Study Guide # 2 (updated)

1. Relative/local extrema; critical points ($\nabla f = \vec{\mathbf{0}}$ or ∇f does not exist); 2^{nd} Derivatives Test; absolute extrema; Max-Min Problems; **Lagrange Multipliers:** *Extremize* $f(\vec{\mathbf{x}})$ subject to a constraint $g(\vec{\mathbf{x}}) = C$, solve the system: $\nabla f = \lambda \nabla g$ and $g(\vec{\mathbf{x}}) = C$.

2. Double integrals; Midpoint Rule for rectangle : $\displaystyle\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\overline{x_i}, \overline{y_j})\,\Delta A$;

3. Type I region $D : \left\{\begin{array}{c} g_1(x) \le y \le g_2(x) \\ a \le x \le b \end{array}\right.$; Type II region $D : \left\{\begin{array}{c} h_1(y) \le x \le h_2(y) \\ c \le y \le d \end{array}\right.$;

iterated integrals over Type I and II regions: $\displaystyle\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$ and

$\displaystyle\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$, respectively; Reversing Order of Integration (regions that are both Type I and Type II); properties of double integrals.

4. Integral inequalities: $\displaystyle mA \le \iint_D f(x,y)\,dA \le MA$, where A = area of D and $m \le f(x,y) \le M$ on D.

5. Change of Variables Formula in Polar Coordinates: if $D : \left\{\begin{array}{c} h_1(\theta) \le r \le h_2(\theta) \\ \alpha \le \theta \le \beta \end{array}\right.$, then

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta,\ r\sin\theta)\,\underset{\uparrow}{r}\,dr\,d\theta.$$

6. Applications of double integrals:

(a) Area of region D is $\displaystyle A(D) = \iint_D dA$

(b) Volume of solid under graph of $z = f(x,y)$, where $f(x,y) \ge 0$, is $\displaystyle V = \iint_D f(x,y)\,dA$

(c) Mass of D is $\displaystyle m = \iint_D \rho(x,y)\,dA$, where $\rho(x,y)$ = density (per unit area); sometimes write $\displaystyle m = \iint_D dm$, where $dm = \rho(x,y)\,dA$.

(d) Moment about the x-axis $\displaystyle M_x = \iint_D y\,\rho(x,y)\,dA$; moment about the y-axis $\displaystyle M_y = \iint_D x\,\rho(x,y)\,dA$.

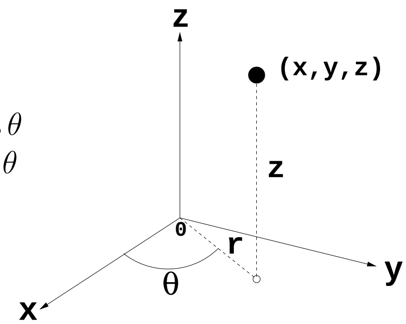
(e) Center of mass $(\overline{x}, \overline{y})$, where $\displaystyle \overline{x} = \frac{M_y}{m} = \frac{\iint_D x\,\rho(x,y)\,dA}{\iint_D \rho(x,y)\,dA}$, $\displaystyle \overline{y} = \frac{M_x}{m} = \frac{\iint_D y\,\rho(x,y)\,dA}{\iint_D \rho(x,y)\,dA}$

Remark: centroid = center of mass when density is constant (this is useful).

7. Elementary solids $E \subset \mathbb{R}^3$ of Type 1, Type 2, Type 3; triple integrals over solids E: $\iiint_E f(x,y,z)\, dV$; volume of solid E is $V(E) = \iiint_E dV$; applications of triple integrals, mass of a solid, moments about the coordinate planes M_{xy}, M_{xz}, M_{yz}, center of mass of a solid $(\overline{x}, \overline{y}, \overline{z})$.

8. Cylindrical Coordinates (r, θ, z):

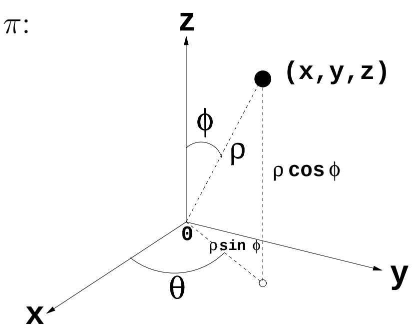
From CC to RC : $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$



Going from RC to CC use $x^2 + y^2 = r^2$ and $\tan\theta = \dfrac{y}{x}$ (make sure θ is in correct quadrant).

