Final review[1]

1. Consider
$$\begin{bmatrix} 2 & r \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ s \end{bmatrix}$$
Determine r, s so that the system is consistent; is inconsistent; has a unique solution; has infinitely many solutions.

   Augmented matrix:
$$\left[\begin{array}{cc|c} 2 & r & 2 \\ 1 & -1 & s \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & s \\ 2 & r & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & s \\ 0 & r+2 & 2-2s \end{array}\right]$$

2. If
$$\begin{bmatrix} a+b & c+d \\ c-d & a-b \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 10 & 2 \end{bmatrix},$$
find a, b, c, d.

3. Show that

   A. If any 2×2 matrices A, B are symmetric, then $AB - BA$ can never be I_2.

   B. For any $n \times n$ matrices A, B, $Trace(A^T A) \geq 0$.

4. Find two different 2×2 matrices so that $A^2 = I_2$; Find two different 2×2 matrices so that $A^2 = O$.

5. For any square matrix A, show $A - A^T$ is skew-symmetric.

6. Let A, B be $n \times n$ symmetric matrices, show that AB is symmetric iff $AB = BA$.

7. Find a row/col echelon form of
$$\begin{bmatrix} -1 & 2 \\ 2 & -1 \\ 2 & -2 \end{bmatrix}$$

8. Solve
$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 0 & 1 \\ 1 & 0 & 2 & 1 \end{bmatrix} x = \begin{bmatrix} 8 \\ 7 \\ 3 \end{bmatrix}.$$
   Reduced row echelon form of the augmented matrix: $\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$

[1] The summary and problems do not necessarily or accurately reflect the contents or depth of the exam. The final exam will consist of multiple choice problems.

9. Assume $r \neq 2$. Then the inverse of $\begin{bmatrix} 1 & 2 \\ r & 4 \end{bmatrix}$ is equal to

A. $\begin{bmatrix} 1 & 2 \\ r & 4 \end{bmatrix}$

B. $\begin{bmatrix} -2 & 1 \\ \frac{r}{2} & -\frac{1}{2} \end{bmatrix}$

C. $\frac{1}{r-2}\begin{bmatrix} 2 & 1 \\ \frac{r}{2} & \frac{1}{2} \end{bmatrix}$

D. $\frac{1}{r-2}\begin{bmatrix} -2 & 1 \\ \frac{r}{2} & -\frac{1}{2} \end{bmatrix}$

E. $\frac{1}{r-2}\begin{bmatrix} -2 & \frac{r}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}$

10. Compute

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

6 terms of $(\pm)a_{1-}a_{2-}a_{3-}$. 6 permutations, even: $123, 231, 312$, odd: $132, 213, 321$.

$$\begin{aligned} \det(A) = & \, a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ & - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}. \end{aligned}$$

$$\begin{aligned} \det(A) &= -\det(A_{r_i \leftrightarrow r_j}) \ (i \neq j) \\ \det(A) &= \frac{1}{k}(A_{kr_i \to r_i}) \ (k \neq 0) \\ \det(A) &= \det(A_{kr_i + r_j \leftrightarrow r_j}) \ (i \neq j) \end{aligned}$$

11. Compute

$$\begin{vmatrix} 1 & 2 & -3 & 4 \\ -4 & 2 & 1 & 3 \\ 1 & 0 & 0 & -3 \\ 2 & 0 & -2 & 3 \end{vmatrix} = 8$$

12. Compute

$$|A| = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{vmatrix} = -6$$

Find $adj(A)$ and A^{-1}.

$$A^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

13. Let A and B be 3×3 matrices, and $\det(A) = 2, \det(B) = 3$. Then $\det(AB^{-1}) =$

A. 6

B. 2

C. 3

D. $\frac{2}{3}$

E. $\frac{3}{2}$

14. Let

$$A = \begin{bmatrix} 2 & a & 3 \\ 5 & b & -1 \\ 1 & c & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad A\mathbf{x} = \mathbf{b}.$$

If $\det(A) \neq 0$, then the first entry of $\mathbf{x}$ is equal to

A. $-b + 17c - 6a$

B. $5/(-b + 17c - 6a)$

C. 0

D. 1

E. $2b - 5a$

15. Let

$$\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

Find x_1.

by Cramer's rule.

$$x_1 = \frac{\begin{vmatrix} 3 & 3 \\ -1 & -1 \end{vmatrix}}{|A|} = 0, \ x_2 = 1$$

16. Find c_1, c_2 so that

$$c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} -5 \\ 6 \end{bmatrix}$$

17. Which of the following V is a vector space?

A. V: the set of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ with $x \leq 0$, with the usual operations in R^2.

B. V: the set of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ with $x + y = 1$, with the usual operations in R^2.

C. V: the set of all matrices $\begin{bmatrix} a & b \\ b+1 & d \end{bmatrix}$ with regular matrix operaitons.

D. V: the set of all positive real numbers, with $\oplus$ defined by $u \oplus v = uv$, and $c \odot u = u^c$.

E. V: all $n \times n$ nonsingular matrices.

18. Is $A = \begin{bmatrix} 5 & 1 \\ -1 & 9 \end{bmatrix}$ in *span* $\left\{ \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} \right\}$? Are these four matrices linearly dependent?

