ \lim_{n\to\infty} b_n = 0 \end{cases}$, then the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges.

5 **RATIO TEST**: Let $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$.

(i) If $L < 1 \Longrightarrow \sum_{n=1}^{\infty} a_n$ converges.

(ii) If $L > 1 \Longrightarrow \sum_{n=1}^{\infty} a_n$ diverges.

(If $L = 1$, then test is inconclusive.)

6 **ROOT TEST**: Let $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$.

(i) If $L < 1 \Longrightarrow \sum_{n=1}^{\infty} a_n$ converges.

(ii) If $L > 1 \Longrightarrow \sum_{n=1}^{\infty} a_n$ diverges.

(If $L = 1$, then test is inconclusive.)

(6) Alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$; estimating alternating series: if $s = \sum_{n=1}^{\infty}(-1)^{n-1}b_n$,
then $|s - s_n| \leq b_{n+1}$ Conditional convergence (i.e.,$\sum a_n$ converges, but $\sum |a_n|$ diverges);
Absolute convergence (i.e. $\sum |a_n|$ converges). NOTE: absolute convergence $\Longrightarrow$ convergence.

(7) Power series about a: $\sum_{n=0}^{\infty} c_n(x-a)^n$; **Radius of Convergence** and **Interval of Convergence**;
usually use Ratio Test to determine **ROC** and **IOC** (for IOC don't forget to check the endpoints!)

(8) Operations on power series; differentiation, integration of power series.

(9) **Taylor Series about a**: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n\,!}\,(x-a)^n$.

(10) **Maclaurin Series**: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n\,!}\,x^n$.

(11) **n^{th}-degree Taylor polynomial**: $T_n = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k\,!}\,(x-a)^k \approx f(x)$, near $x = a$.

(12) **Binomial Series**: $(1+x)^k = \sum_{n=0}^{\infty} \begin{pmatrix} k \\ n \end{pmatrix} x^k$, where $\begin{pmatrix} k \\ n \end{pmatrix} = \frac{n\,!}{k\,!\,(n-k)\,!}$.

(13) **Useful Maclaurin Series**

(a) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots,$ valid for $-1 < x < 1$

(b) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n\,!} = 1 + x + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \cdots,$ valid for $-\infty < x < \infty$

(c) $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)\,!} = x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!} - \frac{x^7}{7\,!} + \cdots,$ valid for $-\infty < x < \infty$

(d) $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)\,!} = 1 - \frac{x^2}{2\,!} + \frac{x^4}{4\,!} - \frac{x^6}{6\,!} + \cdots,$ valid for $-\infty < x < \infty$

(e) $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots,$ valid for $-1 < x < 1$

(f) $(1+x)^k = \sum_{n=0}^{\infty} \begin{pmatrix} k \\ n \end{pmatrix} x^k = 1 + kx + \frac{k(k-1)}{2\,!}\,x^2 + \frac{k(k-1)(k-2)}{3\,!}\,x^3 + \cdots,$ for $-1 < x < 1$