9. Spherical Coordinates (ρ, θ, ϕ), where $0 \le \phi \le \pi$:

From SC to RC : $\begin{cases} x = (\rho\sin\phi)\cos\theta \\ y = (\rho\sin\phi)\sin\theta \\ z = \rho\cos\phi \end{cases}$



Going from RC to SC use $x^2 + y^2 + z^2 = \rho^2$, $\tan\theta = \dfrac{y}{x}$ and $\cos\phi = \dfrac{z}{\rho}$.

10. Triple integrals in <u>Cylindrical Coordinates</u>: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$, $\boldsymbol{dV = r\, dz\, dr\, d\theta}$

$$\iiint_E f(x,y,z)\, dV = \iiint_E f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

$$\uparrow$$

11. Triple integrals in <u>Spherical Coordinates</u>: $\begin{cases} x = (\rho\sin\phi)\cos\theta \\ y = (\rho\sin\phi)\sin\theta \\ z = \rho\cos\phi \end{cases}$, $\boldsymbol{dV = \rho^2 \sin\phi \;\; d\rho\, d\phi\, d\theta}$

$$\iiint_E f(x,y,z)\, dV = \iiint_E f(\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$\uparrow$$

12. Vector fields on $\mathbb{R}^2$ and $\mathbb{R}^3$: $\vec{\mathbf{F}}(x,y) = \langle P(x,y),\ Q(x,y)\rangle$ and $\vec{\mathbf{F}}(x,y,z) = \langle P(x,y),\ Q(x,y),\ R(x,y)\rangle$; $\vec{\mathbf{F}}$ is a conservative vector field if $\vec{\mathbf{F}} = \nabla f$, for some real-valued function f.

13. Line integral of a function $f(x, y)$ along C, parameterized by $x = x(t), \quad y = y(t)$ and $a \leq t \leq b$, is

$$\int_C f(x, y)\ ds = \int_a^b f(x(t),\ y(t))\ \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt\,.$$

(independent of orientation of C, other properties and applications of line integrals of f)

Remarks:

(a) $\int_C f(x, y)\, ds$ is sometimes called the *"line integral of f with respect to arc length"*)

(b) $\int_C f(x, y)\, dx = \int_a^b f(x(t), y(t))\, x'(t)\, dt$

(c) $\int_C f(x, y)\, dy = \int_a^b f(x(t), y(t))\, y'(t)\, dt$

14. Line integral of vector field $\vec{\mathbf{F}}(x, y)$ along C, parameterized by $\vec{\mathbf{r}}(t)$ and $a \leq t \leq b$, is given by

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\, dt\,.$$

(depends on orientation of C, other properties and applications of line integrals of f)

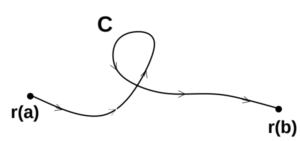
15. Connection between line integral of vector fields and line integral of functions:

$$\int_C \vec{\mathbf{F}}\ \cdot\ d\vec{\mathbf{r}} = \int_C (\vec{\mathbf{F}} \cdot \vec{\mathbf{T}})\, ds$$

where $\vec{\mathbf{T}}$ is the unit tangent vector to the curve C.

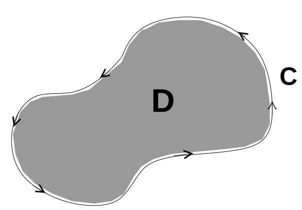
16. If $\vec{\mathbf{F}}(x, y) = P(x, y)\, \vec{\mathbf{i}} + Q(x, y)\, \vec{\mathbf{j}}$, then $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C P(x, y)\, dx + Q(x, y)\, dy$; Work $= \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$.

17. <u>FUNDAMENTAL THEOREM OF CALCULUS FOR LINE INTEGRALS</u>: $\int_C \nabla f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a))$:



18. A vector field $\vec{\mathbf{F}}(x, y) = P(x, y)\, \vec{\mathbf{i}} + Q(x, y)\, \vec{\mathbf{j}}$ is *conservative* (i.e. $\vec{\mathbf{F}} = \nabla f$) if $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$; how to determine a potential function f if $\vec{\mathbf{F}}(\vec{\mathbf{x}}) = \nabla f(\vec{\mathbf{x}})$.

