Spring 2012

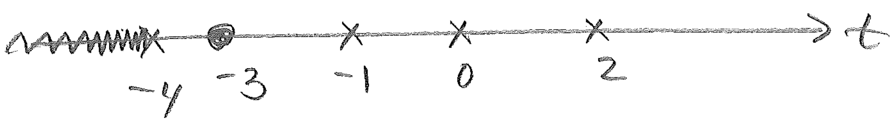
MA 266 - Exam # 1

Name SOLUTIONS **PUID#** ____________________

(5 pts) **1.** Find the largest open interval for which the solution of the following initial value problem is guaranteed to exist: $\begin{cases} t\,(t+1)\,y' + 4\,y = \dfrac{\ln(t+4)}{(t-2)} \\ y(-3) = 2 \end{cases}$.

$p(t) = \frac{4}{t(t+1)}$, $f(t) = \frac{4}{t(t+1)}$

$g(t) = \frac{\ln(t+4)}{t(t+1)(t-2)}$



A. $-4 < t < 0$

B. $-4 < t < -1$

C. $-4 < t < 2$

D. $-\infty < t < -1$

E. $-\infty < t < 0$

(5 pts) **2.** If $y = t^r$ is a solution of $t^2y'' - bty' + by = 0$ $(t > 0)$, then

$y' = rt^{r-1}$

$y'' = r(r-1)t^{r-2}$

$t^2\{r(r-1)t^{r-2}\} - bt\{rt^{r-1}\} + bt^r = 0$

$r(r-1) - br + b = 0$

$r^2 - (1+b)r + b = 0$

$r = \frac{(1+b) \pm \sqrt{(1+b)^2 - 4b}}{2} = \frac{(1+b) \pm 1-b}{2}$

$r = 1$ or $r = b$

A. $r = -b$ or $r = -1$

B. $r = b$ or $r = 1$

C. $r = b + 1$ or $r = 1$

D. $r = \dfrac{b \pm \sqrt{b^2 - 4b}}{2}$

E. $r = \pm 1$

(45 pts) **3.** On the next 3 pages, identify the **TYPE** of differential equation (**First Order Linear**, **Separable**, **Homogeneous**, **Exact**) and then solve it. (If an equation is of more than one type, just list one).

(a) Solve $x\,y' + (x-1)y = 3x^2e^{-x}$ $\quad (x > 0)$. $\quad y' + (1-\frac{1}{x})y = 3xe^{-x}$

$$\mu(x) = e^{\int (1-\frac{1}{x})dx} = e^{x-\ln x} = \frac{e^x}{x}$$

$$y = \frac{1}{\left(\frac{e^x}{x}\right)}\left[\int \left(\frac{e^x}{x}\right)(3x)\,dx + C\right]$$

$$y = \frac{x}{e^x}\left[3x + C\right]$$

TYPE $\boxed{\text{FOL}}$

$$y = \boxed{\frac{x}{e^x}\left[3x + C\right]}$$

3. (cont'd) .

(b) Solve $y\,\dfrac{dy}{dx} = 4\left(\dfrac{y^2-1}{x^2-1}\right)$.

$$\int \frac{y}{y^2 - 1}\,dy = \int \frac{4}{x^2-1}\,dx = \int\left(\frac{2}{x-1} + \frac{-2}{x+1}\right)dx$$

$$\frac{1}{2}\ln|y^2-1| = 2\ln|x-1| - 2\ln|x+1| + C$$

If $y^2-1=0$ i.e. $y=1$ or $y=-1$, then

$y=1$ is a soln since it satisfies d.e.

$y=-1$ " " " " " " "

TYPE SEP

3. (cont'd)

(c) Solve $\begin{cases} (2x+8e^{2y})\,dy + (2x+2y)\,dx = 0 \\ y(1)=0 \end{cases}$

$$\frac{\partial(2x+8e^{2y})}{\partial x} \overset{?}{=} \frac{\partial(2x+2y)}{\partial y}$$

$$2 = 2 \checkmark \qquad \text{EXACT}$$

$$\begin{cases} \dfrac{\partial\psi}{\partial y} = 2x+8e^{2y} \xrightarrow{I_y} \psi = 2xy+4e^{2y}+g(x) \\ \qquad\qquad\qquad\qquad\qquad\qquad \downarrow D_x \\ \dfrac{\partial\psi}{\partial x} = 2x+2y \quad = \quad \dfrac{\partial\psi}{\partial x} = 2y+g'(x) \end{cases}$$

