**Spring 2009**

# MA 16600
## Study Guide - Exam # 3

**(1)** Sequences $\{a_n\}$; limits of sequences; Limit Laws for Sequences; *Squeeze Theorem*; monotone sequences; bounded sequences; *Monotone Sequence Theorem*; **Theorem:** $\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right)$, provided $\lim_{n\to\infty} a_n = L$ and $f$ is continuous at $L$.

**(2)** Infinite series $\sum_{k=1}^{\infty} a_k$; sequence of partial sums $s_n = \sum_{k=1}^{n} a_k$; the infinite series $\sum_{k=1}^{\infty} a_k$ converges to $s$ if $s_n \to s$ and write $\sum_{k=1}^{\infty} a_k = s$.

**(3)** **Sums of Special Series**:

(i) Geometric Series: $\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}$, if $|r| < 1$, i.e., CONVERGES; and DIVERGES if $|r| \geq 1$.

(ii) Telescoping Series: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is an example. The partial sums $s_n$ of this series are

$$s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

The middle terms cancel and thus for this example, $s_n = 1 - \frac{1}{n+1} \to 1$, so $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

**(4)** List of Convergence Tests for Infinite Series $\sum_{n=1}^{\infty} a_n$.

(0) **Divergence Test** `(usually the` $1^{st}$ `test to apply)`

(1) **Integral Test** `(usually one of the last tests to apply, after all others fail)`

(2) **Comparison Test** `(usually compare to a p-Series or Geometric series)`

(3) **Limit Comparison Test** `(usually compare to a p-Series or Geometric series)`

(4) **Alternating Series Test** `(apply only if series is an alternating series)`

(5) **Ratio Test** (useful when $a_n$ contains factorials like $n!$ or when $a_n$ contains terms like $c^n$; NOT useful when $a_n$ is an algebraic function of $n$)

(6) **Root Test** (useful mostly when $a_n$ is of the form $c^n$; one of the last tests to apply)

- Very useful series is the **p-Series**: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ CONVERGES when $p > 1$; DIVERGES when $p \leq 1$.
- A useful inequality: $\ln x < x^m$, for any $m > 0$.

## (5) Statements of Convergence Tests for Infinite Series $\sum_{n=1}^{\infty} a_n$:

**(0) Divergence Test**: If $\lim_{n\to\infty} a_n \neq 0 \Longrightarrow \sum_{n=1}^{\infty} a_n$ DIVERGES.

**(1) Integral Test**: Suppose $f$ is continuous, increasing, and positive on $[1, \infty)$ and $f(n) = a_n$.

(i) If $\int_1^{\infty} f(x)\,dx$ converges $\Longrightarrow \sum_{n=1}^{\infty} a_n$ CONVERGES.

(ii) If $\int_1^{\infty} f(x)\,dx$ diverges $\Longrightarrow \sum_{n=1}^{\infty} a_n$ DIVERGES.

(i.e., $\int_1^{\infty} f(x)\,dx$ CONVERGES if and only if $\sum_{n=1}^{\infty} a_n$ CONVERGES.)

**(2) Comparison Test**: Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms $(a_n,\, b_n > 0)$.

(i) If $a_n \leq b_n$ for all $n$ and $\sum b_n$ converges $\Longrightarrow \sum a_n$ CONVERGES.

(ii) If $a_n \geq b_n$ for all $n$ and $\sum b_n$ diverges $\Longrightarrow \sum a_n$ DIVERGES.

**(3) Limit Comparison Test**: Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms $(a_n,\, b_n > 0)$.
If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where $0 < c < \infty$, then either both series CONVERGE or both DIVERGE.

**(4) Alternating Series Test**: If $\begin{cases} (i)\ \ b_n > 0 \\ (ii)\ \ \{b_n\} \text{ is decreasing} \\ (iii)\ \ \lim_{n\to\infty} b_n = 0 \end{cases}$ then the alternating series

$\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ CONVERGES.

**(5) Ratio Test**: Let $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$.

(i) If $L < 1 \Longrightarrow \sum_{n=1}^{\infty} a_n$ CONVERGES ABSOLUTELY (hence CONVERGES)

(ii) If $L > 1 \Longrightarrow \sum_{n=1}^{\infty} a_n$ DIVERGES.

(If $L = 1$, then test is *inconclusive*.)

**(6) Root Test**: Let $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$.

(i) If $L < 1 \Longrightarrow \sum_{n=1}^{\infty} a_n$ CONVERGES ABSOLUTELY (hence CONVERGES).

(ii) If $L > 1 \Longrightarrow \sum_{n=1}^{\infty} a_n$ DIVERGES.

(If $L = 1$, then test is *inconclusive*.)

**(6)** Alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$; Estimating the sum of an alternating series:

$$\text{If } \sum_{n=1}^{\infty}(-1)^{n-1}b_n = s, \text{ then } |s - s_n| \le b_{n+1}.$$

**(7)** Conditional convergence (i.e.,$\sum a_n$ converges, but $\sum |a_n|$ diverges); Absolute convergence (i.e. $\sum |a_n|$ converges). **Theorem:** Absolute Convergence $\Longrightarrow$ Convergence.

**(8)** Power series about $a$: $\sum_{n=0}^{\infty} c_n(x-a)^n$; **Radius of Convergence** and **Interval of Convergence**; usually use RATIO TEST to determine **ROC** and **IOC** (for IOC don't forget to check the endpoints!)

**(9)** Operations on power series; addition, differentiation, integration, etc of power series.

**(10)** **Taylor Series about $a$**: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$

**(11)** **Maclaurin Series**: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$ (this is just the Taylor Series of $f$ about 0).

**(12)** **$n^{th}$-degree Taylor polynomial**: $T_n = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \approx f(x)$, near $x = a$.

**(13)** **Binomial Series**: $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$, where $\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-(n-1))}{n!}$.

**(14)** **Table of Useful Maclaurin Series**

(a) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots,$ **valid for** $-1 < x < 1$

(b) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$ **valid for** $-\infty < x < \infty$

(c) $\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$ **valid for** $-\infty < x < \infty$

(d) $\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots,$ **valid for** $-\infty < x < \infty$

(e) $\tan^{-1} x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots,$ **valid for** $-1 < x < 1$

(f) $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots,$ for $-1 < x < 1$