19. <u>GREEN'S THEOREM</u>: $\int_C P(x, y)\, dx + Q(x, y)\, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA$ ($C =$ boundary of D):



20. Del Operator: $\dfrac{\partial}{\partial x}\,\vec{\mathbf{i}} + \dfrac{\partial}{\partial y}\,\vec{\mathbf{j}} + \dfrac{\partial}{\partial z}\,\vec{\mathbf{k}}$; if $\vec{\mathbf{F}}(x,y,z) = P(x,y,z)\,\vec{\mathbf{i}} + Q(x,y,z)\,\vec{\mathbf{j}} + R(x,y,z)\vec{\mathbf{k}}$, then

$$\text{curl } \vec{\mathbf{F}} = \nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \quad \text{and} \quad \text{div } \vec{\mathbf{F}} = \nabla \cdot \vec{\mathbf{F}} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

curl $\vec{\mathbf{F}}$ only defined for vector fields on $\mathbb{R}^3$; properties of curl and divergence;

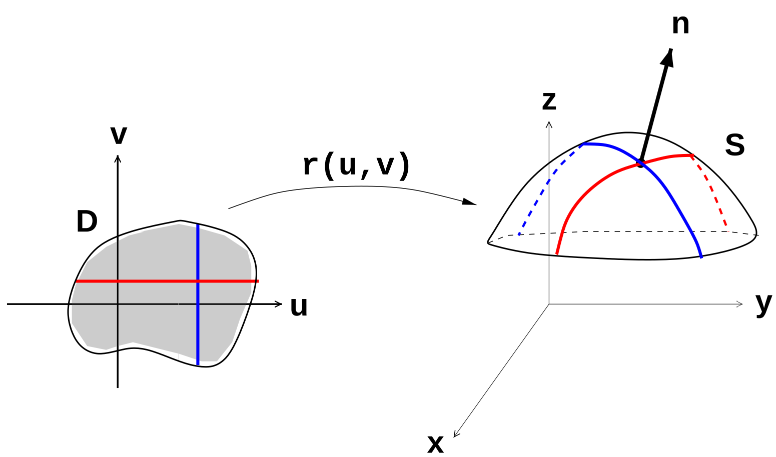
(i) If curl $\vec{\mathbf{F}} = \vec{\mathbf{0}}$, then $\vec{\mathbf{F}}$ is a conservative vector field (i.e., $\vec{\mathbf{F}}(\vec{\mathbf{x}}) = \nabla f(\vec{\mathbf{x}})$).

(ii) If curl $\vec{\mathbf{F}} = \vec{\mathbf{0}}$, then $\vec{\mathbf{F}}$ is *irrotational*; if div $\vec{\mathbf{F}} = 0$, then $\vec{\mathbf{F}}$ is *incompressible*.

(iii) *Laplace's Equation*: $\nabla^2 f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2} = 0.$

(iv) If $\vec{\mathbf{F}}(x,y,z) = P\,\vec{\mathbf{i}} + Q\,\vec{\mathbf{j}} + R\vec{\mathbf{k}}$, then $\nabla^2\vec{\mathbf{F}} = \nabla^2 P\,\vec{\mathbf{i}} + \nabla^2 Q\,\vec{\mathbf{j}} + \nabla^2 R\,\vec{\mathbf{k}}$.

21. Parametric surface S: $\vec{\mathbf{r}}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, where $(u,v) \in D$:



Normal vector to surface S : $\vec{\mathbf{n}} = \vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$; tangent planes and normal lines to parametric surfaces.

22. Surface area of a surface S:

(i) $A(S) = \displaystyle\iint_D |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v|\,dA$

(ii) If S is the graph of $z = f(x,y)$ above D, then $A(S) = \displaystyle\iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA$;

<u>Remark</u>: $dS = |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v|\,dA$ = differential of surface area; while $d\vec{\mathbf{S}} = (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\,dA$

23. The surface integral of $f(x,y,z)$ over the surface S:

(i) $\displaystyle\iint_S f(x,y,z)\,dS = \iint_D f(\vec{\mathbf{r}}(u,v))\,|\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v|\,dA.$

(ii) If S is the graph of $z = h(x,y)$ above D, then

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,h(x,y))\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA.$$

24. The surface integral of $\vec{\mathbf{F}}$ over the surface S (recall, $d\vec{\mathbf{S}} = (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\, dA$):