$$\Rightarrow g'(x) = 2x \Rightarrow g(x) = x^2$$

$$\therefore \psi(x,y) = 2xy+4e^{2y}+x^2$$

Or, $\begin{cases} \dfrac{\partial\psi}{\partial x} = 2x+2y \xrightarrow{I_x} \psi = x^2+2xy+h(y) \\ \qquad\qquad\qquad\qquad\qquad\qquad \downarrow D_y \\ \dfrac{\partial\psi}{\partial y} = 2x+8e^{2y} = \dfrac{\partial\psi}{\partial y} = 2x+h'(y) \end{cases}$

$$\Rightarrow h'(y) = 8e^{2y} \Rightarrow h(y) = 4e^{2y}$$

$\therefore \psi(x,y) = 2xy+4e^{2y}+x^2$, same as above

Soln given implicitly by $\psi(x,y) = C$

i.e. $2xy+4e^{2y}+x^2 = C$

TYPE $\boxed{\text{EXACT}}$

Since $y(1)=0$

$\Rightarrow \quad 0+4+1 = C$

$5 = C$

$$\boxed{2xy+4e^{2y}+x^2 = 5}$$

(10 pts) **4.** Find the *explicit* solution of the initial value problem :
$$\begin{cases} \dfrac{dy}{dx} = \dfrac{2x^2+y^2}{xy} = f(x,y) \\ y(1) = -2 \end{cases}.$$

What is $y(2)$?

Since $f(tx,ty) = f(x,y)$ d.e. is homogeneous.

so $\boxed{y = xv}$ and $\dfrac{dy}{dx} = x\dfrac{dv}{dx} + v$

Eqn transforms to $x\dfrac{dv}{dx} + v = \dfrac{2x^2 + x^2v^2}{x^2 v} = \dfrac{2+v^2}{v} = \dfrac{2}{v} + v$

$\Rightarrow x\dfrac{dv}{dx} = \dfrac{2}{v} \Rightarrow \int v\,dv = \int \dfrac{2}{x}\,dx$

$\dfrac{v^2}{2} = 2\ln|x| + C_1 \Rightarrow v^2 = 4\ln|x| + C$

$\dfrac{y^2}{x^2} = 4\ln|x| + C$, since $y(1) = -2$

$\Rightarrow 4 = 0 + C$ so $\dfrac{y^2}{x^2} = 4\ln|x| + 4$

$\therefore y^2 = 4x^2\ln|x| + 4x^2$

Since $y(1) = -2$, $y = -\sqrt{4x^2\ln|x| + 4x^2}$

$y = \boxed{-\sqrt{4x^2\ln|x| + 4x^2}}$

$y(2) = \boxed{-\sqrt{16\ln 2 + 16}}$

(5 pts) **5.** An object of mass 10 kg is projected directly upward from the ground at a speed of 200 m/sec. If the drag force due to air resistance is $50v^2$ and assuming the positive direction is upward, then the initial value problem describing this motion is :

($g = 9.8$ m/sec^2)

A. $10\,\dfrac{dv}{dt} = -(98 + 50v^2), \quad v(0) = 200$

B. $10\,\dfrac{dv}{dt} = 98 - 50v^2, \quad v(0) = 200$

C. $10\,\dfrac{dv}{dt} = -98 + 50v^2, \quad v(0) = 200$

D. $10\,\dfrac{dv}{dt} = 9.8 + 50v^2, \quad v(0) = 200$

E. $10\,\dfrac{dv}{dt} = -9.8 - 50|v|^2, \quad v(0) = 200$

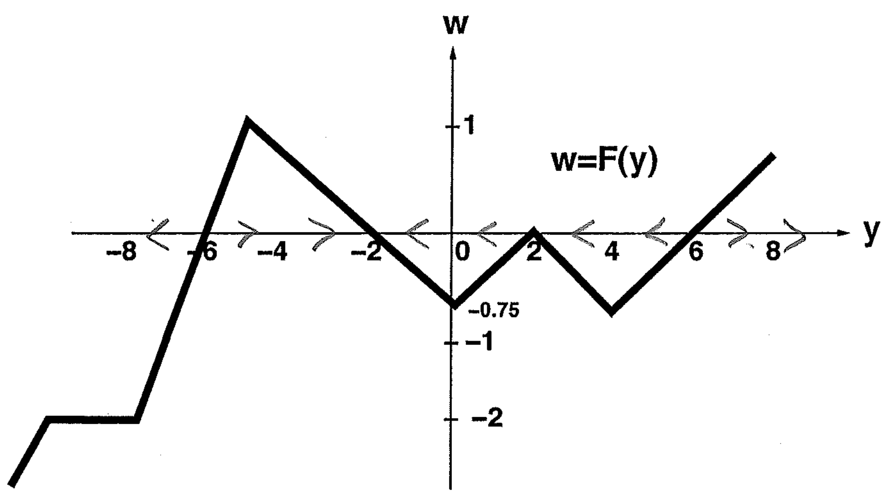
+ direction m

F_g F_d

$m\dfrac{dv}{dt} = -mg - 50v^2$

$10\dfrac{dv}{dt} = -98 - 50v^2$

(10 pts) **6.** Find all equilibrium solutions of the autonomous differential equation $\dfrac{dy}{dt} = F(y)$ and classify their stability (**Unstable, Stable, Semistable**) of each equilibrium solution if the graph of $F(y)$ is as shown below:



Equilibrium Solution	Stability
$y = -6$	unstable
$y = -2$	stable
$y = 2$	semistable
$y = 6$	unstable
$y =$	
$y =$	

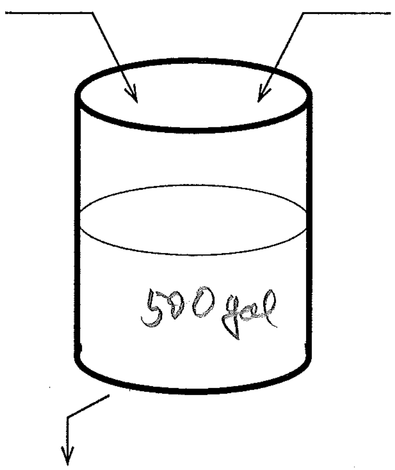
(5 pts) **7.** A 1000 gallon tank is initially half full of pure water with 25 lbs of salt dissolved in it. A solution with a salt concentration of 3 lb/gal is pumped into the tank at a rate of 10 gal/min. Pure water is also pumped into the tank through a second pump at a rate of 2 gal/min. The well-stirred mixture pumped out at 20 gal/min. If $N(t)$ is the number of lbs (pounds) of salt in the tank at time t, the initial value problem describing this process is

$r_{i_2} = 2$ gal/min

$c_{i_2} = 0$

$r_{i_1} = 10$ gal/min

$c_{i_1} = 3$ lb/gal



$r_o = 20$ gal/min

$C_o = \dfrac{N}{500-8t}$

$\therefore \dfrac{dN}{dt} = \{r_{i_1}c_{i_1} + r_{i_2}c_{i_2}\} - r_o C_o$

$\dfrac{dN}{dt} = (30+0) - \dfrac{20N}{500-8t}$

$N(0) = 25$

A. $\begin{cases} \dfrac{dN}{dt} = 36 - \dfrac{20\,N(t)}{500-8t} \\ N(0) = 25 \end{cases}$

B. $\begin{cases} \dfrac{dN}{dt} = 36 - \dfrac{20\,N(t)}{1000-8t} \\ N(0) = 25 \end{cases}$

C. $\begin{cases} \dfrac{dN}{dt} = 30 - \dfrac{20\,N(t)}{500+8t} \\ N(0) = 25 \end{cases}$

D. $\begin{cases} \dfrac{dN}{dt} = 30 - \dfrac{20\,N(t)}{500-8t} \\ N(0) = 25 \end{cases}$

E. $\begin{cases} \dfrac{dN}{dt} = 30 - \dfrac{20\,N(t)}{500-10t} \\ N(0) = 25 \end{cases}$

(15 pts) **8.** Find the solution of the initial value problem $\begin{cases} y'' + 2y' - 3y = 0 \\ y(0) = y_0 \\ y'(0) = 2 \end{cases}$.

For what value(s) of y_0 will the solution be <u>bounded</u> for $0 < t < \infty$?

C.E. $r^2 + 2r - 3 = 0 \Rightarrow (r+3)(r-1) = 0$

$\therefore\ y = c_1 e^{-3t} + c_2 e^{t}$ is general soln to d.e.

$y_0 = y(0) = c_1 + c_2 \Rightarrow c_1 = y_0 - c_2$

$2 = y'(0) = -3c_1 + c_2$

so

$c_2 = 2 + 3c_1 = 2 + 3(y_0 - c_2)$

$c_2 = 2 + 3y_0 - 3c_2$

$4c_2 = 2 + 3y_0$

$c_2 = \dfrac{2+3y_0}{4}$ ✓

and $c_1 = y_0 - \left(\dfrac{2+3y_0}{4}\right) = \left(\dfrac{y_0 - 2}{4}\right)$ ✓

$y = \boxed{\left(\dfrac{y_0 - 2}{4}\right)e^{-3t} + \left(\dfrac{2+3y_0}{4}\right)e^{t}}$

Solution bounded when $y_0 = \boxed{-2/3}$