19. For what values of c are the vectors $[-1, 0, -1]$, $[2, 1, 2]$, $[1, 1, c]$ linearly independent?

20. Find a basis for

$$V = span \left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 11 \\ 10 \\ 7 \end{bmatrix}, \begin{bmatrix} 7 \\ 6 \\ 4 \end{bmatrix} \right\}.$$

Find a basis for V and a basis for $V^{\perp}$.

For V, use the reduced row echelon form:

$$\begin{bmatrix} 1 & 3 & 11 & 7 \\ 2 & 2 & 10 & 6 \\ 2 & 1 & 7 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

21. Find a basis for V: all vectors of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ where $b = a + c$.

22. Find a basis for all 2×2 symmetric matrices.

23. Find a basis for

$$span \left\{ t^3 + t^2 + 2t + 1, t^3 - 3t + 1, t^2 + t + 2, t + 1, t^3 + 1 \right\}$$

24. Find a basis for the solution space of

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & -1 & 1 \end{bmatrix} x = 0$$

25. Decide the row/col rank and nullity of

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & 6 & -8 & 1 \\ 2 & -1 & 0 & 1 \end{bmatrix}$$

26. For what r does

$$\begin{bmatrix} 1 & r \\ 2 & 1 \end{bmatrix} x = 0$$

have a nontrivial solution?

27. For a $m \times n$ matrix, discuss the max/min possible rank, nullity, and number of linearly independent rows/cols.

28. Find the angle between $\left[-1/\sqrt{2}, 0, a\right], [0, -1, b]$. For what a, b are they orthogonal? Orthonormal?

29. For what a, b is $[a, b, 2]$ orthogonal to both $[2, 1, 1]$ and $[1, 0, 1]$?

30. Orthonormalize $[2, 1, 1]$ and $[1, 0, 1]$.

31. Find an orthonormal basis for the subspace of all vectors of the form $[a, a+b, b]^T$.

32. Let V be a subspace of $\mathbb{R}^3$ spanned by $\left\{ \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\}$. Find a basis for $V^\perp$. For $b = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$, find $proj_V b$ and the distance between b and V.

33. Find a least squares soln to $Ax = b$

$$A = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 7 \\ 0 \\ 14 \\ 7 \end{pmatrix}$$

34. Find the least squares fitting for the points $(1, 2), (2, 5), (3, 4), (4, 4)$ using

    A. Straight line.

    B. Quadratic polynomial.

35. Which of the following functions are linear transformations:

    A. $L : R_2 \to R_3$ defined by $L([u_1, u_2]) = [u_1 + 1, u_2, u_1 + u_2]$.

    B. $L : R_3 \to R_3$ defined by $L([u_1, u_2, u_3]) = [1, u_2, u_1]$.

    C. $L : R_3 \to R_3$ defined by $L([u_1, u_2, u_3]) = [0, u_3, u_2]$.

    D. $L : M_{nn} \to M_{nn}$ defined by $L(A) = A^T$.

    E. $L : M_{nn} \to R$ defined by $L(A) = \det(A)$.

F. $L: M_{nn} \to R$ defined by $L(A) = A^{-1}$.

G. $L: M_{22} \to R$ defined by $L(A) = a_{11} + a_{22}$.

36. Find the standard matrix representation for $L: R^2 \to R^2$ defined by $L\left([u_1, u_2]^T\right) = [u_1 - 3u_2, 2u_1 - u_2, 2u_2]^T$.

37. A linear transformation $L: M_{22} \to M_{22}$ satisfies

$$L\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}, \ L\left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 5 & 1 \\ 2 & 3 \end{bmatrix},$$
$$L\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 3 & 4 \\ 2 & 4 \end{bmatrix}, \ L\left(\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Find $L\left(\begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix}\right)$.

38. Consider the matrices

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{bmatrix},$$
$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & -4 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

A. Find the eigenvalues, eigenvectors, basis for the eigenspaces.

B. Decide if they are diagonalizable or not.

C. If yes, find P such that $P^{-1}AP = D$ is diagonal.

D. Can P be orthogonal?

39. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$. Find A^{-1}.

40. Let $B = P^{-1}AP$. Show that if x is an eigenvector of A then $P^{-1}x$ is an eigenvector of B.

41. Consider $A = \begin{bmatrix} 2+2i & -1+3i \\ -2 & 1-i \end{bmatrix}$, (or $\begin{bmatrix} 3 & -1+3i \\ -1-3i & 1 \end{bmatrix}$)

A. Find A^{-1}.

B. Find $|A|$.

C. Solve $Ax = b$ where $b = \begin{bmatrix} 2+i \\ 2-i \end{bmatrix}$.

D. Find the eigenvalues and eigenvectors of A.

42. Let $A = \begin{bmatrix} -2 & -2 & 3 \\ 0 & -2 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

A. Is A diagonalizable? If yes, diagonalize it.

B. Solve $x' = Ax$.

Eigenvectors/eigenvalues: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \leftrightarrow -2, \left\{ \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\} \leftrightarrow 2, \left\{ \begin{bmatrix} -7 \\ -2 \\ 1 \end{bmatrix} \right\} \leftrightarrow -3.$

C. For $x' = Ax$, if $x(0) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, find $x(1)$.