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_D \vec{\mathbf{F}} \cdot (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\, dA.$$

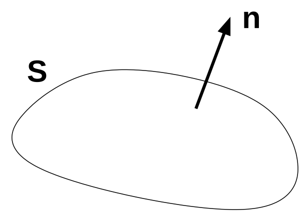
$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_S (\vec{\mathbf{F}} \cdot \vec{\mathbf{n}})\, dS = \iint_D \vec{\mathbf{F}} \cdot (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\, dA.$$

(i) Connection between surface integral of a vector field and a function:

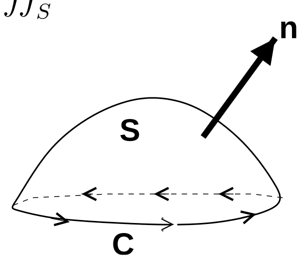
$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_S (\vec{\mathbf{F}} \cdot \vec{\mathbf{n}})\, dS.$$

(The above gives another way to compute $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$)

(ii) $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_S (\vec{\mathbf{F}} \cdot \vec{\mathbf{n}})\, dS =$ *flux* of $\vec{\mathbf{F}}$ across the surface S.



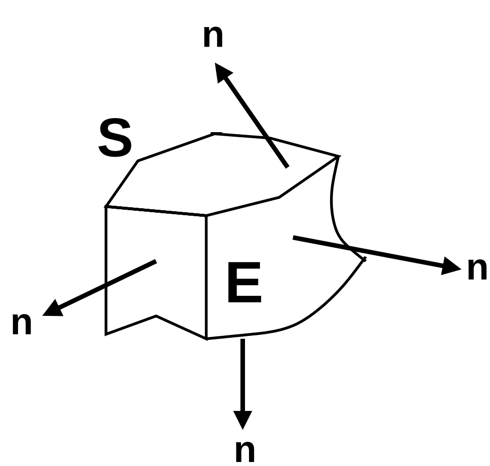
25. STOKES' THEOREM: $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_S \operatorname{curl} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$ (recall, $\operatorname{curl} \vec{\mathbf{F}} = \nabla \times \vec{\mathbf{F}}$).



$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} =$ *circulation* of $\vec{\mathbf{F}}$ around C.

26. THE DIVERGENCE THEOREM/GAUSS' THEOREM: $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iiint_E \operatorname{div} \vec{\mathbf{F}}\ dV$

(recall, $\operatorname{div} \vec{\mathbf{F}} = \nabla \cdot \vec{\mathbf{F}}$).



27. Summary of Line Integrals and Surface Integrals:

LINE INTEGRALS	SURFACE INTEGRALS
$C : \vec{\mathbf{r}}(t)$, where $a \le t \le b$	$S : \vec{\mathbf{r}}(u,v)$, where $(u,v) \in D$
$ds = \lvert\vec{\mathbf{r}}\,'(t)\rvert\, dt =$ differential of arc length	$dS = \lvert\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\rvert\, dA =$ differential of surface area
$\displaystyle\int_C ds =$ length of C	$\displaystyle\iint_S dS =$ surface area of S
$\displaystyle\int_C f(x,y,z)\,ds = \int_a^b f(\vec{\mathbf{r}}(t))\,\lvert\vec{\mathbf{r}}\,'(t)\rvert\,dt$ (independent of orientation of C)	$\displaystyle\iint_S f(x,y,z)\,dS = \iint f(\vec{\mathbf{r}}(u,v))\,\lvert\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\rvert\,dA$ (independent of normal vector $\vec{\mathbf{n}}$)
$d\vec{\mathbf{r}} = \vec{\mathbf{r}}\,'(t)\,dt$	$d\vec{\mathbf{S}} = (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\ dA$
$\displaystyle\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}\,'(t)\,dt$ (depends on orientation of C)	$\displaystyle\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_D \vec{\mathbf{F}}(\vec{\mathbf{r}}(u,v)) \cdot (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\ dA$ (depends on normal vector $\vec{\mathbf{n}}$)
$\displaystyle\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \left(\vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\right)\ ds$ The *circulation* of $\vec{\mathbf{F}}$ around C	$\displaystyle\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_S \left(\vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\right)\ dS$ The *flux* of $\vec{\mathbf{F}}$ across S in direction $\vec{\mathbf{n}}$